

3 章 積分法 解答

1 節 不定積分

練習 1

$$(1) \quad \frac{1}{4}x^4 + C$$

$$(2) \quad \frac{5}{8}x^{\frac{8}{5}} + C$$

$$(3) \quad -\frac{1}{3x^3} + C$$

$$(4) \quad \frac{4}{7}x\sqrt[4]{x^3} + C$$

$$(5) \quad \frac{2}{5}t^2\sqrt{t} + C$$

$$(6) \quad 2\sqrt{x} + C$$

練習 2

$$\begin{aligned}(1) \quad \int (2x + 3) dx &= 2 \int x dx + 3 \int dx \\ &= 2 \cdot \frac{1}{2}x^2 + 3x + C \\ &= x^2 + 3x + C\end{aligned}$$

$$\begin{aligned}(2) \quad \int (x^2 + x - 1) dx &= \int x^2 dx + \int x dx - \int dx \\ &= \frac{1}{3}x^3 + \frac{1}{2}x^2 - x + C\end{aligned}$$

$$(3) \quad \int (-3) dx = -3 \int dx = -3x + C$$

$$\begin{aligned}(4) \quad \int (-6x^2 + 8x - 5) dx &= -6 \int x^2 dx + 8 \int x dx - 5 \int dx \\ &= -6 \cdot \frac{1}{3}x^3 + 8 \cdot \frac{1}{2}x^2 - 5x + C \\ &= -2x^3 + 4x^2 - 5x + C\end{aligned}$$

$$\begin{aligned}(5) \quad \int (x + 3)(3x - 1) dx &= \int (3x^2 + 8x - 3) dx \\ &= x^3 + 4x^2 - 3x + C\end{aligned}$$

練習 3

$$(1) \quad \text{与式} = \int \left(\frac{3}{x} - \frac{1}{x^2} \right) dx = 3 \log |x| + \frac{1}{x} + C$$

$$(2) \quad \text{与式} = \int \left(\sqrt{x} - \frac{2}{\sqrt{x}} \right) dx = \frac{2}{3}x\sqrt{x} - 4\sqrt{x} + C$$

$$\begin{aligned}(3) \quad \text{与式} &= \int \frac{x + 2\sqrt{x} + 1}{x} dx = \int \left(1 + \frac{2}{\sqrt{x}} + \frac{1}{x} \right) dx \\ &= x + 4\sqrt{x} + \log x + C\end{aligned}$$

練習 4

$$(1) \quad \text{与式} = -\cos x - 5 \sin x + C$$

$$(2) \quad \text{与式} = \int \left(\frac{1}{\cos^2 x} - 2 \right) dx = \tan x - 2x + C$$

$$(3) \quad \begin{aligned} \text{与式} &= \int \left(\frac{1}{\cos^2 x} + \frac{2}{\sin^2 x} \right) dx \\ &= \tan x - \frac{2}{\tan x} + C \end{aligned}$$

$$(4) \quad \begin{aligned} \text{与式} &= \int \frac{\cos^2 x}{\sin^2 x} dx = \int \frac{1 - \sin^2 x}{\sin^2 x} dx \\ &= \int \left(\frac{1}{\sin^2 x} - 1 \right) dx = -\frac{1}{\tan x} - x + C \end{aligned}$$

練習 5

$$(1) \quad \text{与式} = \frac{5^x}{\log 5} + C$$

$$(2) \quad \text{与式} = 2e^x - \frac{2^x}{\log 2} + C$$

$$(3) \quad \text{与式} = 10^x - \frac{x^5}{5} + C$$

練習 6

$$(1) \quad \int e^{3x} dx = \frac{1}{3} e^{3x} + C$$

$$(2) \quad \int \sin 2x dx = -\frac{1}{2} \cos 2x + C$$

$$(3) \quad \int e^{-\frac{1}{2}x} dx = -2e^{-\frac{1}{2}x} + C$$

練習 7

$$(1) \quad 3x - 5 = t \quad \text{とおくと}$$

$$x = \frac{t+5}{3}, \quad dx = \frac{1}{3} dt$$

$$\begin{aligned} \text{与式} &= \int t^4 \cdot \frac{1}{3} dt = \frac{1}{15} t^5 + C \\ &= \frac{1}{15} (3x - 5)^5 + C \end{aligned}$$

$$(2) \quad 5x + 1 = t \quad \text{とおくと}$$

$$x = \frac{t-1}{5}, \quad dx = \frac{1}{5} dt$$

$$\begin{aligned} \text{与式} &= \int \sqrt{t} \cdot \frac{1}{5} dt = \frac{2}{15} t^{\frac{3}{2}} + C \\ &= \frac{2}{15} (5x + 1) \sqrt{5x + 1} + C \end{aligned}$$

