

3 章 積分法 解答

1 - 1 不定積分

A 問題

143

$$(1) \int x dx = \frac{1}{2} x^2 + C$$

$$(2) \int x^4 dx = \frac{1}{5} x^5 + C$$

$$(3) \int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + C = \frac{2}{3} x\sqrt{x} + C$$

$$(4) \int t^3 \sqrt{t} dt = \int t^{\frac{4}{3}} dt = \frac{1}{\frac{4}{3}+1} t^{\frac{4}{3}+1} + C = \frac{3}{7} t^2 \sqrt[3]{t} + C$$

$$(5) \int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{1}{-3+1} x^{-3+1} + C = \frac{1}{2x^2} + C$$

$$(6) \int \frac{1}{\sqrt[3]{x}} dx = \int x^{-\frac{1}{3}} dx = \frac{1}{-\frac{1}{3}+1} x^{-\frac{1}{3}+1} + C = \frac{3}{2} \sqrt[3]{x^2} + C$$

144

$$(1) \int dx = x + C$$

$$(2) \int (-6x + 5) dx = -6 \cdot \frac{1}{2} x^2 + 5x + C \\ = -3x^2 + 5x + C$$

$$(3) \int (3x^2 - 4x + 1) dx = 3 \cdot \frac{1}{3} x^3 - 4 \cdot \frac{1}{2} x^2 + x + C \\ = x^3 - 2x^2 + x + C$$

$$(4) \int x(x + 3) dx = \int (x^2 + 3x) dx \\ = \frac{1}{3} x^3 + \frac{3}{2} x^2 + C$$

$$(5) \int (x - 2)(2x - 3) dx = \int (2x^2 - 7x + 6) dx \\ = \frac{2}{3} x^3 - \frac{7}{2} x^2 + 6x + C$$

$$(6) \int (y - 2)^2 dy = \int (y^2 - 4y + 4) dy \\ = \frac{1}{3} y^3 - 2y^2 + 4y + C$$

$$(7) \int (t + a)(t - a) dt = \int (t^2 - a^2) dt \\ = \frac{1}{3} t^3 - a^2 t + C$$

$$(8) \int (x + 1)^2 dx - \int (x - 1)^2 dx \\ = \int \{(x + 1)^2 - (x - 1)^2\} dx \\ = \int 4x dx = 2x^2 + C$$

145

$$(1) \int \frac{x^3+1}{x^2} dx = \int \left(x + \frac{1}{x^2} \right) dx = \frac{1}{2} x^2 - \frac{1}{x} + C$$

$$(2) \int \frac{(x+2)^2}{x} dx = \int \left(x + 4 + \frac{4}{x} \right) dx = \frac{1}{2} x^2 + 4x + 4 \log(x) + C$$

$$(3) \int \frac{(x-1)^2}{\sqrt{x}} dx = \int \left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} \right) dx = \frac{2}{5} x^{\frac{5}{2}} - \frac{4}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

$$= \frac{2}{5} x^2 \sqrt{x} - \frac{4}{3} x \sqrt{x} + 2\sqrt{x} + C$$

146

$$(1) \int \left(\sin x + \frac{3}{\cos^2 x} \right) dx = -\cos x + 3 \tan x + C$$

$$(2) \int \left(2 + \frac{1}{\tan x} \right) \sin x dx = \int (2 \sin x + \cos x) dx = -2 \cos x + \sin x + C$$

$$(3) \int e^{x-1} dx = e^{x-1} + C$$

$$(4) \int (3^x \log 3 - 1) = \frac{3x}{\log 3} \cdot \log 3 - x + C = 3^x - x + C$$

$$(5) \int 5^{1-x} dx = \frac{5^{1-x}}{\log 5} + C$$

147

$$(1) \int \sin \frac{1}{3} x dx = -3 \cos \frac{1}{3} x + C$$

$$(2) \int \cos \pi x dx = \frac{1}{\pi} \sin \pi x + C$$

$$(3) \int e^{-2x} dx = -\frac{1}{2} e^{-2x} + C$$

$$(4) \int e^{\frac{x}{2}} dx = 2e^{\frac{x}{2}} + C$$

148

$$(1) \int \frac{7}{3-5x} dx$$

$$3-5x=t \text{ とおくと } x = \frac{3-t}{5}$$

$$\frac{dx}{dt} = -\frac{1}{5} \text{ より } dx = -\frac{1}{5} dt$$

$$\text{与式} = \int \frac{1}{t} \left(-\frac{1}{5} \right) dt = -\frac{1}{5} \log|t| + C$$

$$= -\frac{1}{5} \log|3-5x| + C$$

$$(2) \int x(2x-1)^3 dx$$

$$2x-1=t \text{ とおく } \text{ と } x=\frac{t+1}{2}, \quad dx=\frac{1}{2} dt$$

$$\begin{aligned} \text{与式} &= \int \frac{t+1}{2} \cdot t^3 \cdot \frac{1}{2} dt = \frac{1}{4} \int (t^4 + t^3) dt \\ &= \frac{1}{4} \left(\frac{1}{5} t^5 + \frac{1}{4} t^4 \right) + C \\ &= \frac{1}{20} t^5 + \frac{1}{16} t^4 + C \\ &= \frac{1}{80} t^4 (4t+5) + C = \frac{1}{80} (2x-1)^4 (8x+1) + C \end{aligned}$$

$$(3) \int \frac{x}{\sqrt{x-4}} dx$$

$$x-4=t \text{ とおく } \text{ と } x=t+4, \quad dx=dt$$

$$\begin{aligned} \text{与式} &= \int \frac{t+4}{\sqrt{t}} dt = \int \left(t^{\frac{1}{2}} + 4t^{-\frac{1}{2}} \right) dt = \frac{2}{3} t^{\frac{3}{2}} + 8t^{\frac{1}{2}} + C \\ &= \frac{2}{3} (x-4) \sqrt{x-4} + 8\sqrt{x-4} + C = \frac{2}{3} (x+8) \sqrt{x-4} \end{aligned}$$

$$(4) \int \frac{x}{(x+2)^2} dx$$

$$x+2=t \text{ とおく } \text{ と } x=t-2, \quad dx=dt$$

$$\begin{aligned} \text{与式} &= \int \frac{t-2}{t^2} dt = \int \left(\frac{1}{t} - \frac{2}{t^2} \right) dt = \log|t| + 2t^{-1} + C \\ &= \log|x+2| + \frac{2}{x+2} + C \end{aligned}$$

$$(5) \int \sqrt[3]{2x-5} dx$$

$$2x-5=t \text{ とおく } \text{ と } 2 \frac{dx}{dt} = 1 \text{ より } dx = \frac{1}{2} dt$$

$$\begin{aligned} \text{与式} &= \int \sqrt[3]{t} \cdot \frac{1}{2} dt = \frac{1}{2} \int t^{\frac{1}{3}} dt = \frac{3}{8} t^{\frac{4}{3}} + C \\ &= \frac{3}{8} (2x-5)^{\frac{4}{3}} \sqrt[3]{2x-5} + C \end{aligned}$$

$$(6) \int \frac{1}{(2+3x)^3} dx$$

$$2+3x=t \text{ とおく } \text{ と } 3 \frac{dx}{dt} = 1 \text{ より } dx = \frac{1}{3} dt$$

$$\begin{aligned} \text{与式} &= \int \frac{1}{t^3} \cdot \frac{1}{3} dt = \frac{1}{3} \int t^{-3} dt = -\frac{1}{6} t^{-2} + C \\ &= -\frac{1}{6(2+3x)^2} + C \end{aligned}$$

$$(1) \int (x^2 - 4x + 1)^2 (x - 2) dx$$

$$x^2 - 4x + 1 = t \text{ とおく } \text{と } (2x - 4) dx = dt \text{ より } (x - 2) dx = \frac{1}{2} dt$$

$$\text{与式} = \int t^2 \cdot \frac{1}{2} dt = \frac{1}{6} t^3 + C = \frac{1}{6} (x^2 - 4x + 1)^3 + C$$

