

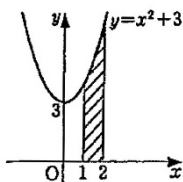
2節 積分法の応用

A 問題

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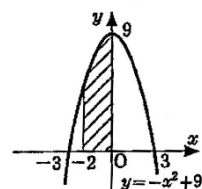
- (1) 求める面積は右図の斜線部分だから

$$\begin{aligned} S &= \int_1^2 (x^2 + 3) dx \\ &= \left[\frac{1}{3} x^3 + 3x \right]_1^2 \\ &= \left(\frac{8}{3} + 6 \right) - \left(\frac{1}{3} + 3 \right) \\ &= \frac{7}{3} + 3 = \frac{16}{3} \end{aligned}$$



- (2) 求める面積は右図の斜線部分だから

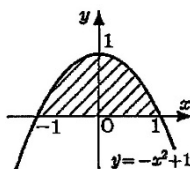
$$\begin{aligned} S &= \int_{-2}^0 (-x^2 + 9) dx \\ &= \left[-\frac{1}{3} x^3 + 9x \right]_{-2}^0 \\ &= 0 - \left(\frac{8}{3} - 18 \right) = \frac{46}{3} \end{aligned}$$



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- (1) 求める面積は右図の斜線部分だから

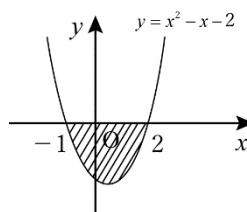
$$\begin{aligned} S &= \int_{-1}^1 (-x^2 + 1) dx \\ &= \left[-\frac{1}{3} x^3 + x \right]_{-1}^1 \\ &= \left(-\frac{1}{3} + 1 \right) - \left(\frac{1}{3} - 1 \right) = \frac{4}{3} \end{aligned}$$



- (2) $y = (x+1)(x-2)$ より

求める面積は右の斜線部分だから

$$\begin{aligned} S &= -\int_{-1}^2 (x^2 - x - 2) dx \\ &= -\left[\frac{1}{3} x^3 - \frac{1}{2} x^2 - 2x \right]_{-1}^2 \\ &= -\left(\frac{8}{3} - 2 - 4 \right) + \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) \\ &= \frac{9}{2} \end{aligned}$$



別解 教科書 p.135 練習 30 の積分計算を使うと

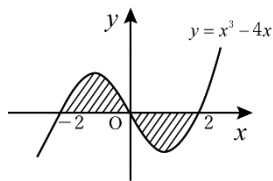
$$\begin{aligned} (1) \quad \int_{-1}^1 (-x^2 + 1) dx &= \int_{-1}^1 (x+1)(x-1) dx \\ &= \frac{(1+1)^3}{6} = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} (2) \quad \int_{-1}^2 (x^2 - x - 2) dx &= -\int_{-1}^2 (x+1)(x-2) dx \\ &= -\frac{(2+1)^3}{6} = \frac{9}{2} \end{aligned}$$

(3) $y = x^3 - 4x = x(x+2)(x-2)$ より

求める面積は右の斜線部分だから

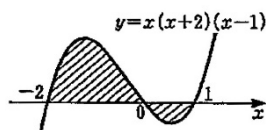
$$\begin{aligned} S &= -2 \int_0^2 (x^3 - 4x) dx \\ &= -2 \left[\frac{1}{4} x^4 - 2x \right]_0^2 \\ &= -2(4 - 8) = 8 \end{aligned}$$



(4) $y = x(x+2)(x-1)$ より

求める面積は右の斜線部分だから

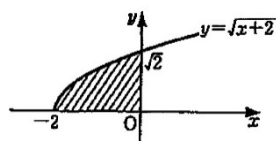
$$\begin{aligned} S &= \int_{-2}^0 (x^3 + x^2 - 2x) dx \\ &= - \int_0^1 (x^3 + x^2 - 2x) dx = \frac{37}{12} \end{aligned}$$



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(1) 求める面積は下の斜線部分だから

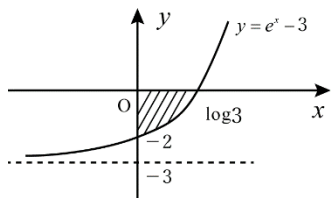
$$\begin{aligned} S &= \int_{-2}^0 \sqrt{x+2} dx = \left[\frac{2}{3} (x+2)^{\frac{3}{2}} \right]_{-2}^0 \\ &= \frac{4\sqrt{2}}{3} \end{aligned}$$



(2) 交点の x 座標は

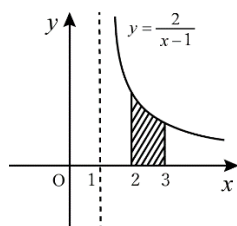
$$e^x - 3 = 0, \quad e^x = 3 \quad \text{より} \quad x = \log 3$$

求める面積は下の斜線部分だから



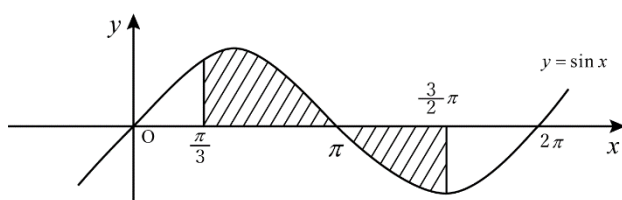
$$\begin{aligned} S &= - \int_0^{\log 3} (e^x - 3) dx = - \left[e^x - 3x \right]_0^{\log 3} \\ &= (-e^{\log 3} + 3 \log 3) + 1 = 3 \log 3 - 2 \end{aligned}$$

(3) 求める面積は下の斜線部分だから



$$S = \int_2^3 \frac{2}{x-1} dx = \left[2 \log(x-1) \right]_2^3 = 2 \log 2$$

(4) 求める面積は下の斜線部分だから



$$\begin{aligned} S &= \int_{\frac{\pi}{3}}^{\pi} \sin x \, dx - \int_{\pi}^{\frac{3}{2}\pi} \sin x \, dx \\ &= \left[-\cos x \right]_{\frac{\pi}{3}}^{\pi} - \left[-\cos x \right]_{\pi}^{\frac{3}{2}\pi} \\ &= 1 - \left(-\frac{1}{2} \right) + 1 = \frac{5}{2} \end{aligned}$$

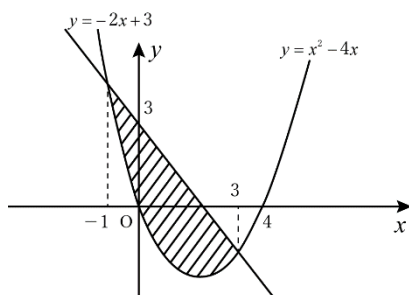
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(1) 交点の x 座標は

$$x^2 - 4x = -2x + 3 \text{ より } (x+1)(x-3) = 0$$

$$\therefore x = -1, 3$$

求める面積は下の斜線部分だから



$$\begin{aligned} S &= \int_{-1}^3 \left\{ (-2x+3) - (x^2-x) \right\} dx \\ &= \int_{-1}^3 (-x^2 + 2x + 3) dx = \left[-\frac{1}{3}x^3 + x^2 + 3x \right]_{-1}^3 \\ &= (-9+9+9) - \left(\frac{1}{3} + 1 - 3 \right) = \frac{32}{3} \end{aligned}$$