練習 8

$$(1) \quad \text{与式} = \frac{1}{2} \log |2x + 3| + C$$

$$(2) \quad \text{与式} = 2 \sin \left(\frac{1}{2} x - 5 \right) + C$$

$$(3) \quad \text{与式} = -\frac{1}{3} e^{-3x+2} + C$$

練習 9

$$2x + 1 = t \quad \text{すなわち} \quad x = \frac{t-1}{2} \quad \text{とおくと} \quad \frac{dx}{dt} = \frac{1}{2} \quad \text{より}$$

$$\begin{aligned} \text{与式} &= \int \frac{t-1}{2} \cdot \sqrt{t} \cdot \frac{1}{2} dt = \int \frac{t\sqrt{t} - \sqrt{t}}{4} dt \\ &= \frac{1}{4} \int \left(t^{\frac{3}{2}} - t^{\frac{1}{2}} \right) dt = \frac{1}{4} \left(\frac{2}{5} t^{\frac{5}{2}} - \frac{2}{3} t^{\frac{3}{2}} \right) + C \\ &= \frac{1}{30} (3t - 5) t^{\frac{3}{2}} + C = \frac{1}{15} (3x - 1) (2x + 1) \sqrt{2x + 1} + C \end{aligned}$$

練習 10

$$(1) \quad \sqrt{x-1} = t \quad \text{すなわち} \quad x = t^2 + 1 \quad \text{とおくと} \quad \frac{dx}{dt} = 2t \quad \text{より}$$

$$\begin{aligned} \text{与式} &= \int (t^2 + 3) \cdot t \cdot 2t \, dt = \int (2t^4 + 6t^2) \, dt \\ &= \frac{2}{5} t^5 + 2t^3 + C = \frac{2}{5} t^3 (t^2 + 5) + C = \frac{2}{5} (x+4)(x-1)\sqrt{x-1} + C \end{aligned}$$

$$(2) \quad \sqrt{x-1} = t \quad \text{すなわち} \quad x = t^2 + 1 \quad \text{とおくと} \quad \frac{dx}{dt} = 2t \quad \text{より}$$

$$\begin{aligned} \text{与式} &= \int \frac{2(t^2 + 1)}{t} \cdot 2t \, dt = 4 \int (t^2 + 1) \, dt \\ &= 4 \left(\frac{1}{3} t^3 + t \right) + C = \frac{4}{3} t (t^2 + 3) + C = \frac{4}{3} (x+2)\sqrt{x-1} + C \end{aligned}$$

$$(\text{別解}) \quad x-1 = t \quad \text{とおくと} \quad \frac{dx}{dt} = 1 \quad \text{より}$$

$$\begin{aligned} (1) \quad \text{与式} &= \int (t+3)\sqrt{t} \, dt = \frac{2}{5} t^{\frac{5}{2}} + 2t^{\frac{3}{2}} + C \\ &= \frac{2}{5} t^{\frac{3}{2}} (t+5) + C = \frac{2}{5} (x+4)(x-1)\sqrt{x-1} + C \end{aligned}$$

$$\begin{aligned} (2) \quad \text{与式} &= \int \frac{2(t+1)}{\sqrt{t}} \, dt = 2 \left(\frac{2}{3} t^{\frac{3}{2}} + 2t^{\frac{1}{2}} \right) + C \\ &= \frac{4}{3} t^{\frac{1}{2}} (t+3) + C = \frac{4}{3} (x+2)\sqrt{x-1} + C \end{aligned}$$

練習 11

$$(1) \quad x^2 + 5x + 1 = t \quad \text{とおくと} \quad (2x+5) \, dx = dt$$

$$\text{与式} = \int t^3 \, dt = \frac{1}{4} t^4 + C = \frac{1}{4} (x^2 + 5x + 1)^4 + C$$

$$(2) \quad \cos x = t \quad \text{とおくと} \quad -\sin x \, dx = dt$$

よって

$$\text{与式} = \int t^5 (-dt) = -\frac{1}{6} t^6 + C = -\frac{1}{6} \cos^6 x + C$$

$$(3) \quad x^2 = t \quad \text{とおくと} \quad 2x \, dx = dt$$

よって

$$\text{与式} = \int e^t \cdot \frac{1}{2} \, dt = \frac{1}{2} e^t + C = \frac{1}{2} e^{x^2} + C$$

$$(4) \quad 2e^x - 1 = t \quad \text{とおくと} \quad 2e^x \, dx = dt$$

よって

$$\text{与式} = \int t^2 \cdot \frac{1}{2} \, dt = \frac{1}{6} t^3 + C = \frac{1}{6} (2e^x - 1)^3 + C$$