$$(2) \int \sin^4 \theta \cos \theta d\theta$$

$$\sin \theta = t \text{ とおく } \text{と } \cos \theta d\theta = dt$$

$$\text{与式} = \int t^4 dt = \frac{1}{5} t^5 + C = \frac{1}{5} \sin^5 \theta + C$$

$$(3) \int x \sqrt{x^2 + 1} dx$$

$$x^2 + 1 = t \text{ とおく } \text{と } 2x dx = dt \text{ より } x dx = \frac{1}{2} dt$$

$$\text{与式} = \int \sqrt{t} \cdot \frac{1}{2} dt = \frac{1}{3} t^{\frac{3}{2}} + C = \frac{1}{3} (x^2 + 1) \sqrt{x^2 + 1} + C$$

$$(4) \int x e^{1-x^2} dx$$

$$1 - x^2 = t \text{ とおく } \text{と } -2x dx = dt \text{ より } x dx = -\frac{1}{2} dt$$

$$\text{与式} = \int e^t \cdot \left(-\frac{1}{2} dt\right) = -\frac{1}{2} e^t + C = -\frac{1}{2} e^{1-x^2} + C$$

$$(5) \int \frac{(\log x)^2}{x} dx$$

$$\log x = t \text{ とおく } \text{と } \frac{1}{x} dx = dt$$

$$\text{与式} = \int t^2 dt = \frac{1}{3} t^3 + C = \frac{1}{3} (\log x)^3 + C$$

$$(1) \int \frac{2x-1}{x^2-x} dx = \int \frac{(x^2-x)'}{x^2-x} dx = \log |x^2-x| + C$$

$$(2) \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \frac{(e^x + e^{-x})'}{e^x + e^{-x}} dx = \log(e^x + e^{-x}) + C$$

$$(3) \int \frac{\cos x}{1 + \sin x} dx = \int \frac{(1 + \sin x)'}{1 + \sin x} dx = \log(1 + \sin x) + C$$

$$\begin{aligned}
(1) \quad & \int (x+2) \sin x \, dx \\
&= \int (x+2)(-\cos x)' \, dx = -(x+2)\cos x + \int \cos x \, dx \\
&= -(x+2)\cos x + \sin x + C \\
(2) \quad & \int x e^{2x} \, dx = \int x \left(\frac{1}{2} e^{2x} \right)' \, dx = \frac{1}{2} x e^{2x} - \frac{1}{2} \int e^{2x} \, dx \\
&= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C \\
(3) \quad & \int x^2 \log x \, dx = \int \left(\frac{1}{3} x^3 \right)' \log x \, dx \\
&= \frac{1}{3} x^3 \log x - \int \frac{1}{3} x^3 \cdot \frac{1}{x} \, dx = \frac{1}{3} x^3 \log x - \frac{1}{9} x^3 + C
\end{aligned}$$

$$\begin{aligned}
(1) \quad & \int x^2 e^x \, dx = \int x^2 (e^x)' \, dx = x^2 e^x - \int 2x e^x \, dx \\
&= x^2 e^x - 2 \int x (e^x)' \, dx \\
&= x^2 e^x - 2x e^x + 2 \int e^x \, dx \\
&= x^2 e^x - 2x e^x + 2e^x + C \\
&= (x^2 - 2x + 2) e^x + C \\
(2) \quad & \int x^2 \cos x \, dx = \int x^2 (\sin x)' \, dx = x^2 \sin x - \int 2x \sin x \, dx \\
&= x^2 \sin x - 2 \int x (-\cos x)' \, dx \\
&= x^2 \sin x + 2x \cos x - 2 \int \cos x \, dx \\
&= x^2 \sin x + 2x \cos x - 2 \sin x + C \\
(3) \quad & \int (\log x)^2 \, dx = \int x' (\log x)^2 \, dx = x (\log x)^2 - \int x 2 (\log x) \cdot \frac{1}{x} \, dx \\
&= x (\log x)^2 - 2 \int \log x \, dx \\
&= x (\log x)^2 - 2 \int x' \log x \, dx \\
&= x (\log x)^2 - 2x \log x + 2 \int x \cdot \frac{1}{x} \, dx \\
&= x (\log x)^2 - 2x \log x + 2x + C
\end{aligned}$$

$$(1) \int \frac{x^2}{x+1} dx = \int \left(x-1 + \frac{1}{x+1} \right) dx = \frac{1}{2} x^2 - x + \log|x+1| + C$$

$$(2) \int \frac{1}{x(x-2)} dx = \frac{1}{2} \int \left(\frac{1}{x-2} - \frac{1}{x} \right) dx = \frac{1}{2} (\log|x-2| - \log|x|) + C$$

$$= \frac{1}{2} \log \left| \frac{x-2}{x} \right| + C$$

$$(3) \frac{3x+5}{(x-1)(x+3)} = \frac{9}{x-1} + \frac{b}{x+3} \quad \text{とおく。}$$

両辺に $(x-1)(x+3)$ を掛けて

$$3x+5 = a(x+3) + b(x-1) = (a+b)x + 3a-b$$

$$a+b=3, \quad 3a-b=5 \quad \text{より } a=2, \quad b=1$$

$$\text{与式} = \int \left(\frac{2}{x-1} + \frac{1}{x+3} \right) dx = 2 \log|x-1| + \log|x+3| + C$$

$$= \log(x-1)^2 |x+3| + C$$

$$(1) \int 2 \sin^2 x dx = \int 2 \cdot \frac{1-\cos 2x}{2} dx = \int (1-\cos 2x) dx$$

$$= x - \frac{1}{2} \sin 2x + C$$

$$(2) \int \sin 3x \cos x dx = \frac{1}{2} \int (\sin 4x + \sin 2x) dx = -\frac{1}{8} \cos 4x - \frac{1}{4} \cos 2x + C$$

$$(3) \int \cos x \cos 5x dx = \frac{1}{2} \int (\cos 6x + \cos 4x) dx = \frac{1}{2} \sin 6x + \frac{1}{8} \sin 4x + C$$

$$(4) \int (1+\cos x)^2 dx = \int (1+2\cos x + \cos^2 x) dx$$

$$= \int \left(1+2\cos x + \frac{1+\cos 2x}{2} \right) dx$$

$$= \frac{3}{2} x + 2 \sin x + \frac{1}{4} \sin 2x + C$$

$$(1) \int x^3 \sqrt{x+1} dx$$

$$x+1=t \text{ とおくと } dx=dt$$

$$\begin{aligned} \text{与式} &= \int (t-1)^3 \sqrt{t} dt = \int \left(t^{\frac{1}{3}} - t^{\frac{4}{3}} \right) dt = \frac{3}{7} t^{\frac{7}{3}} - \frac{3}{4} t^{\frac{4}{3}} + C \\ &= \frac{3}{7} t^2 \sqrt[3]{t} - \frac{3}{4} t^3 \sqrt{t} + C = \frac{3}{7} (x+1)^2 \sqrt[3]{x+1} - \frac{3}{4} (x+1) \sqrt{x+1} + C \end{aligned}$$

$$(2) \int \frac{1}{e^x - 2} dx$$

$$e^x = t \text{ とおくと } \frac{dt}{dx} = e^x \text{ より } dx = \frac{dt}{e^t} = \frac{dt}{t}$$

$$\begin{aligned} \text{与式} &= \int \frac{1}{t-2} \cdot \frac{1}{t} dt = \frac{1}{2} \int \left(\frac{1}{t-2} - \frac{1}{t} \right) dt = \frac{1}{2} (\log|t-2| - \log|t|) + C \\ &= \frac{1}{2} \log \left| \frac{t-2}{t} \right| + C = \frac{1}{2} \log \left| \frac{e^x - 2}{e^x} \right| + C \end{aligned}$$

$$(3) \int (2x-1)e^{x^2-x+3} dx$$

$$x^2 - x + 3 = t \text{ とおくと } \frac{dt}{dx} = 2x-1 \text{ より } (2x-1)dx = dt$$

$$\text{与式} = \int e^t dt = e^t + C = e^{x^2-x+3} + C$$

$$(4) \int \sin^5 \theta d\theta = \int \sin^4 \theta \sin \theta d\theta = \int (1 - \cos^2 \theta)^2 \sin \theta d\theta$$