別解

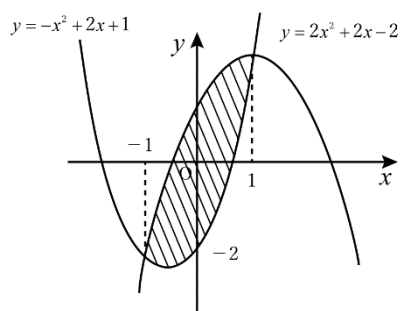
$$S = -\int_{-1}^3 (x+1)(x-3) dx = \frac{(3+1)^4}{6} = \frac{32}{3}$$

(2) 交点の x 座標は

$$2x^2 + 2x - 2 = -x^2 + 2x + 1 \text{ より } 3x^2 = 3$$

$$\therefore x = \pm 1$$

求める面積は下の斜線部分だから



$$\begin{aligned} S &= \int_{-1}^1 \{ (-x^2 + 2x + 1) - (2x^2 + 2x - 2) \} dx \\ &= \int_{-1}^1 (-3x^2 + 3) dx = \left[-x^3 + 3x \right]_{-1}^1 \\ &= (1+3) - (-1-3) = 4 \end{aligned}$$

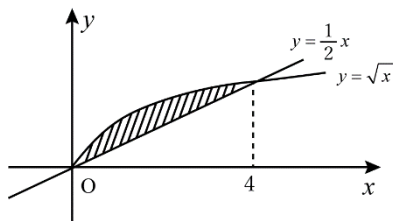
別解

$$S = -3 \int_{-1}^1 (x+1)(x-1) dx = \frac{3(1+1)^3}{6} = 4$$

(3) 交点の x 座標は

$$\sqrt{x} = \frac{1}{2}x, \quad 4x = x^2 \text{ より } x = 0, \quad 4$$

求める面積は下の斜線部分だから



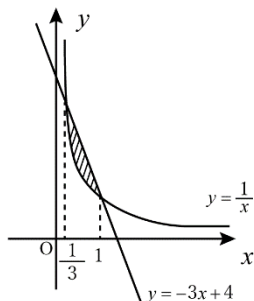
$$\begin{aligned} S &= \int_0^4 \left(\sqrt{x} - \frac{1}{2}x \right) dx \\ &= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^2 \right]_0^4 \\ &= \frac{16}{3} - 4 = \frac{4}{3} \end{aligned}$$

(4) 交点の x 座標は

$$\frac{1}{x} = -3x + 4 \text{ より } 3x^2 - 4x + 1 = 0$$

$$(3x-1)(x-1) = 0 \quad \therefore x = \frac{1}{3}, 1$$

求める面積は下図の斜線部分だから



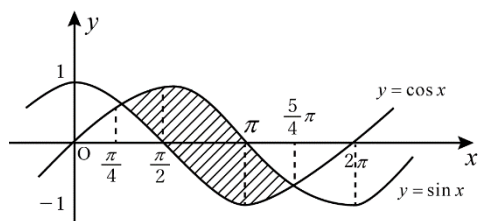
$$\begin{aligned} S &= \int_{\frac{1}{3}}^1 \left(-3x + 4 - \frac{1}{x} \right) dx \\ &= \left[-\frac{3}{2}x^2 + 4x - \log x \right]_{\frac{1}{3}}^1 \\ &= \left(-\frac{3}{2} + 4 \right) - \left(-\frac{1}{6} + \frac{4}{3} - \log \frac{1}{3} \right) \\ &= \frac{4}{3} - \log 3 \end{aligned}$$

(5) 交点の x 座標は

$$\sin x = \cos x \text{ より } \sqrt{2} \sin \left(x - \frac{\pi}{4} \right) = 0$$

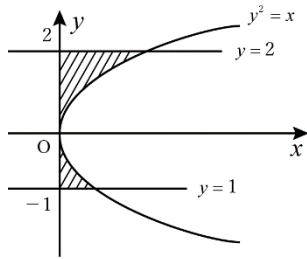
$$x - \frac{\pi}{4} = 0, \pi \quad \therefore x = \frac{\pi}{4}, \frac{5}{4}\pi$$

求める面積は下図の斜線部分だから



$$\begin{aligned} S &= \int_{\frac{\pi}{4}}^{\frac{5}{4}\pi} (\sin x - \cos x) dx = \left[-\cos x - \sin x \right]_{\frac{\pi}{4}}^{\frac{5}{4}\pi} \\ &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) - \left(-\frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{2} \right) = 2\sqrt{2} \end{aligned}$$

- (1) 求める面積は下図の斜線部分だから

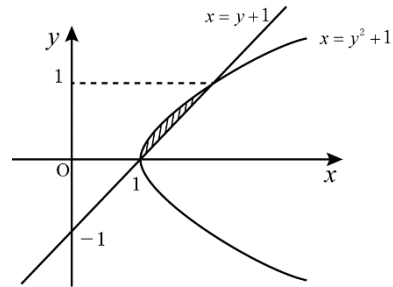


$$\begin{aligned}
 S &= \int_0^2 y^2 dy + \int_{-1}^0 y^2 dy \\
 &= \left[\frac{1}{3} y^3 \right]_0^2 + \left[\frac{1}{3} y^3 \right]_{-1}^0 \\
 &= \frac{8}{3} + \frac{1}{3} = 3
 \end{aligned}$$

- (2) 交点の
- y
- 座標は

$$y^2 + 1 = y + 1 \text{ より } y(y-1) = 0 \quad \therefore y = 0, 1$$

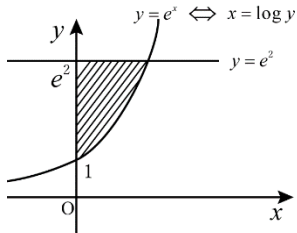
求める面積は下図の斜線部分だから



$$\begin{aligned}
 S &= \int_0^1 \{(y+1) - (y^2+1)\} dy \\
 &= \left[\frac{1}{2} y^2 - \frac{1}{3} y^3 \right]_0^1 \\
 &= \frac{1}{6}
 \end{aligned}$$

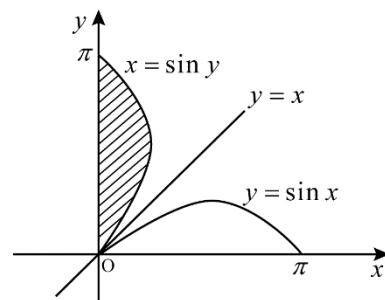
- (3)
- $y = e^x$
- は
- $x = \log y$

求める面積は下図の斜線部分だから



$$\begin{aligned}
 S &= \int_1^{e^2} \log y dy \\
 &= \left[y \log y - y \right]_1^{e^2} \\
 &= (2e^2 - e^2) - (-1) = e^2 + 1
 \end{aligned}$$

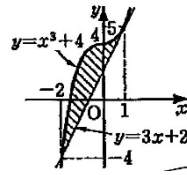
- (4) 求める面積は下図の斜線部分だから



$$\begin{aligned}
 S &= \int_0^\pi \sin y dy \\
 &= \left[-\cos y \right]_0^\pi \\
 &= 1 - (-1) = 2
 \end{aligned}$$

- 187 $y' = 3x^2$ より, 接線の方程式は $y = 3x + 2$
 $x^3 + 4 = 3x + 2$ より, 共有点の x 座標は
 $(x-1)^2(x+2) = 0$
 $\therefore x = 1, -2$