$$\cos \theta = t \text{ とおくと } \frac{dt}{d\theta} = -\sin \theta \text{ より } dt = -\sin \theta d\theta$$

$$\begin{aligned} \text{与式} &= \int (1-t^2)^2 (-dt) = -\int (t^4 - 2t^2 + 1) dt \\ &= -\frac{1}{5} t^5 + \frac{2}{3} t^3 - t + C = -\frac{1}{5} \cos^5 \theta + \frac{2}{3} \cos^3 \theta - \cos \theta + C \end{aligned}$$

$$(5) \int \frac{\cos^3 x}{\sqrt{\sin x}} dx - \int \frac{(1 - \sin^2 \theta) \cos \theta}{\sqrt{\sin \theta}} d\theta$$

$$\sin \theta = t \text{ とおくと } \frac{dt}{d\theta} = \cos \theta \text{ より } \cos \theta d\theta = dt$$

$$\begin{aligned} \text{与式} &= \int \frac{(1-t^2)}{\sqrt{t}} dt = \int \left(t^{\frac{1}{2}} - t^{\frac{3}{2}} \right) dt = 2t^{\frac{1}{2}} - \frac{2}{5} t^{\frac{5}{2}} + C \\ &= 2\sqrt{\sin x} - \frac{2}{5} \sin^2 x \sqrt{\sin x} + C \end{aligned}$$

$$(6) \quad \int \frac{\log x}{x(\log x + 1)} dx$$

$$\log x = t \text{ とおくと } \frac{dt}{dx} = \frac{1}{x} \text{ より } \frac{1}{x} dx = dt$$

$$\begin{aligned} \text{与式} &= \int \frac{t}{t+1} dt = \int \left(1 - \frac{1}{t+1} \right) dt = t - \log|t+1| + C \\ &= \log x - \log|\log x + 1| + C \end{aligned}$$

B 問題

156

$$(1) \quad \sin x = t \text{ とおくと } \frac{dt}{dx} = \cos x \text{ より}$$

$$\cos x dx = dt$$

$$\begin{aligned} \int \frac{\cos x}{4 - \sin^2 x} dx &= \int \frac{dt}{4 - t^2} = \frac{1}{4} \int \left(\frac{1}{t+2} - \frac{1}{t-2} \right) dt \\ &= \frac{1}{4} \{ \log|t+2| - \log|t-2| \} + C \\ &= \frac{1}{4} \log \left| \frac{t+2}{t-2} \right| + C \\ &= \frac{1}{4} \log \left| \frac{\sin x + 2}{\sin x - 2} \right| + C \\ &= \frac{1}{4} \log \left(\frac{2 + \sin x}{2 - \sin x} \right) + C \end{aligned}$$

$$(2) \quad \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1 - \sin^2 x} dx$$

$$\sin x = t \text{ とおけば } \frac{dt}{dx} = \cos x \text{ より}$$

$$\cos x dx = dt$$

$$\begin{aligned} \int \frac{1}{\cos x} dx &= \int \frac{1}{1-t^2} dt = \frac{1}{2} \int \left(\frac{1}{t+1} - \frac{1}{t-1} \right) dt \\ &= \frac{1}{2} (\log|t+1| - \log|t-1|) + C \\ &= \frac{1}{2} \log \left| \frac{x+1}{x-1} \right| + C \\ &= \frac{1}{2} \log \left(\frac{1 + \sin x}{1 - \sin x} \right) + C \end{aligned}$$

$$(3) \quad e^x + 1 = t \text{ とおくと } \frac{dt}{dx} = e^x \text{ より}$$

$$dx = \frac{1}{e^x} dt = \frac{1}{t-1} dt$$

$$\begin{aligned} \int \frac{e^{3x}}{(e^x + 1)^2} dx &= \int \frac{(t-1)^3}{t^2(t-1)} dt \\ &= \int \frac{(t^2 - 2t + 1)(t-1)}{t^2(t-1)} dt = \int \left(1 - \frac{2}{t} + \frac{1}{t^2} \right) dt \\ &= t - 2 \log|t| - \frac{1}{t} + C \\ &= e^x - 2 \log(e^x + 1) - \frac{1}{e^x + 1} + C \end{aligned}$$

$$\begin{aligned}
 (1) \quad \int \frac{x}{\sqrt{2x+1}-\sqrt{x+1}} dx &= \int \frac{x(\sqrt{2x+1}+\sqrt{x+1})}{(2x+1)-(x+1)} dx \\
 &= \int (\sqrt{2x+1}+\sqrt{x+1}) dx = \frac{2}{3} \cdot \frac{1}{2} (2x+1)^{\frac{3}{2}} + \frac{2}{3} (x+1)^{\frac{3}{2}} + C \\
 &= \frac{1}{3} (2x+1)\sqrt{2x+1} + \frac{2}{3} (x+1)\sqrt{x+1} + C
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \int \frac{2x^2+x-1}{x^3+1} dx &= \int \frac{\cancel{(x+1)}(2x-1)}{\cancel{(x+1)}-(x^2-x+1)} dx \\
 &= \int \frac{(x^2-x+1)}{x^2-x+1} dx = \log(x^2-x+1) + C
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int \left(\tan x + \frac{1}{\tan x} \right)^2 dx &= \int \left(\tan^2 x + 2 + \frac{1}{\tan^2 x} \right) dx \\
 &= \int \left(1 + \tan^2 x + 1 + \frac{1}{\tan^2 x} \right) dx = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx \\
 &= \tan x - \frac{1}{\tan x} + C
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int \cos^4 x dx &= \int (\cos^2 x)^2 dx = \int \left(\frac{1+\cos 2x}{2} \right)^2 dx \\
 &= \frac{1}{4} \int (1 + 2\cos 2x + \cos^2 2x) dx \\
 &= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2} \right) dx \\
 &= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C
 \end{aligned}$$

別解 $\int \cos^4 x dx = \int \cos^3 x \cos x dx = \int \cos^3 x (\sin x)' dx$

$$\begin{aligned}
 &= \cos^3 x \sin x - \int (\cos^3 x)' \sin x dx \\
 &= \cos^3 x \sin x - \int (-3\cos^2 x \sin x) \sin x dx \\
 &= \cos^3 x \sin x + 3 \int (\sin x \cos x)^2 dx \\
 &= \cos^3 x \sin x + \frac{3}{4} \int \sin^2 2x dx \\
 &= \cos^3 x \sin x + \frac{3}{4} \int \frac{1-\cos 4x}{2} dx \\
 &= \cos^3 x \sin x + \frac{3}{8} x - \frac{3}{32} \sin 4x + C \quad \text{でもよい。}
 \end{aligned}$$

(参考) 別解の答えは次のように変形することによって
同じであることがわかる。

$$\begin{aligned}
 & \frac{3}{8}x + (\sin x \cos x) \cos^2 x - \frac{3}{32} \sin 4x + C \\
 &= \frac{3}{8}x + \frac{1}{2} \sin 2x \cdot \frac{1 + \cos 2x}{2} - \frac{3}{32} \sin 4x + C \\
 &= \frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{4} \sin 2x \cos 2x - \frac{3}{32} \sin 4x + C \\
 &= \frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{8} \sin 4x - \frac{3}{32} \sin 4x + C \\
 &= \frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C
 \end{aligned}$$

158

$$\frac{x^2+4x-1}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$$

両辺に $(x^2+1)(x+2)$ を掛けて

$$\begin{aligned}
 x^2+4x-1 &= (ax+b)(x+2) + c(x^2+1) \\
 &= (a+c)x^2 + (2a+b)x + 2b+c
 \end{aligned}$$

$a+c=1$, $2a+b=4$, $2b+c=-1$ より

$a=2$, $b=0$, $c=-1$

このとき

$$\begin{aligned}
 \text{与式} &= \int \frac{2x}{x^2+1} dx - \int \frac{1}{x+2} dx \\
 &= \int \frac{(x^2+1)}{x^2+1} dx - \int \frac{1}{x+2} dx \\
 &= \log(x^2+1) - \log(x+2) + C
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad \int e^{-x} \sin x \, dx &= \int (-e^{-x})' \sin x \, dx = -e^{-x} \sin x + \int e^{-x} \cos x \, dx \\
 &= -e^{-x} \sin x + \int (-e^{-x})' \cos x \, dx \\
 &= -e^{-x} \sin x + -e^{-x} \cos x - \int e^{-x} \sin x \, dx
 \end{aligned}$$

$$2 \int e^{-x} \sin x \, dx = -e^{-x} \sin x - e^{-x} \cos x$$

$$\text{よって, } \int e^{-x} \sin x \, dx = -\frac{1}{2} e^{-x} (\sin x + \cos x) + C$$

$$\begin{aligned}
 (2) \quad \int \log(x + \sqrt{x^2 + 1}) \, dx &= \int (x)' \log(x + \sqrt{x^2 + 1}) \, dx \\
 &= x \log(x + \sqrt{x^2 + 1}) - \int x \left\{ \log(x + \sqrt{x^2 + 1}) \right\}' \, dx
 \end{aligned}$$

$$\begin{aligned}
 \text{ここで } \left\{ \log(x + \sqrt{x^2 + 1}) \right\}' &= \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} \\
 &= \frac{1}{\sqrt{x^2 + 1}}
 \end{aligned}$$

$$\begin{aligned}
 \text{よって } \int \log(x + \sqrt{x^2 + 1}) \, dx &= x \log(x + \sqrt{x^2 + 1}) - \int \frac{x}{\sqrt{x^2 + 1}} \, dx \\
 &= x \log(x + \sqrt{x^2 + 1}) - \sqrt{x^2 + 1} + C
 \end{aligned}$$

160 点 (x, y) における接線の傾きは $f'(x)$ だから

$$f'(x) = 6x^2 - 2x + 3$$

両辺積分して

$$\begin{aligned}
 f(x) &= \int (6x^2 - 2x + 3) \, dx \\
 &= 2x^3 - x^2 + 3x + C
 \end{aligned}$$

ここで, 点 $(-1, 3)$ を通るから $f(-1) = 3$ より

$$-2 - 1 - 3 + C = 3 \quad \therefore C = 9$$

したがって $f(x) = 2x^3 - x^2 + 3x + 9$

$$\therefore y = 2x^3 - x^2 + 3x + 9$$

発展問題

$$161 \quad x + \sqrt{x^2 + 1} = t \text{ より } \frac{dt}{dx} = 1 + \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}} \cdot 2x = \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}$$

$$\frac{dx}{\sqrt{x^2 + 1}} = \frac{dt}{\sqrt{x^2 + 1} + x} = \frac{dt}{t}$$

$$\text{与式} = \int \frac{dt}{t} = \log|t| + C = \log|x + \sqrt{x^2 + 1}| + C$$

162

$$\begin{aligned} (1) \quad \int \cos^n x \, dx &= \int \cos^{n-1} x \cos x \, dx = \int \cos^{n-1} x (\sin x) \, dx \\ &= \cos^{n-1} x \sin x - \int (n-1) \cos^{n-2} x (-\sin^2 x) \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx \end{aligned}$$

$$\therefore n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx$$

$$\text{よって, } \int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$\begin{aligned} (2) \quad \int \tan^n x \, dx &= \int \tan^{n-2} x \tan^2 x \, dx = \int \tan^{n-2} x \left(\frac{1}{\cos^2 x} - 1 \right) dx \\ &= - \int \tan^{n-2} x \, dx + \int \tan^{n-2} x \frac{1}{\cos^2 x} \, dx \\ &= - \int \tan^{n-2} x \, dx + \int \tan^{n-2} x (\tan x)' \, dx \\ &= - \int \tan^{n-2} x \, dx + \tan^{n-1} x - \int (n-2) \tan^{n-3} x \cdot \frac{1}{\cos^2 x} \cdot \tan x \, dx \\ &= \tan^{n-1} x - \int \tan^{n-2} x \, dx - (n-2) \int \tan^{n-2} x (1 + \tan^2 x) \, dx \\ &= \tan^{n-1} x - (n-1) \int \tan^{n-2} x \, dx - (n-2) \int \tan^n x \, dx \end{aligned}$$

$$\therefore (n-1) \int \tan^n x \, dx = \tan^{n-1} x - (n-1) \int \tan^{n-2} x \, dx$$

$$\text{よって, } \int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

A 問題

163

$$\begin{aligned}
 (1) \quad \int_0^1 (3x^2 - 1) dx &= \left[x^3 - x \right]_0^1 \\
 &= (1 - 1) - 0 = 0
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \int_1^4 (x - 2)(2x + 1) dx &= \int_1^4 (2x^2 - 3x - 2) dx \\
 &= \left[\frac{2}{3} x^3 - \frac{3}{2} x^2 - 2x \right]_1^4 \\
 &= \frac{2}{3} (64 - 1) - \frac{3}{2} (16 - 1) - 2(4 - 1) \\
 &= 42 - \frac{45}{2} - 6 = \frac{27}{2}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int_{-3}^2 (x - 1)^2 dx &= \int_{-3}^2 (x^2 - 2x + 1) dx \\
 &= \left[\frac{1}{3} x^3 - x^2 + x \right]_{-3}^2 \\
 &= \left(\frac{8}{3} - 4 + 2 \right) - (-9 - 9 - 3) = \frac{65}{3}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_2^1 (6x^2 + 2x - 1) dx &= \left[2x^3 + x^2 - x \right]_2^1 \\
 &= (2 + 1 - 1) - (16 + 4 - 2) \\
 &= -16
 \end{aligned}$$

$$\begin{aligned}
 (\text{別解}) \quad \int_2^1 (6x^2 + 2x - 1) dx &= - \int_1^2 (6x^2 + 2x - 1) dx \\
 &= - \left[2x^3 + x^2 - x \right]_1^2 \\
 &= - \{ (16 + 4 - 2) - (2 + 1 - 1) \} \\
 &= -16
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \int_1^3 (4x^3 + 2x) dx &= \left[x^4 + x^2 \right]_1^3 \\
 &= (81 + 9) - (1 + 1) = 88
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad \int_{-1}^2 (x^4 + 3x^2 + 2) dx &= \left[\frac{1}{5} x^5 + x^3 + 2x \right]_{-1}^2 \\
 &= \left(\frac{32}{5} + 8 + 4 \right) - \left(-\frac{1}{5} - 1 - 2 \right) \\
 &= \frac{108}{5}
 \end{aligned}$$

164

$$(1) \quad \int_0^4 x\sqrt{x} dx = \int_0^4 x^{\frac{3}{2}} dx = \left[\frac{2}{5} x^{\frac{5}{2}} \right]_0^4 = \frac{64}{5}$$

$$(2) \quad \int_1^e \frac{1}{x} dx = [\log x]_1^e = 1$$

$$\begin{aligned}
 (3) \quad \int_{-1}^0 \frac{x+3}{x+2} dx &= \int_{-1}^0 \left(\frac{1}{x+2} + 1 \right) dx \\
 &= \left[\log |x+2| + x \right]_{-1}^0 \\
 &= \log 2 + 1
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_4^5 \frac{1}{(x-3)(x-2)} dx &= \int_4^5 \left(\frac{1}{x-3} - \frac{1}{x-2} \right) dx \\
 &= \left[\log \left| \frac{x-3}{x-2} \right| \right]_4^5 \\
 &= \log \frac{2}{3} - \log \frac{1}{2} = \log \frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \int_0^8 \frac{x-5}{\sqrt[3]{x}} dx &= \int_0^8 \left(x^{\frac{2}{3}} - 5x^{-\frac{1}{3}} \right) dx \\
 &= \left[\frac{3}{5} x^{\frac{5}{3}} - \frac{15}{2} x^{\frac{2}{3}} \right]_0^8 \\
 &= -\frac{54}{5}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad \int_0^1 \frac{dx}{\sqrt{x+1} - \sqrt{x}} &= \int_0^1 (\sqrt{x+1} + \sqrt{x}) dx \\
 &= \left[\frac{2}{3} (x+1)^{\frac{3}{2}} + \frac{2}{3} x^{\frac{3}{2}} \right]_0^1 \\
 &= \frac{4\sqrt{2}}{3}
 \end{aligned}$$