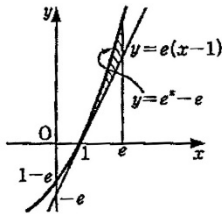
$$\begin{aligned} \text{よって } S &= \int_{-2}^1 \{ (x^3 + 4) - (3x + 2) \} dx \\ &= \left[\frac{x^4}{4} - \frac{3}{2}x^2 + 2x \right]_{-2}^1 = \frac{27}{4} \end{aligned}$$



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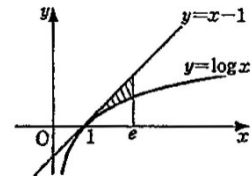
- (1) $y = e^x - e$ より $y' = e^x$
 $x = 1$ のとき $y' = e$ となるから
接線の方程式は $y = e(x-1)$
求める面積は下図の斜線部分だから

$$\begin{aligned} S &= \int_1^e \{ (e^x - e) - e(x-1) \} dx \\ &= \int_1^e (e^x - ex) dx = \left[e^x - \frac{1}{2}ex^2 \right]_1^e \\ &= \left(e^e - \frac{1}{2}e^3 \right) - \left(e - \frac{1}{2}e \right) \\ &= e^e - \frac{1}{2}e^3 - \frac{1}{2}e \end{aligned}$$



- (2) $y = \log x$ より $y' = \frac{1}{x}$
 $x = 1$ のとき $y' = 1$ となるから
接線の方程式は $y = x - 1$
求める面積は下図の斜線部分だから

$$\begin{aligned} S &= \int_1^e \{ (x-1) - \log x \} dx \\ &= \left[\frac{1}{2}x^2 - x \right]_1^e - \left[x \log x - x \right]_1^e \\ &= \left(\frac{1}{2}e^2 - e \right) - \left(\frac{1}{2} - 1 \right) - (e \log e - e + 1) \\ &= \frac{1}{2}e^2 - e - \frac{1}{2} \end{aligned}$$



$$(1) \quad \frac{x^2}{9} + y^2 = 1 \text{ より } y = \pm \sqrt{1 - \frac{x^2}{9}}$$

$$S = 4 \int_0^3 \sqrt{1 - \frac{x^2}{9}} dx = \frac{4}{3} \int_0^3 \sqrt{9 - x^2} dx$$

$$= \int_0^3 \sqrt{9 - x^2} dx \text{ は半径 } 3 \text{ の円の面積の } \frac{1}{4}$$

を表すから

$$S = \frac{4}{3} \cdot \frac{1}{4} \cdot 3^2 \pi = 3\pi$$

$$(2) \quad y^2 = x^2(1-x) \text{ より } y = \pm x\sqrt{1-x}$$

$$S = 2 \int_0^1 x\sqrt{1-x} dx$$

$1-x=t$ とおくと

$dx = -dt$ だから

x	$0 \rightarrow 1$
t	$1 \rightarrow 0$

$$S = 2 \int_0^1 (1-t)\sqrt{t} dt = 2 \int_0^1 \left(t^{\frac{1}{2}} - t^{\frac{3}{2}} \right) dt$$

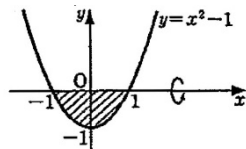
$$= 2 \left[\frac{2}{3} t^{\frac{3}{2}} - \frac{2}{5} t^{\frac{5}{2}} \right]_0^1 = 2 \left(\frac{2}{3} - \frac{2}{5} \right) = \frac{8}{15}$$

(1) 求める体積は下図の斜線部分だから

$$V = \pi \int_{-1}^1 (x^2 - 1)^2 dx$$

$$= 2\pi \int_0^1 (x^4 - 2x^2 + 1) dx$$

$$= 2\pi \left[\frac{x^5}{5} - \frac{2}{3} x^3 + x \right]_0^1 = \frac{16}{15} \pi$$

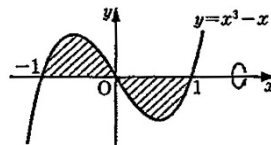


(2) 求める体積は下図の斜線部分だから

$$V = \pi \int_{-1}^1 (x^3 - x)^2 dx$$

$$= 2\pi \int_0^1 (x^6 - 2x^4 + x^2) dx$$

$$= 2\pi \left[\frac{x^7}{7} - \frac{2}{5} x^5 + \frac{x^3}{3} \right]_0^1 = \frac{16}{105} \pi$$

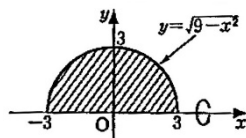


(3) 求める体積は右図の斜線部分だから

$$V = \pi \int_{-3}^3 (\sqrt{9 - x^2})^2 dx$$

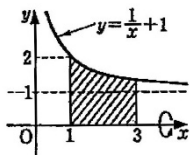
$$= 2\pi \int_0^3 (9 - x^2) dx$$

$$= 2\pi \left[9x - \frac{x^3}{3} \right]_0^3 = 36\pi$$



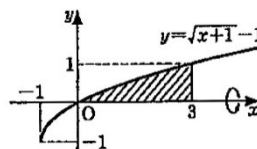
(1) 求める体積は下図の斜線部分だから

$$\begin{aligned}
 V &= \pi \int_1^3 \left(\frac{1}{x} + 1 \right)^2 dx \\
 &= \pi \int_1^3 \left(\frac{1}{x^2} + \frac{2}{x} + 1 \right) dx \\
 &= \pi \left[-\frac{1}{x} + 2 \log x + x \right]_1^3 = \frac{8 + 6 \log 3}{3} \pi
 \end{aligned}$$



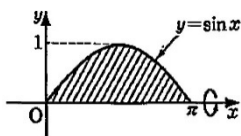
(2) 求める体積は下図の斜線部分だから

$$\begin{aligned}
 V &= \pi \int_0^3 \left(\sqrt{x+1} - 1 \right)^2 dx \\
 &= \pi \int_0^3 \left(x + 2 - 2\sqrt{x+1} \right) dx \\
 &= \pi \left[\frac{x^2}{2} + 2x - \frac{4}{3} (x+1)^{3/2} \right]_0^3 = \frac{7}{6} \pi
 \end{aligned}$$



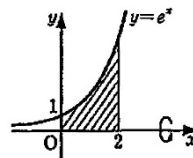
(3) 求める体積は下図の斜線部分だから

$$\begin{aligned}
 V &= \pi \int_0^\pi \sin^2 x \, dx \\
 &= \pi \int_0^\pi \frac{1 - \cos 2x}{2} \, dx \\
 &= \pi \left[\frac{1}{2} x - \frac{1}{4} \sin 2x \right]_0^\pi = \frac{1}{2} \pi^2
 \end{aligned}$$

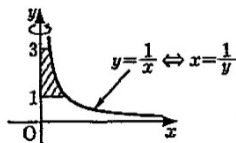


(4) 求める体積は下図の斜線部分だから

$$\begin{aligned}
 V &= \pi \int_0^2 e^{2x} \, dx \\
 &= \pi \left[\frac{1}{2} e^{2x} \right]_0^2 \\
 &= \frac{e^4 - 1}{2} \pi
 \end{aligned}$$

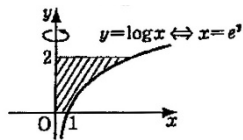
(1) $y = \frac{1}{x}$ より $x = \frac{1}{y}$

$$\begin{aligned}
 V &= \pi \int_1^3 x^2 \, dy = \pi \int_1^3 \frac{1}{y^2} \, dy \\
 &= \pi \left[-\frac{1}{y} \right]_1^3 = \frac{2}{3} \pi
 \end{aligned}$$