165

$$\begin{aligned}
 (1) \quad \int_0^{\frac{\pi}{4}} \sin^2 x dx &= \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 - \cos 2x) dx \\
 &= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \int_0^{\frac{\pi}{3}} \tan^2 x dx &= \int_0^{\frac{\pi}{3}} \left(\frac{1}{\cos^2 x} - 1 \right) dx \\
 &= \left[\tan x - x \right]_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int_{-2}^2 (e^x + e^{-x})^2 dx &= \int_{-2}^2 (e^{2x} + 2 + e^{-2x}) dx \\
 &= \left[\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} \right]_{-2}^2 \\
 &= e^4 - \frac{1}{e^4} + 8
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_0^1 (5^x + e^x) dx &= \left[\frac{5^x}{\log 5} + e^x \right]_0^1 \\
 &= \frac{4}{\log 5} + e - 1
 \end{aligned}$$

$$(5) \quad \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - \sin x \right) dx = \left[\tan x + \cos x \right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}$$

$$\begin{aligned}
 (6) \quad \int_0^{\frac{\pi}{3}} \frac{\sin 2x}{\cos x} dx &= \int_0^{\frac{\pi}{3}} \frac{2 \sin x \cos x}{\cos x} dx \\
 &= \left[-2 \cos x \right]_0^{\frac{\pi}{3}} = 1
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad & \int_0^1 (1-e^x)^2 dx + \int_0^1 (1+e^x)^2 dx \\
 &= \int_0^1 \left\{ (1-2e^x+e^{2x}) + (1+2e^x+e^{2x}) \right\} dx \\
 &= 2 \int_0^1 (1+e^{2x}) dx = 2 \left[x + \frac{1}{2} e^{2x} \right]_0^1 \\
 &= 2 \left\{ \left(1 + \frac{1}{2} e^2 \right) - \frac{1}{2} \right\} = 1 + e^2
 \end{aligned}
 \qquad
 \begin{aligned}
 (2) \quad & \int_1^e \log(x^2+4x) dx - \int_1^e (x+4) dx \\
 &= \int_1^e \left\{ \log(x^2+4x) - \log(x+4) \right\} dx \\
 &= \int_1^e \log \frac{x(x+4)}{x+4} dx = \int_1^e \log x dx \\
 &= \int_1^e x \log x dx = [x \log x]_1^e - \int_1^e x \cdot \frac{1}{x} dx \\
 &= e - [x]_1^e = e - (e-1) = 1
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad & \text{与式} = \int_1^2 (x^3 - 2x) dx - \int_1^2 (x^3 + x^2 + 5) dx \\
 &= \int_1^2 (-3x^2 - 5) dx = [-x^3 - 5x]_1^2 = (-8+10) - (-1-5) = -12
 \end{aligned}$$

$$(2) \quad \text{与式} = \int_0^{\frac{\pi}{12}} \sin 2x dx - \int_0^{\frac{\pi}{12}} 2 \sin x \cos x dx = \int_0^{\frac{\pi}{12}} (\sin 2x - \sin 2x) dx = 0$$

$$\begin{aligned}
 (3) \quad & \text{与式} = \int_0^1 (e^x + e^{-x}) dx + \int_1^2 (e^x + e^{-x}) dx \\
 &= \int_0^2 (e^x + e^{-x}) dx = [e^x - e^{-x}]_0^2 = e^2 - e^{-2} = e^2 - \frac{1}{e^2}
 \end{aligned}$$

$$(4) \quad \text{与式} = \int_0^{\frac{\pi}{3}} \sin^2 x + \int_0^{\frac{\pi}{3}} \cos^2 x dx = \int_0^{\frac{\pi}{3}} (\sin^2 x + \cos^2 x) dx = \int_0^{\frac{\pi}{3}} dx = [x]_0^{\frac{\pi}{3}} = \frac{\pi}{3}$$

(1) $1-x=t$ とおくと $\frac{dt}{dx} = -1$ より

$$dx = -dt$$

x	0	$\rightarrow$	1
t	1	$\rightarrow$	0

$$\begin{aligned} \int_0^1 (x+1)\sqrt{1-x} dx &= \int_1^0 (2-t)\sqrt{t} (-dt) = \int_0^1 \left(2t^{\frac{1}{2}} - t^{\frac{3}{2}} \right) dt \\ &= \left[\frac{4}{3} t^{\frac{3}{2}} - \frac{2}{5} t^{\frac{5}{2}} \right]_0^1 = \frac{4}{3} - \frac{2}{5} = \frac{14}{15} \end{aligned}$$

(2) $x^2-1=t$ とおくと $\frac{dt}{dx} = 2x$ より

$$x dx = \frac{1}{2} dt$$

x	1	$\rightarrow$	3
t	0	$\rightarrow$	8

$$\int_1^3 x\sqrt{x^2-1} dx = \int_0^8 \sqrt{t} \cdot \frac{1}{2} dt = \frac{1}{2} \left[\frac{2}{3} t^{\frac{3}{2}} \right]_0^8 = \frac{16}{3} \sqrt{2}$$

(3) $x^2+1=t$ とおくと $\frac{dt}{dx} = 2x$ より

$$x dx = \frac{1}{2} dt$$

x	-1	$\rightarrow$	0
t	2	$\rightarrow$	1

$$\int_{-1}^0 x(x^2+1)^3 dx = \int_2^1 t^3 \cdot \frac{1}{2} dt = \frac{1}{2} \left[\frac{1}{4} t^4 \right]_2^1 = \frac{1}{8} (1-16) = -\frac{15}{8}$$

$$\begin{aligned}
 (1) \quad \int_0^{\frac{\pi}{2}} \sin 3x \cos x \, dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin 4x + \sin 2x) \, dx \quad (2) \\
 &= \frac{1}{2} \left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} \right]_0^{\frac{\pi}{2}} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$t = \cos x \text{ とおくと } \frac{dt}{dx} = -\sin x \text{ より}$$

$$\sin x \, dx = -dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow \frac{\pi}{3} \\ \hline t & 1 \rightarrow \frac{1}{2} \\ \hline \end{array}$$

$$\begin{aligned}
 \int_0^{\frac{\pi}{3}} \sin x \cos^2 x \, dx &= -\int_1^{\frac{1}{2}} t^2 \, dt \\
 &= \int_{\frac{1}{2}}^1 t^2 \, dt = \left[\frac{1}{3} t^3 \right]_{\frac{1}{2}}^1 = \frac{7}{24}
 \end{aligned}$$

$$(3) \quad e^x = t \text{ とおくと } e^x \frac{dx}{dt} = 1 \text{ より}$$

$$e^x dx = dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow 1 \\ \hline t & 1 \rightarrow e \\ \hline \end{array}$$

$$\begin{aligned}
 \int_0^1 \frac{e^{2x}}{e^x + 1} \, dx &= \int_1^e \frac{t}{t+1} \, dt \\
 &= \int_1^e \left(1 - \frac{1}{t+1} \right) \, dt \\
 &= \left[t - \log |t+1| \right]_1^e \\
 &= e - 1 + \log \frac{2}{e+1}
 \end{aligned}$$

$$(4) \quad e^x = t \text{ とおくと } e^x \frac{dx}{dt} = 1 \text{ より}$$

$$e^x dx = dt \quad \begin{array}{|c|c|} \hline x & 1 \rightarrow 2 \\ \hline t & e \rightarrow e^2 \\ \hline \end{array}$$

$$\begin{aligned}
 \int_1^2 \frac{1}{e^x - 1} \, dx &= \int_1^2 \frac{e^x}{e^x(e^x - 1)} \, dx \\
 &= \int_e^{e^2} \frac{1}{t(t-1)} \, dt = \int_e^{e^2} \left(\frac{1}{t-1} - \frac{1}{t} \right) \, dt \\
 &= \left[\log \left| \frac{t-1}{t} \right| \right]_e^{e^2} = \log \frac{e^2 - 1}{e^2} - \log \frac{e-1}{e} \\
 &= \log \frac{e+1}{e}
 \end{aligned}$$

$$(5) \quad \log x = t \text{ とおくと } \frac{1}{x} \frac{dx}{dt} = 1 \text{ より}$$

$$\frac{1}{x} dx = dt \quad \begin{array}{|c|c|} \hline x & e^2 \rightarrow e^3 \\ \hline t & 2 \rightarrow 3 \\ \hline \end{array}$$

$$\begin{aligned}
 \int_{e^2}^{e^3} \frac{1}{x \log x} \, dx &= \int_2^3 \frac{1}{t} \, dt = \left[\log |t| \right]_2^3 \\
 &= \log 3 - \log 2 = \log \frac{3}{2}
 \end{aligned}$$

$$(6) \quad x^2 = t \text{ とおくと } 2x \frac{dx}{dt} = 1 \text{ より}$$

$$x \, dx = \frac{1}{2} dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow 1 \\ \hline t & 0 \rightarrow 1 \\ \hline \end{array}$$

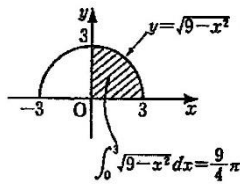
$$\begin{aligned}
 \int_0^1 x e^{x^2} \, dx &= \frac{1}{2} \int_0^1 e^t \, dt \\
 &= \frac{1}{2} \left[e^t \right]_0^1 = \frac{1}{2} (e - 1)
 \end{aligned}$$

(1) $x = 3 \sin \theta$ とおくと $\frac{dx}{d\theta} = 3 \cos \theta$ より

$$dx = 3 \cos \theta d\theta$$

x	$0 \rightarrow 3$
θ	$0 \rightarrow \frac{\pi}{2}$

$$\begin{aligned} \int_0^3 \sqrt{9-x^2} dx &= \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2 \theta} \cdot 3 \cos \theta d\theta \\ &= 9 \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 \theta} \cdot \cos \theta d\theta \\ &= 9 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta = \frac{9}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta \\ &= \frac{9}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}} = \frac{9}{4} \pi \end{aligned}$$



(2) $x = 3 \sin \theta$ とおくと $\frac{dx}{d\theta} = 3 \cos \theta$ より

$$dx = 3 \cos \theta d\theta$$

$$x = 0 \text{ のとき } \sin \theta = 0 \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow \frac{3}{2} \\ \hline \theta & 0 \rightarrow \frac{\pi}{6} \\ \hline \end{array}$$

よって $\theta = 0$

$$x = \frac{3}{2} \text{ のとき } \sin \theta = \frac{1}{2}$$

よって $\theta = \frac{\pi}{6}$

$$\begin{aligned} \int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} &= \int_0^{\frac{\pi}{6}} \frac{3 \cos \theta}{\sqrt{9-9\sin^2 \theta}} d\theta \\ &= \int_0^{\frac{\pi}{6}} \frac{3 \cos \theta}{3\sqrt{1-\sin^2 \theta}} d\theta \\ &= \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6} \end{aligned}$$

(3) $x = \sqrt{2} \tan \theta$ とおくと $\frac{dx}{d\theta} = \frac{\sqrt{2}}{\cos^2 \theta}$ より

$$dx = \frac{\sqrt{2}}{\cos^2 \theta} d\theta$$

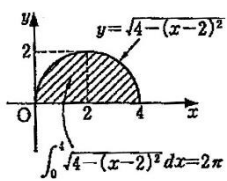
$$x = 0 \text{ のとき } \tan \theta = 0 \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow \sqrt{6} \\ \hline \theta & 0 \rightarrow \frac{\pi}{3} \\ \hline \end{array}$$

よって $\theta = 0$

$$x = \sqrt{6} \text{ のとき } \tan \theta = \sqrt{3}$$

よって $\theta = \frac{\pi}{3}$

$$\begin{aligned} \int_0^{\sqrt{6}} \frac{1}{x^2+2} dx &= \int_0^{\frac{\pi}{3}} \frac{1}{2 \tan^2 \theta + 2} \cdot \frac{\sqrt{2}}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{3}} \frac{1}{2(\tan^2 \theta + 1)} \cdot \frac{\sqrt{2}}{\cos^2 \theta} d\theta \\ &= \frac{\sqrt{2}}{2} \int_0^{\frac{\pi}{3}} \cos^2 \theta \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \frac{\sqrt{2}}{2} \int_0^{\frac{\pi}{3}} d\theta = \frac{\sqrt{2}}{6} \pi \end{aligned}$$



(4) $\int_1^2 \frac{dx}{x^2-2x+2} = \int_1^2 \frac{1}{(x-1)^2+1} dx$ より

$$x-1 = \tan \theta \text{ とおくと}$$

$$\frac{dx}{d\theta} = \frac{1}{\cos^2 \theta} \text{ より } dx = \frac{1}{\cos^2 \theta} d\theta$$

$$\begin{array}{|c|c|} \hline x & 1 \rightarrow 2 \\ \hline \theta & 0 \rightarrow \frac{\pi}{4} \\ \hline \end{array} \quad \text{だから}$$

$$\begin{aligned} \int_1^2 \frac{1}{(x-1)^2+1} dx &= \int_0^{\frac{\pi}{4}} \frac{1}{\tan^2 \theta + 1} \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \cos^2 \theta \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{4} \end{aligned}$$

$$(5) \quad \int_0^4 \sqrt{4x - x^2} dx = \int_0^4 \sqrt{4 - (x-2)^2} dx \quad \text{より} \quad x-2 = 2 \sin \theta \quad \text{とおくと}$$

$$\frac{dx}{d\theta} = 2 \cos \theta \quad \text{より} \quad dx = 2 \cos \theta d\theta$$

x	$0 \rightarrow 4$
θ	$-\frac{\pi}{2} \rightarrow \frac{\pi}{2}$

だから

$$\begin{aligned} \int_0^4 \sqrt{4 - (x-2)^2} dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4 - \sin^2 \theta} \cdot 2 \cos \theta d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{1 - \sin^2 \theta} \cdot 2 \cos \theta d\theta = 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta \\ &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta = 2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\pi \end{aligned}$$

$$(6) \quad x = \tan \theta \quad \text{とおくと}$$

$$\frac{dx}{d\theta} = \frac{1}{\cos^2 \theta} \quad \text{より} \quad dx = \frac{1}{\cos^2 \theta} d\theta$$

x	$0 \rightarrow 1$
θ	$0 \rightarrow \frac{\pi}{4}$

だから

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1+x^2}} &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 \theta}} \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 \theta}} \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \cos \theta \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\cos \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos \theta}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta \end{aligned}$$

$$\text{ここで } t = \sin \theta \quad \text{とおくと}$$

$$\frac{dt}{d\theta} = \cos \theta \quad \text{より} \quad \cos \theta d\theta = dt$$