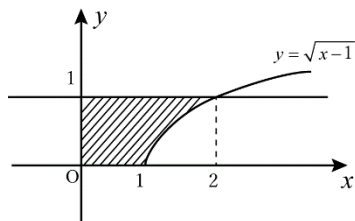
(2) $y = \log x$ と $x = e^y$

$$\begin{aligned} V &= \pi \int_0^2 x^2 dy = \pi \int_0^2 e^{2y} dy \\ &= \pi \left[\frac{1}{2} e^{2y} \right]_0^2 = \frac{e^4 - 1}{2} \pi \end{aligned}$$



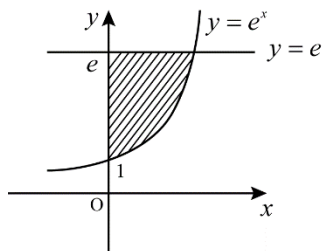
(3) $y = \sqrt{x-1}$ と $x = y^2 + 1$

$$\begin{aligned} V &= \pi \int_0^1 x^2 dy = \pi \int_0^1 (y^2 + 1)^2 dy \\ &= \pi \int_0^1 (y^4 + 2y^2 + 1) dy = \pi \left[\frac{1}{5} y^5 + \frac{2}{3} y^3 + y \right]_0^1 \\ &= \pi \left(\frac{1}{5} + \frac{2}{3} + 1 \right) = \frac{28}{15} \pi \end{aligned}$$



(4) $y = e^x$ と $x = \log y$

$$\begin{aligned} V &= \pi \int_1^e x^2 dy = \pi \int_1^e (\log y)^2 dy \\ &= \pi \int_1^e y' (\log y)^2 dy = \pi \left[y (\log y)^2 \right]_1^e - \pi \int_1^e y \cdot 2 (\log y) \cdot \frac{1}{y} dy \\ &= \pi e - \pi \int_1^e 2 \log y dy = \pi e - 2\pi [y \log y - y]_1^e \\ &= \pi e - 2\pi (-1) = (e + 2) \pi \end{aligned}$$



$$(1) \quad \text{与式} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sqrt{\frac{k}{n}} = \int_0^1 \sqrt{x} \, dx = \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^1 = \frac{2}{3}$$

$$\begin{aligned} (2) \quad \text{与式} &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{n+1}{n} + \frac{n+2}{n} + \frac{n+3}{n} + \cdots + \frac{n+n}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \left(1 + \frac{1}{n} \right) + \left(1 + \frac{2}{n} \right) + \left(1 + \frac{3}{n} \right) + \cdots + \left(1 + \frac{n}{n} \right) \right\} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(1 + \frac{k}{n} \right) = \int_0^1 (1+x) \, dx \\ &= \left[\frac{1}{2} (1+x)^2 \right]_0^1 = \frac{3}{2} \end{aligned}$$

$$(3) \quad \text{与式} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{\frac{k}{n}} = \int_0^1 e^x \, dx = \left[e^x \right]_0^1 = e - 1$$

(1) $0 \leq x \leq 1$ のとき $x > x^3$ だから

$$x+1 \geq x^3+1 \geq 1$$

逆数をとると

$$\frac{1}{x+1} \leq \frac{1}{x^3+1} \leq 1$$

となる。よって、示された。

(2) (1)の不等式の等号は $x=0$, 1 のときだけだから

$$\int_0^1 \frac{1}{x+1} \, dx < \int_0^1 \frac{1}{x^3+1} \, dx < \int_0^1 dx$$

が成り立つ。

$$\left[\log(x+1) \right]_0^1 < \int_0^1 \frac{1}{x^3+1} \, dx < \left[x \right]_0^1$$

よって、 $\log 2 < \int_0^1 \frac{1}{x^3+1} \, dx < 1$ が成り立つ。

195 自然数 k に対して $k \leq x \leq k+1$ のとき

$$\frac{1}{k+1} \leq \frac{1}{x} \text{ より } \frac{1}{(k+1)^2} < \frac{1}{x^2}$$

等号が成り立つのは $x=k+1$ のときだけだから

$$\int_k^{k+1} \frac{1}{(k+1)^2} dx < \int_k^{k+1} \frac{1}{x^2} dx$$

すなわち

$$\frac{1}{(k+1)^2} < \int_k^{k+1} \frac{1}{x^2} dx$$

$k=1, 2, 3, \dots, n-1$ として、辺々加えると

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} < \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x^2} dx = \int_1^n \frac{1}{x^2} dx$$

$$\text{右辺} = \int_1^n \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^n = -\frac{1}{n} + 1$$

$$\text{よって, } \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} < 1 - \frac{1}{n}$$

が成り立つ。

B 問題

196 $y = x^2 \dots \textcircled{1}$ とすると $y' = 2x$

①上の点 (t, t^2) における接線の方程式は $y - t^2 = 2t(x - t)$

$$\therefore y = 2tx - t^2 \dots \textcircled{2}$$

これが点 $(1, -3)$ を通るためには

$$-3 = 2t - t^2$$

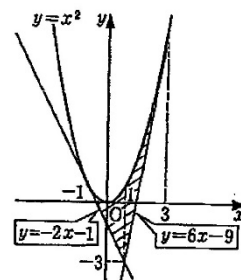
$$t^2 - 2t - 3 = 0$$

$$(t+1)(t-3) = 0$$

$$\therefore t = -1, 3$$

$t = -1$ のとき②から $y = -2x - 1$

$t = 3$ のとき②から $y = 6x - 9$



また求める面積は右図の斜線部分になるから

$$\text{求める面積を } S \text{ とすると } S = \int_{-1}^1 \{x^2 - (-2x - 1)\} dx + \int_1^3 \{x^2 - (6x - 9)\} dx$$

$$= \int_{-1}^1 (x^2 + 2x + 1) dx + \int_1^3 (x^2 - 6x + 9) dx$$

$$= 2 \left[\frac{1}{3} x^3 + x \right]_0^1 + \left[\frac{1}{3} x^3 - 3x^2 + 9x \right]_1^3$$

$$= 2 \left\{ \left(\frac{1}{3} + 1 \right) - 0 \right\} + (9 - 27 + 27) - \left(\frac{1}{3} - 3 + 9 \right)$$

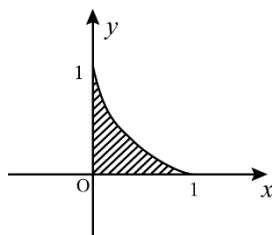
$$= \frac{16}{3}$$

197 $\sqrt{x} + \sqrt{y} = 1$ より $\sqrt{y} = 1 - x$ として、両辺 2 乗する。

$$y = 1 - 2\sqrt{x} + x$$

求める面積は、右図の斜線部分だから

$$\begin{aligned} S &= \int_0^1 (1 - 2\sqrt{x} + x) dx \\ &= \left[x - \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 \right]_0^1 \\ &= 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6} \end{aligned}$$



198 交点の x 座標

$$2 \cos x = 3 \tan x$$

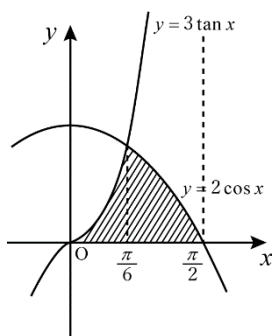
$$2 \cos^2 x = 3 \sin x$$

$$2 \sin^2 x + 3 \sin x - 2 = 0$$

$$(2 \sin x - 1)(\sin x + 2) = 0$$

$$\sin x = \frac{1}{2}$$

$$0 \leq x \leq \frac{\pi}{2} \text{ より } x = \frac{\pi}{6}$$



求める面積は右図の斜線部分だから

$$\begin{aligned} S &= \int_0^{\frac{\pi}{6}} 3 \tan x dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2 \cos x dx \\ &= 3 \int_0^{\frac{\pi}{6}} \frac{(-\cos x)'}{\cos x} dx + 2 \left[\sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= 3 \left[\log(\cos x) \right]_0^{\frac{\pi}{6}} + 2 \left(1 - \frac{1}{2} \right) \\ &= 1 - 3 \log \frac{\sqrt{3}}{2} \end{aligned}$$

(1) グラフはともに原点に関して対称であり、

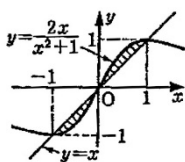
交点の x 座標は

$$\frac{2x}{x^2+1} = x \quad \text{より} \quad x(x+1)(x-1) = 0$$

$$\therefore x = 0, \pm 1$$

求める面積は下図の斜線部分

$$\begin{aligned} S &= 2 \int_0^1 \left(\frac{2x}{x^2+1} - x \right) dx \\ &= 2 \left[\log(x^2+1) - \frac{1}{2}x^2 \right]_0^1 = 2 \log 2 - 1 \end{aligned}$$

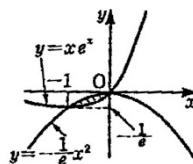
(2) 交点の x 座標は

$$xe^x = -\frac{1}{e}x^2 \quad \text{より} \quad x(e^{x+1} + x) = 0$$

$$\therefore x = 0, -1$$

求める面積は下図の斜線部分

$$\begin{aligned} S &= \int_{-1}^0 \left(-\frac{1}{e}x^2 - xe^x \right) dx \\ &= \left[-\frac{1}{3e}x^3 \right]_{-1}^0 - \left[xe^x \right]_{-1}^0 + \int_{-1}^0 e^x dx \\ &= -\frac{1}{3e} - \frac{1}{e} + \left[e^x \right]_{-1}^0 = 1 - \frac{7}{3e} \end{aligned}$$



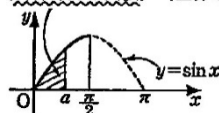
$$200 \quad 2 \int_0^a \sin x \, dx = \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$2 \left[-\cos x \right]_0^a = \left[-\cos x \right]_0^{\frac{\pi}{2}}$$

$$-2 \cos a + 2 = 1 \quad \cos a = \frac{1}{2}$$

$$0 < a < \frac{\pi}{2} \quad \text{より} \quad a = \frac{\pi}{3}$$

2 × (斜線部分の面積) = (全体の面積)



201

切り口の断面積を $S(x)$ とおくと

$$S(x) = \frac{1}{2} \cdot \sin x \cdot \sin x \cdot \sin 60^\circ = \frac{\sqrt{3}}{4} \sin^2 x$$

$$\text{よって} \quad V = \int_0^\pi \frac{\sqrt{3}}{4} \sin^2 x \, dx$$

$$= \frac{\sqrt{3}}{4} \int_0^\pi \frac{1 - \cos 2x}{2} dx$$

$$= \frac{\sqrt{3}}{8} \left[x - \frac{1}{2} \sin 2x \right]_0^\pi = \frac{\sqrt{3}}{8} \pi$$

- (1) $y = x^2 - 4 \cdots \textcircled{1}$, $y = 3x \cdots \textcircled{2}$ のグラフは
 下のようになり, $\textcircled{1}$, $\textcircled{2}$ の $y < 0$ の部分を
 x 軸に対称に折り返して考える。

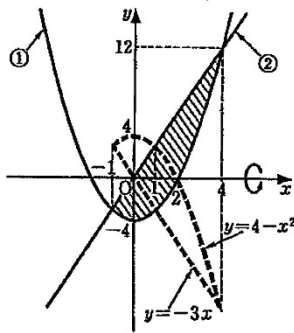
$\textcircled{1}$, $\textcircled{2}$ の交点は

$$x^2 - 4 = 3x \text{ より } (x+1)(x-4) = 0$$

$$\therefore x = -1, 4$$

また, $y = x^2 - 4$ は y 軸に関して対称だから

$$\begin{aligned} V &= 2\pi \int_0^1 (4 - x^2)^2 dx - \pi \int_{-1}^0 (-3x)^2 dx \\ &\quad + \pi \int_1^4 (3x)^2 dx - \pi \int_2^4 (x^2 - 4)^2 dx \\ &= 2\pi \left[\frac{x^5}{5} - \frac{8}{3}x^3 + 16x \right]_0^1 - \pi \left[3x^3 \right]_{-1}^0 \\ &\quad + \pi \left[3x^3 \right]_1^4 - \pi \left[\frac{x^5}{5} - \frac{8}{3}x^3 + 16x \right]_2^4 = 132\pi \end{aligned}$$



- (2) $y = \sin x \cdots \textcircled{1}$, $y = \sin 2x \cdots \textcircled{2}$ のグラフは
 下のようになり, $\textcircled{2}$ の $y < 0$ の部分を
 x 軸に対称に折り返して考える。

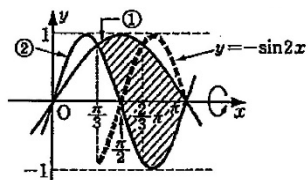
$\textcircled{1}$, $\textcircled{2}$ の交点は

$$\sin x = \sin 2x \text{ より } \sin x (2 \cos x - 1) = 0$$

$$\sin x = 0, \cos x = \frac{1}{2} \quad \frac{\pi}{3} \leq x \leq \pi \text{ より}$$

$$x = \frac{\pi}{3}, \pi$$

$$\begin{aligned} V &= \pi \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \sin^2 x \, dx + \pi \int_{\frac{2}{3}\pi}^{\pi} \sin^2 2x \, dx \\ &\quad - \pi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2 2x \, dx \\ &= \frac{\pi}{2} \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} (1 - \cos 2x) \, dx + \frac{\pi}{2} \int_{\frac{2}{3}\pi}^{\pi} (1 - \cos 4x) \, dx \\ &\quad - \frac{\pi}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 - \cos 4x) \, dx \\ &= \frac{\pi}{2} \left\{ \left[x - \frac{1}{2} \sin 2x \right]_{\frac{\pi}{3}}^{\frac{2}{3}\pi} + \left[x - \frac{1}{4} \sin 4x \right]_{\frac{2}{3}\pi}^{\pi} \right. \\ &\quad \left. - \left[x - \frac{1}{4} \sin 4x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \right\} = \frac{2\pi^2 + 3\sqrt{3}\pi}{8} \end{aligned}$$