θ	$0 \rightarrow \frac{\pi}{4}$
t	$0 \rightarrow \frac{1}{\sqrt{2}}$

だから

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta &= \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{1-t^2} dt \\ &= \int_0^{\frac{1}{\sqrt{2}}} \frac{-1}{(t+1)(t-1)} dt = \frac{1}{2} \int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{t+1} - \frac{1}{t-1} \right) dt \\ &= \frac{1}{2} \left[\log \left| \frac{t+1}{t-1} \right| \right]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{2} \log \frac{\sqrt{2}+1}{\sqrt{2}-1} \\ &= \frac{1}{2} \log (\sqrt{2}+1)^2 = \log (\sqrt{2}+1) \end{aligned}$$

$$\begin{aligned}
 (1) \quad \int_0^{\pi} x \sin x \, dx &= \int_0^{\pi} x (-\cos x)' \, dx \\
 &= \left[x(-\cos x) \right]_0^{\pi} + \int_0^{\pi} \cos x \, dx \\
 &= \pi + \left[\sin x \right]_0^{\pi} = \pi
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \int_{-1}^0 x e^{-x} \, dx &= \int_{-1}^0 x (-e^{-x})' \, dx \\
 &= \left[x(-e^{-x}) \right]_{-1}^0 + \int_{-1}^0 e^{-x} \, dx \\
 &= -e - \left[e^{-x} \right]_{-1}^0 = -1
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int_0^2 \log(x+1) \, dx &= \int_0^2 (x+1)' \cdot \log(x+1) \, dx \\
 &= \left[(x+1) \log(x+1) \right]_0^2 - \int_0^2 \frac{x+1}{x+1} \, dx \\
 &= 3 \log 3 - 2
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_1^e x \log x \, dx &= \left[\frac{x^2}{2} \log x \right]_1^e - \int_1^e \frac{1}{2} x \, dx \\
 &= \frac{1}{2} e^2 - \frac{1}{4} \left[x^2 \right]_1^e = \frac{1}{4} (e^2 + 1)
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \int_0^{\frac{\pi}{2}} (x+1) \cos x \, dx &= \int_0^{\frac{\pi}{2}} (x+1) (\sin x)' \, dx \\
 &= \left[(x+1) \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx \\
 &= \frac{\pi}{2} + 1 + \left[\cos x \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad \int_{\frac{1}{e}}^1 x^2 \log x \, dx &= \int_{\frac{1}{e}}^1 \left(\frac{1}{3} x^3 \right)' \log x \, dx \\
 &= \left[\frac{1}{3} x^3 \log x \right]_{\frac{1}{e}}^1 - \frac{1}{3} \int_{\frac{1}{e}}^1 x^2 \, dx \\
 &= \frac{1}{3e^3} - \frac{1}{9} \left[x^3 \right]_{\frac{1}{e}}^1 = \frac{1}{9} \left(\frac{4}{e^3} - 1 \right)
 \end{aligned}$$

$$(1) \quad F'(x) = x \sin x$$

$$\begin{aligned}
 (2) \quad F'(x) &= \int_0^x \sin t \, dt + x \sin x \\
 &= \left[-\cos t \right]_0^x + x \sin x \\
 &= 1 - \cos x + x \sin x
 \end{aligned}$$

$$(3) \quad F'(x) = (2x \sin 2x) \times (2x)' = 4x \sin 2x$$

$$\begin{aligned}
 (4) \quad F'(x) &= (3x \sin 3x) \times (3x)' - (x \sin x) \times (x)' \\
 &= 9x \sin 3x - x \sin x
 \end{aligned}$$

B 問題

173

$$(1) \quad \int_1^{e^2} |\log x - 1| dx$$

$$= \int_1^e (1 - \log x) dx + \int_e^{e^2} (\log x - 1) dx$$

$$\begin{aligned} \text{ここで, } \int \log x dx &= \int x' \log x dx = x \log x - \int x \cdot \frac{1}{x} dx \\ &= x \log x - x + c \end{aligned}$$

$$\begin{aligned} \therefore \text{与式} &= \left[x - x \log x + x \right]_1^e + \left[x \log x - x - x \right]_e^{e^2} \\ &= (2e - e) - 2 + (2e^2 - 2e^2) - (e - 2e) \\ &= 2e - 2 \end{aligned}$$

$$(2) \quad |\sin x - \cos x| = \sqrt{2} \sin \left(x - \frac{\pi}{4} \right) \text{ だから}$$

$$0 \leq x \leq \frac{\pi}{4} \text{ のとき } \sin \left(x - \frac{\pi}{4} \right) \leq 0$$

$$\frac{\pi}{4} \leq x \leq \pi \text{ のとき } \sin \left(x - \frac{\pi}{4} \right) \geq 0 \text{ である。}$$

$$\begin{aligned} \therefore \text{与式} &= \sqrt{2} \int_0^{\frac{\pi}{4}} \sin \left(x - \frac{\pi}{4} \right) dx + \sqrt{2} \int_{\frac{\pi}{4}}^{\pi} \sin \left(x - \frac{\pi}{4} \right) dx \\ &= -\sqrt{2} \left[-\cos \left(x - \frac{\pi}{4} \right) \right]_0^{\frac{\pi}{4}} + \sqrt{2} \left[-\cos \left(x - \frac{\pi}{4} \right) \right]_{\frac{\pi}{4}}^{\pi} \\ &= \left(\sqrt{2} - \sqrt{2} \cdot \frac{\sqrt{2}}{2} \right) + \left(\sqrt{2} \cdot \frac{\sqrt{2}}{2} + \sqrt{2} \right) = 2\sqrt{3} \end{aligned}$$

$$(1) \quad \sin^2 x \cos^3 x = \sin^2 x (1 - \sin^2 x) \cos x$$

よ り $\sin x = t$ とおくと

$$\frac{dt}{dx} = \cos x \quad \text{よ り} \quad \cos x dx = dt$$

x	$0 \rightarrow \frac{\pi}{2}$
t	$0 \rightarrow 1$

だから

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x dx &= \int_0^{\frac{\pi}{2}} \sin^2 x (1 - \sin^2 x) \cos x dx \\ &= \int_0^1 t^2 (1 - t^2) dt = \int_0^1 (t^2 - t^4) dt \\ &= \left[\frac{1}{3} t^3 - \frac{1}{5} t^5 \right]_0^1 = \frac{2}{15} \end{aligned}$$

$$(2) \quad \int_0^{\frac{\pi}{3}} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{3}} \frac{\cos x}{\cos^2 x} dx = \int_0^{\frac{\pi}{3}} \frac{\cos x}{1 - \sin^2 x} dx$$

$\sin x = t$ とおくと $\frac{dt}{dx} = \cos x$ よ り

$$\cos x dx = dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow \frac{\pi}{3} \\ \hline t & 0 \rightarrow \frac{\sqrt{3}}{2} \\ \hline \end{array} \quad \text{だから}$$

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \frac{dx}{\cos x} &= \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1-t^2} dt = \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{(1+t)(1-t)} dt \\ &= \frac{1}{2} \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{1}{1+t} + \frac{1}{1-t} \right) dt = \frac{1}{2} \left[\log(1+t) - \log(1-t) \right]_0^{\frac{\sqrt{3}}{2}} \\ &= \frac{1}{2} \left(\log \frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}} \right) = \frac{1}{2} \log \frac{2+\sqrt{3}}{2-\sqrt{3}} = \frac{1}{2} \log (2+\sqrt{3})^2 \\ &= \log (2+\sqrt{3}) \end{aligned}$$

$$\begin{aligned}
 (1) \quad \int_0^1 (1-x^2) e^x dx &= \int_0^1 (1-x^2) (e^x)' dx \\
 &= \left[(1-x^2) e^x \right]_0^1 + 2 \int_0^1 x e^x dx \\
 &= -1 + 2 \int_0^1 x e^x dx
 \end{aligned}$$

$$\begin{aligned}
 \text{また} \quad \int_0^1 x e^x dx &= \int_0^1 x (e^x)' dx \\
 &= \left[x e^x \right]_0^1 - \int_0^1 e^x dx = e - \left[e^x \right]_0^1 = 1
 \end{aligned}$$