$$x^2 + (y-b)^2 = a^2 \text{ より}$$

$$y = b \pm \sqrt{a^2 - x^2}$$

$$y_1 = b + \sqrt{a^2 - x^2}, \quad y_2 = b - \sqrt{a^2 - x^2}$$

とおくと

$$V = \pi \int_{-a}^a y_1^2 dx - \pi \int_{-a}^a y_2^2 dx$$

$$= 2\pi \int_0^a (y_1^2 - y_2^2) dx$$

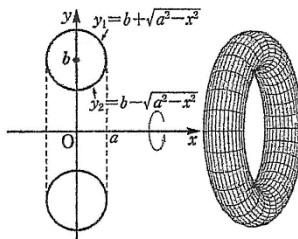
$$= 2\pi \int_0^a 4b\sqrt{a^2 - x^2} dx$$

$$= 8\pi b \int_0^a \sqrt{a^2 - x^2} dx$$

ここで, $\int_0^a \sqrt{a^2 - x^2} dx$ は原点を中心とし,

$$\text{半径 } a \text{ の円の } \frac{1}{4} \text{ の面積を表すから, } \int_0^a \sqrt{a^2 - x^2} dx = \frac{1}{4} \pi a^2$$

$$\text{よって, } V = 8\pi b \times \frac{1}{4} \pi a^2 = 2\pi^2 a^2 b$$

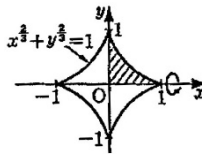


- 204 曲線 $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ は右図のようなアステロイドと呼ばれる曲線である。

$$y^{\frac{2}{3}} = 1 - x^{\frac{2}{3}} \text{ より } y^2 = \left(1 - x^{\frac{2}{3}}\right)^3$$

$$V = \pi \int_0^1 y^2 dx = \pi \int_0^1 \left(1 - x^{\frac{2}{3}}\right)^3 dx = \pi \int_0^1 \left(1 - 3x^{\frac{2}{3}} + 3x^{\frac{4}{3}} - x^2\right) dx$$

$$= \pi \left[x - \frac{9}{5} x^{\frac{5}{3}} + \frac{9}{7} x^{\frac{7}{3}} - \frac{1}{3} x^3 \right]_0^1 = \frac{16}{105} \pi$$



$$(1) \quad 2 \sin x = 1 \quad \text{とすると} \quad \sin x = \frac{1}{2}$$

$$0 \leq x \leq \pi \quad \text{より} \quad x = \frac{\pi}{6}, \frac{5}{6}\pi$$

グラフは直線 $x = \frac{\pi}{2}$ に関して対称

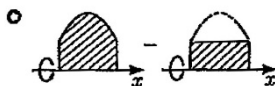
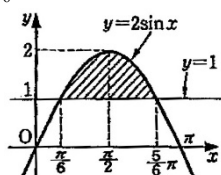
であるから、求める体積 V は

$$V = 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2 \sin x)^2 dx - 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1^2 dx$$

$$= 8\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1 - \cos 2x}{2} dx - 2\pi \left[x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 8\pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{2}{3}\pi^2$$

$$= \frac{(2\pi + 3\sqrt{3})\pi}{3}$$



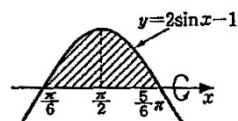
$$(2) \quad V = 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2 \sin x - 1)^2 dx$$

$$= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (4 \sin^2 x - 4 \sin x + 1) dx$$

$$= 2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \{ 2(1 - \cos 2x) - 4 \sin x + 1 \} dx$$

$$= 2\pi \left[-\sin 2x + 4 \cos x + 3x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= (2\pi - 3\sqrt{3})\pi$$



関数 $y = x^3$ と直線 $y = x$ の交点は

$A(1, 1)$, また $OA = \sqrt{2}$ である。

$0 \leq x \leq 1$ として, $y = x^3$ 上の点 $P(x, x^3)$ から直線 $y = x$ に垂線 PH を下ろし $PH = l$, $OH = t$ とおくと, 右図で $\triangle PHQ$ は二等辺三角形で

$$l = PH = \frac{PQ}{\sqrt{2}} = \frac{x - x^3}{\sqrt{2}}$$

また, $\triangle ORQ$ で $\frac{x}{t+l} = \cos 45^\circ = \frac{1}{\sqrt{2}}$ より

$$t = \sqrt{2}x - l = \sqrt{2}x - \frac{x - x^3}{\sqrt{2}} = \frac{x + x^3}{\sqrt{2}}$$

したがって $\frac{dt}{dx} = \frac{1+2x^2}{\sqrt{2}}$

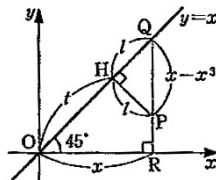
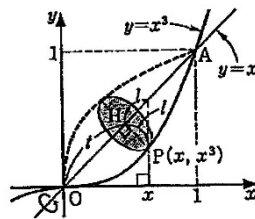
$$dt = \frac{1+2x^2}{\sqrt{2}} dx$$

t	$0 \rightarrow \sqrt{2}$
x	$0 \rightarrow 1$

断面積は πl^2

よって $V = \int_0^1 \pi l^2 dt$

$$\begin{aligned} &= \pi \int_0^1 \left(\frac{x - x^3}{\sqrt{2}} \right)^2 \cdot \frac{1+2x^2}{\sqrt{2}} dx \\ &= \frac{\pi}{2\sqrt{2}} \int_0^1 (2x^5 - 3x^4 + x^2) dx \\ &= \frac{\pi}{2\sqrt{2}} \left[\frac{1}{3} x^6 - \frac{3}{5} x^5 + \frac{1}{3} x^3 \right]_0^1 \\ &= \frac{\pi}{2\sqrt{2}} \left(\frac{1}{3} - \frac{3}{5} + \frac{1}{5} \right) = \frac{\sqrt{2}}{60} \pi \end{aligned}$$



3 章の問題

1

$$\begin{aligned}
 (1) \quad \int \sqrt[3]{x} \log x \, dx &= \int \left(\frac{3}{4} x^{\frac{4}{3}} \right)' \log x \, dx \\
 &= \frac{3}{4} x^{\frac{4}{3}} \log x - \int \frac{3}{4} x^{\frac{4}{3}} (\log x)' \, dx \\
 &= \frac{3}{4} x^{\frac{4}{3}} \log x - \int \frac{3}{4} x^{\frac{1}{3}} \, dx \\
 &= \frac{3}{4} x^{\frac{4}{3}} \log x - \frac{9}{16} x^{\frac{4}{3}} + C \\
 &= \frac{3}{4} x \sqrt[3]{x} \left(\log x - \frac{3}{4} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \int x 2^x \, dx &= \int \left(\frac{2^x}{\log 2} \right)' \cdot x \, dx \\
 &= \frac{x \cdot 2^x}{\log 2} - \int \frac{2^x}{\log 2} (x)' \, dx \\
 &= \frac{x \cdot 2^x}{\log 2} - \frac{1}{\log 2} \int 2^x \, dx \\
 &= \frac{x \cdot 2^x}{\log 2} - \frac{2^x}{(\log 2)^2} + C
 \end{aligned}$$

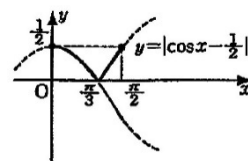
$$\begin{aligned}
 (3) \quad \int_0^\pi \sqrt{1 - \cos x} \, dx &= \int_0^\pi \sqrt{1 - \left(1 - 2 \sin^2 \frac{x}{2} \right)} \, dx \\
 &= \int_0^\pi \sqrt{2} \sin \frac{x}{2} \, dx \\
 &= \sqrt{2} \left[-2 \cos \frac{x}{2} \right]_0^\pi = 2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_0^1 \frac{1}{e^x + 1} \, dx \quad e^x + 1 = t \quad \text{とおくと} \\
 e^x = t - 1 \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow 1 \\ \hline t & 0 \rightarrow e + 1 \\ \hline \end{array} \\
 e^x dx = dt \\
 \therefore dx = \frac{dt}{e^x} = \frac{dt}{t-1}
 \end{aligned}$$

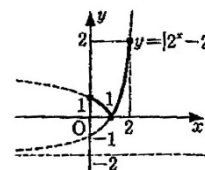
$$\begin{aligned}
 (\text{与式}) &= \int_2^{e+1} \frac{dt}{t(t-1)} = \int_2^{e+1} \left(\frac{1}{t-1} - \frac{1}{t} \right) dt \\
 &= \left[\log |t-1| - \log |t| \right]_2^{e+1} \\
 &= \log e - \log(e+1) + \log 2 = \log \frac{2e}{e+1}
 \end{aligned}$$

2

$$\begin{aligned}
 (1) \quad \int_0^{\frac{\pi}{2}} \left| \cos x - \frac{1}{2} \right| \, dx &= \int_0^{\frac{\pi}{3}} \left(\cos x - \frac{1}{2} \right) \, dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{2} - \cos x \right) \, dx \\
 &= \left[\sin x - \frac{1}{2} x \right]_0^{\frac{\pi}{3}} + \left[\frac{1}{2} x - \sin x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) + \left(\frac{\pi}{4} - 1 \right) - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{2} \right) = \sqrt{3} - 1 - \frac{\pi}{12}
 \end{aligned}$$



$$\begin{aligned}
 (2) \quad \int_0^2 |2^x - 2| \, dx &= \int_0^1 (2 - 2^x) \, dx + \int_1^2 (2^x - 2) \, dx \\
 &= \left[2x - \frac{2^x}{\log 2} \right]_0^1 + \left[\frac{2^x}{\log 2} - 2x \right]_1^2 \\
 &= \left(2 - \frac{2}{\log 2} \right) + \frac{1}{\log 2} + \left(\frac{4}{\log 2} - 4 \right) - \left(\frac{2}{\log 2} - 2 \right) = \frac{1}{\log 2}
 \end{aligned}$$



$$\begin{aligned}
 (1) \quad \int_0^1 x \cos \pi x \, dx &= \int_0^1 x \left(\frac{1}{\pi} \sin \pi x \right)' dx \\
 &= \left[x \frac{1}{\pi} \sin \pi x \right]_0^1 - \frac{1}{\pi} \int_0^1 \sin \pi x \, dx \\
 &= 0 + \frac{1}{\pi^2} \left[\cos \pi x \right]_0^1 = -\frac{2}{\pi^2}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad I &= \int_0^1 \left\{ \cos^2 \pi x - (ax + b) \right\}^2 dx \\
 &= \int_0^1 \left\{ \cos^2 \pi x - 2(ax + b) \cos \pi x + (ax + b)^2 \right\} dx
 \end{aligned}$$

$$\begin{aligned}
 \text{ここで} \quad \int_0^1 \cos^2 \pi x \, dx &= \int_0^1 \frac{1 + \cos 2\pi x}{2} dx \\
 &= \frac{1}{2} \left[x + \frac{1}{2\pi} \sin 2\pi x \right]_0^1 = \frac{1}{2} \\
 \int_0^1 2(ax + b) \cos \pi x \, dx &= 2a \int_0^1 x \cos \pi x \, dx + 2b \int_0^1 \cos \pi x \, dx \\
 &= 2a \cdot \left(-\frac{2}{\pi^2} \right) + 2b \left[\frac{1}{\pi} \sin \pi x \right]_0^1 = -\frac{4a}{\pi^2}
 \end{aligned}$$

$$\begin{aligned}
 \int_0^1 (ax + b)^2 dx &= \int_0^1 (a^2 x^2 + 2abx + b^2) dx \\
 &= \left[\frac{a^2}{3} x^3 + abx^2 + b^2 x \right]_0^1 = \frac{a^2}{3} + ab + b^2
 \end{aligned}$$

$$\text{よって} \quad I = \frac{1}{2} + \frac{4a}{\pi^2} + \frac{a^2}{3} + ab + b^2$$

$$\begin{aligned}
 (3) \quad I &= \left(b + \frac{a}{2} \right)^2 + \frac{a^2}{12} + \frac{4a}{\pi^2} + \frac{1}{2} \\
 &= \left(b + \frac{a}{2} \right)^2 + \frac{1}{12} \left(a + \frac{24}{\pi^2} \right)^2 + \frac{1}{2} - \frac{48}{\pi^4}
 \end{aligned}$$

これより $b + \frac{a}{2} = 0$ かつ $a + \frac{24}{\pi^2} = 0$ のとき、最小値をとる。

したがって $a = -\frac{24}{\pi^2}$, $b = \frac{12}{\pi^2}$ のとき 最小値 $\frac{1}{2} - \frac{48}{\pi^4}$

$$\begin{aligned}
 (1) \quad f(x) &= x^2 - \int_0^x (x-t)f'(t)dt \\
 &= x^2 - x \int_0^x f'(t)dt + \int_0^x tf'(t)dt
 \end{aligned}$$

$$f(0) = 0^2 - 0 + \int_0^0 tf'(t)dt = 0$$

$$\begin{aligned}
 f'(x) &= 2x - \int_0^x f'(t)dt - xf'(x) + xf'(x) \\
 &= 2x - \left[f(t) \right]_0^x = 2x - f(x) \quad (\text{証明終})
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad (e^x f(x))' &= e^x f(x) + e^x f'(x) \\
 &= e^x f(x) + e^x (2x - f(x)) \\
 &= e^x f(x) + 2xe^x - e^x f(x) = 2xe^x \quad (\text{証明終})
 \end{aligned}$$

$$(3) \quad (e^x f(x))' = 2xe^x \quad \text{の両辺を } x \text{ で積分すると}$$

$$\int (e^x f(x))' dx = \int 2xe^x dx$$

$$e^x f(x) = 2 \int (e^x)' x dx$$

$$= 2e^x x - 2 \int e^x \cdot 1 dx = 2xe^x - 2e^x + C$$

$$\therefore f(x) = 2x - 2 + Ce^{-x}$$

$$f(0) = 0 \quad \text{より} \quad f(0) = -2 + Ce^0 = 0$$

$$\therefore C = 2$$

$$\text{よって} \quad f(x) = 2x - 2 + 2e^{-x}$$

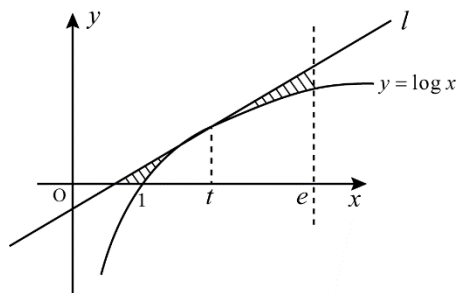
(1) $y = \log x$ より $y' = \frac{1}{x}$

l の方程式は

$$y - \log t = \frac{1}{t}(x - t)$$

$$\therefore y = \frac{1}{t}x + \log t - 1$$

(2) 求める面積は下図の斜線部分だから



$$\begin{aligned} S(t) &= \int_1^e \left(\frac{1}{t}x + \log t - 1 - \log x \right) dx \\ &= \left[\frac{1}{2t}x^2 + (\log t - t)x - (x \log x - x) \right]_1^e \\ &= \frac{1}{2t}(e^2 - 1) + (\log t - t)(e - 1) - 1 \\ &= \frac{e^2 - 1}{2t} + (e - 1)\log t - e \end{aligned}$$