$$\text{よって} \quad \int_0^1 (1-x^2) e^x dx = -1 + 2 \cdot 1 = 1$$

$$\begin{aligned}
 (2) \quad \int_0^{\frac{\pi}{2}} x \sin^2 x dx &= \int_0^{\frac{\pi}{2}} x \cdot \frac{1-\cos 2x}{2} dx \\
 &= \int_0^{\frac{\pi}{2}} x \cdot \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right)' dx \\
 &= \left[x \cdot \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) \right]_0^{\frac{\pi}{2}} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(x - \frac{1}{2} \sin 2x \right) dx \\
 &= \frac{\pi^2}{8} - \frac{1}{2} \left[\frac{1}{2} x^2 + \frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{2}} = \frac{1}{16} \pi^2 + \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int_0^{\log 2} x e^{2x} dx &= \int_0^{\log 2} x \left(\frac{1}{2} e^{2x} \right)' dx \\
 &= \left[\frac{1}{2} x e^{2x} \right]_0^{\log 2} - \int_0^{\log 2} \frac{1}{2} e^{2x} dx \\
 &= \frac{1}{2} (\log 2) e^{2 \log 2} - \frac{1}{4} \left[e^{2x} \right]_0^{\log 2} \\
 &= \frac{1}{2} (\log 2) e^{\log 4} - \frac{1}{4} e^{2 \log 2} + \frac{1}{4} \\
 &= \frac{1}{2} (\log 2) \cdot 4 - \frac{1}{4} \cdot 4 + \frac{1}{4} = 2 \log 2 - \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_1^e x (\log x)^2 dx &= \int_1^e \left(\frac{x^2}{2} \right)' (\log x)^2 dx \\
 &= \left[\frac{x^2}{2} (\log x)^2 \right]_1^e - \int_1^e \frac{x^2}{2} \cdot 2 (\log x) \frac{1}{x} dx \\
 &= \frac{1}{2} e^2 - \int_1^e x \log x dx \\
 &= \frac{1}{2} e^2 - \int_1^e \left(\frac{x^2}{2} \right)' \log x dx \\
 &= \frac{1}{2} e^2 - \left[\frac{x^2}{2} \cdot \log x \right]_1^e + \int_1^e \frac{x^2}{2} \cdot \frac{1}{x} dx \\
 &= \frac{1}{2} e^2 - \frac{1}{2} e^2 + \left[\frac{1}{4} x^2 \right]_1^e = \frac{1}{4} (e^2 - 1)
 \end{aligned}$$

$$\begin{aligned}
I &= \int_0^1 \left\{ \frac{1}{2} \log(x^2 + 1) \right\}' \log(x^2 + 1) dx \\
&= \left[\frac{1}{2} \log(x^2 + 1) \cdot \log(x^2 + 1) \right]_0^1 \\
&\quad - \int_0^1 \frac{1}{2} \log(x^2 + 1) \cdot \frac{2x}{x^2 + 1} dx \\
&= \frac{1}{2} (\log 2)^2 - \int_0^1 \frac{x}{x^2 + 1} \log(x^2 + 1) dx \\
&\text{したがって} \\
I &= \frac{1}{2} (\log 2)^2 - I \\
2I &= \frac{1}{2} (\log 2)^2 \quad \text{よって} \quad I = \frac{1}{4} (\log 2)^2
\end{aligned}$$

$$\begin{aligned}
\log(x^2 + 1) = t \quad \text{とおくと} \quad \frac{2x}{x^2 + 1} \cdot \frac{dx}{dt} = 1 \quad \text{より} \\
\frac{x}{x^2 + 1} dx = \frac{1}{2} dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow 1 \\ \hline t & 0 \rightarrow \log 2 \\ \hline \end{array} \\
I = \int_0^1 \log(x^2 + 1) \cdot \frac{x}{x^2 + 1} dx \\
= \int_0^{\log 2} t \left(\frac{1}{2} dt \right) = \frac{1}{2} \int_0^{\log 2} t dt \\
= \frac{1}{2} \left[\frac{1}{2} t^2 \right]_0^{\log 2} = \frac{1}{4} (\log 2)^2
\end{aligned}$$

$$\begin{aligned}
(1) \quad F(x) &= \int_0^x (x-t) \cos t \, dt \\
&= x \int_0^x \cos t \, dt - \int_0^x t \cos t \, dt \quad \text{より} \\
F'(x) &= \int_0^x \cos t \, dt + x \cos x - x \cos x \\
&= \left[\sin t \right]_0^x = \sin x
\end{aligned}$$

$$(2) \quad F(x) = \int_0^x t \sin(x-t) \, dt \quad \text{において}$$

$$\begin{aligned}
x-t=u \quad \text{とおくと} \quad t=x-u \\
\frac{du}{dt} = -1 \quad \text{より} \quad dt = (-1) du \quad \begin{array}{|c|c|} \hline t & 0 \rightarrow x \\ \hline u & x \rightarrow 0 \\ \hline \end{array} \\
F(x) = \int_x^0 (x-u) \sin u \cdot (-1) du \\
= \int_0^x (x-u) \sin u \, du \\
= x \int_0^x \sin u \, du - \int_0^x u \sin u \, du
\end{aligned}$$

$$\begin{aligned}
\text{よって} \quad F'(x) &= \int_0^x \sin u \, du + x \sin x - x \sin x \\
&= \left[-\cos u \right]_0^x = -\cos x + 1
\end{aligned}$$

$$\begin{aligned}
(3) \quad F(x) &= \int_0^x e^t \log \frac{x+1}{t+1} dt \\
&= \int_0^x e^t \{ \log(x+1) - \log(t+1) \} dt \\
&= \log(x+1) \int_0^x e^t dt - \int_0^x e^t \log(t+1) dt \\
\text{よって} \quad F'(x) &= \frac{1}{x+1} \int_0^x e^t dt + e^x \log(x+1) \\
&\quad - e^x \log(x+1) \\
&= \frac{1}{x+1} [e^t]_0^x = \frac{e^x - 1}{x+1}
\end{aligned}$$

$$178 \quad f(x) = x \int_0^x \sin^2 t \, dt - \int_0^x t \sin^2 t \, dt \quad \text{より}$$

$$f'(x) = \int_0^x \sin^2 t \, dt + x \sin^2 x - x \sin^2 x$$

$$= \int_0^x \sin^2 t \, dt$$

$$\text{ゆえに } f''(x) = \sin^2 x$$

179

$$\int_0^x (x-t) f(t) \, dt = a \cos x - 2 \cdots \textcircled{1}$$

$$x \int_0^x f(t) \, dt - \int_0^x t f(t) \, dt = a \cos x - 2$$

両辺を x で微分すると

$$\int_0^x f(t) \, dt + x f(x) - x f(x) = -a \sin x$$

$$\int_0^x f(t) \, dt = -a \sin x$$

さらに両辺を x で微分すると

$$f(x) = -a \cos x \cdots \textcircled{2}$$

また, ①において $x=0$ とおくと

$$0 = a \cos 0 - 2 \quad \text{より } a = 2$$

$$a = 2 \text{ を } \textcircled{2} \text{ に代入して } f(x) = -2 \cos x$$

180

$$(1) \quad \int_{-a}^a f(x) \, dx = \int_{-a}^0 f(x) \, dx + \int_0^a f(x) \, dx$$

$$= \int_{-a}^0 f(x) \, dx \quad \text{について}$$

$$x = -t \quad \text{とおくと}$$

$$dx = -dt$$

x	$-a \rightarrow 0$
t	$a \rightarrow 0$

$$\int_{-a}^0 f(x) \, dx = \int_a^0 f(-t)(-dt) = \int_0^a f(-t) \, dt$$

$$\therefore \int_{-a}^a f(x) \, dx = \int_0^a f(x) \, dx + \int_0^a f(-x) \, dx$$

が成り立つ。

$$(2) \quad \int_0^1 f(1-x) \, dx \quad \text{について}$$

$$1-x = t \quad \text{とおくと}$$

$$dx = -dt$$

x	$0 \rightarrow 1$
t	$1 \rightarrow 0$

$$\int_0^1 f(1-x) \, dx = \int_1^0 f(t)(-dt) = \int_0^1 f(t) \, dt$$

$$\therefore \int_0^1 \{f(x) + f(1-x)\} \, dx = 2 \int_0^1 f(x) \, dx$$

が成り立つ。

(1) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$ において

$x = \frac{\pi}{2} - t$ とおくと

x	$0 \rightarrow \frac{\pi}{2}$
t	$\frac{\pi}{2} \rightarrow 0$

$dx = -dt$ だから

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx &= \int_{\frac{\pi}{2}}^0 \frac{\sin\left(\frac{\pi}{2} - t\right)}{\sin\left(\frac{\pi}{2} - t\right) + \cos\left(\frac{\pi}{2} - t\right)} (-dt) \\ &= \int_{\frac{\pi}{2}}^0 \frac{\cos t}{\cos t + \sin t} (-dt) = \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t} dt \end{aligned}$$

よって, $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$

(2) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} dx = \left[x \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$$