(3) $S'(t) = -\frac{e^2 - 1}{2t^2} + \frac{e - 1}{t} = \frac{e - 1}{2t^2}(2t - e - 1)$

$$S'(t) = 0 \text{ より } t = \frac{e + 1}{2}$$

増減表は次のようになる。

t	1	...	$\frac{e+1}{2}$	...	e
$S'(t)$		-	0	+	
$S(t)$		↘	極小	↗	

よって、 $t = \frac{e+1}{2}$ のとき $S(t)$ は最小になる。

$P(t, \cos t)$ とおくと, $y = \cos x$ より $y' = -\sin x$

P における接線の方程式は

$$y - \cos t = -\sin t(x - t)$$

$$y = -(\sin t)x + t \sin t + \cos t$$

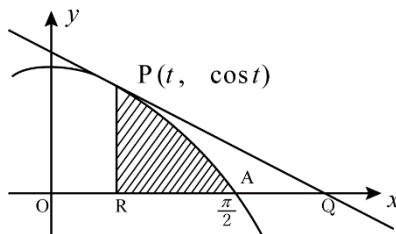
Q の x 座標は $y = 0$ として

$$x = t + \frac{\cos t}{\sin t}$$

$$\triangle PQR = \frac{1}{2} QR \cdot PR$$

$$= \frac{1}{2} \cdot \frac{\cos t}{\sin t} \cdot \cos t$$

$$= \frac{\cos^2 t}{2 \sin t}$$



右図の斜線部分の図形 APR の面積は

$$\int_t^{\frac{\pi}{2}} \cos x \, dx = \left[\sin x \right]_t^{\frac{\pi}{2}} = 1 - \sin t$$

(図形 APQ) : (図形 APR) = 1 : 2 だから

$$(\text{図形 APR}) = \frac{2}{3} \triangle PQR$$

$$\therefore 1 - \sin t = \frac{2}{3} \cdot \frac{\cos^2 t}{2 \sin t}$$

$$3(\sin t - \sin^2 t) = 1 - \sin^2 t$$

$$2 \sin^2 t - 3 \sin t + 1 = 0$$

$$(\sin t - 1)(2 \sin t - 1) = 0$$

$$0 < t < \frac{\pi}{2} \text{ だから } \sin t = \frac{1}{2}$$

$$\therefore t = \frac{\pi}{6} \text{ よって, } P\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$$

7

- (1) 楕円を x 軸のまわりに回転してできる
立体の体積を V_1 , 円柱 (半径 b , 高さ $2a$)
の体積を V_2 とすると

$$V_1 = \pi \int_{-2}^2 y^2 dx = 2\pi \int_0^2 \left(1 - \frac{x^2}{4}\right) dx$$

$$= 2\pi \left[x - \frac{x^3}{12} \right]_0^2 = 2\pi \left(2 - \frac{2}{3} \right) = \frac{8}{3} \pi$$

$$V_2 = \pi b^2 \cdot 2a = 2ab^2 \pi$$

ここで, 点 (a, b) は $\frac{x^2}{4} + y^2 = 1$ 上の点だから

$$\frac{a^2}{4} + b^2 = 1 \quad \therefore b^2 = 1 - \frac{a^2}{4} \quad \text{より}$$

$$V_2 = 2a \left(1 - \frac{a^2}{4} \right) \pi = 2 \left(a - \frac{a^3}{4} \right) \pi$$

$$\text{よって } V = V_1 - V_2 = \frac{8}{3} \pi - 2 \left(a - \frac{a^3}{4} \right) \pi$$

$$V = \left(\frac{8}{3} - 2a + \frac{a^3}{2} \right) \pi \quad (0 < a < 2)$$

$$(2) \quad V' = \left(-2 + \frac{3}{2} a^2 \right) \pi$$

$$= \frac{3}{2} \left(a - \frac{2\sqrt{3}}{3} \right) \left(a + \frac{2\sqrt{3}}{3} \right) \pi$$

a	0	...	$\frac{2\sqrt{3}}{3}$	...	2
V'			0		
V			極小		

$$a = \frac{2\sqrt{3}}{3} \quad \text{のとき}$$

$$V = \left(\frac{8}{3} - \frac{4\sqrt{3}}{3} + \frac{4\sqrt{3}}{9} \right) \pi$$

$$= \frac{8(3 - \sqrt{3})}{9} \pi$$

$$\text{増減表より最小値は } a = \frac{2\sqrt{3}}{3} \quad \text{のとき}$$

$$\frac{8(3 - \sqrt{3})}{9} \pi$$

8

- (1) 2 曲線の交点の x 座標は

$$e^x = ne^{-x} \quad \text{より}$$

$$e^{2x} = n \quad \Leftrightarrow \quad 2x = \log n$$

$$\therefore x = \frac{1}{2} \log n$$

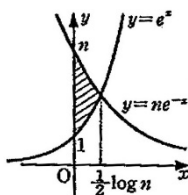
$$S_n = \int_0^{\frac{1}{2} \log n} (ne^{-x} - e^x) dx$$

$$= \left[-ne^{-x} - e^x \right]_0^{\frac{1}{2} \log n}$$

$$= \left(-ne^{-\frac{1}{2} \log n} - e^{\frac{1}{2} \log n} \right) - (-n - 1)$$

$$= -n \cdot n^{-\frac{1}{2}} - n^{\frac{1}{2}} + n + 1 = n - 2\sqrt{n} + 1$$

$$\therefore S_n = (\sqrt{n} - 1)^2$$



$$(2) \quad \lim_{n \rightarrow \infty} (S_{n+1} - S_n)$$

$$= \lim_{n \rightarrow \infty} \left\{ (\sqrt{n+1} - 1)^2 - (\sqrt{n} - 1)^2 \right\}$$

$$= \lim_{n \rightarrow \infty} (\sqrt{n+1} + \sqrt{n} - 2)(\sqrt{n+1} - \sqrt{n})$$

$$= \frac{(\sqrt{n+1} + \sqrt{n} - 2)(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

と変形すると

$$= \lim_{n \rightarrow \infty} \left(1 - \frac{2}{\sqrt{n+1} + \sqrt{n}} \right) = 1$$

$$1 \leq 1+x^2 \leq 2 \quad \text{より} \quad \frac{1}{2} \leq \frac{1}{1+x^2} \leq 1$$

各辺に x^n を掛けて

$$\frac{1}{2} x^n \leq \frac{x^n}{1+x^2} \leq x^n$$

等号は $x=0$, 2 のときだけ成り立つから

$$\int_0^1 \frac{1}{2} x^n \leq \int_0^1 \frac{x^n}{1+x^2} dx \leq \int_0^1 x^n dx$$

$$\left[\frac{1}{2(n+1)} x^{n+1} \right]_0^1 \leq \int_0^1 \frac{x^n}{1+x^2} dt \leq \left[\frac{1}{n+1} x^n \right]_0^1$$

$$\text{よって, } \frac{1}{2(n+1)} \leq \int_0^1 \frac{x^n}{1+x^2} dx \leq \frac{1}{n+1}$$

が成り立つ。