練習 12

$$f(x) = t \text{ とおくと } f'(x)dx = dt$$

よって, $\alpha \neq -1$ のとき

$$\int \{f(x)\}^\alpha f'(x)dx = \int t^\alpha dt = \frac{1}{\alpha+1} t^{\alpha+1} + C = \frac{1}{\alpha+1} \{f(x)\}^{\alpha+1} + C$$

練習 13

$$(1) \text{ 与式} = \int \frac{(x^2-1)'}{x^2-1} \times \frac{1}{2} dx = \frac{1}{2} \log|x^2-1| + C$$

$$(2) \text{ 与式} = \int \frac{\cos x}{\sin x} dx = \int \frac{(\sin x)'}{\sin x} dx = \log|\sin x| + C$$

$$(3) \text{ 与式} = \int \frac{(e^x-1)'}{e^x-1} dx = \log|e^x-1| + C$$

$$(4) \text{ 与式} = \int \frac{(\log x)'}{\log x} dx = \log|\log x| + C$$

練習 14

$$(1) \text{ 与式} = \int x(-\cos x)' dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C$$

$$(2) \text{ 与式} = \int x \left(\frac{1}{2} \sin 2x \right)' dx = \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x dx = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C$$

$$(3) \text{ 与式} = \int x(e^x)' dx = x e^x - \int e^x dx = x e^x - e^x + C$$

$$(4) \text{ 与式} = \int x e^{-x} dx = \int x(-e^{-x})' dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x} + C$$

練習 15

$$(1) \text{ 与式} = \int (x)' \log 3x dx = x \log 3x - \int x \cdot \frac{3}{3x} dx = x \log 3x - \int dx = x \log 3x - x + C$$

$$(2) \text{ 与式} = \int \left(\frac{1}{2} x^2 \right)' \log x dx = \frac{1}{2} x^2 \log x - \int \frac{1}{2} x^2 \cdot \frac{1}{x} dx \\ = \frac{1}{2} x^2 \log x - \frac{1}{2} \int x dx = \frac{1}{2} x^2 \log x - \frac{1}{4} x^2 + C$$

$$(3) \text{ 与式} = \int (x+1)' \log(x+1) dx \\ = (x+1) \log(x+1) - \int (x+1) \cdot \frac{1}{x+1} dx \\ = (x+1) \log(x+1) - \int dx = (x+1) \log(x+1) - x + C$$

練習 16

$$\begin{aligned}
 \int e^x \cos x dx &= \int (e^x)' \cos x dx \\
 &= e^x \cos x - \int e^x (\cos x)' dx = e^x \cos x + \int e^x \sin x dx \\
 &= e^x \cos x + \int (e^x)' \sin x dx = e^x \cos x + \left\{ e^x \sin x - \int e^x (\sin x)' dx \right\} \\
 \therefore \int e^x \cos x dx &= e^x \cos x + e^x \sin x - \int e^x \cos x dx
 \end{aligned}$$

したがって

$$2 \int e^x \cos x dx = e^x (\sin x + \cos x) + C_1$$

$$\text{よって } \int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + C \quad (\text{ただし, } C = \frac{1}{2} C_1)$$

練習 17

(1) 右の割り算より

$$\text{与式} = \int \left(x + 2 + \frac{5}{x-2} \right) dx = \frac{1}{2} x^2 + 2x + 5 \log |x-2| + C$$

$$\begin{array}{r}
 x+2 \\
 x-2 \overline{) x^2 + 1} \\
 \underline{x^2 - 2x} \\
 2x+1 \\
 \underline{2x-4} \\
 5
 \end{array}$$

$$(2) \text{ 与式} = \int \frac{1}{x} - \frac{1}{x+1} dx = \log |x| - \log |x+1| + C = \log \left| \frac{x}{x+1} \right| + C$$

$$(3) \frac{-x+7}{(x-2)(x+3)} = \frac{a}{x-2} + \frac{b}{x+3} \text{ とおく。}$$

両辺に $(x-2)(x+3)$ を掛けて

$$\begin{aligned}
 -x+7 &= a(x+3) + b(x-2) \\
 &= (a+b)x + 3a - 2b
 \end{aligned}$$

両辺の係数を比較して $a+b=-1, 3a-2b=7$

これより $a=1, b=-2$

したがって

$$\begin{aligned}
 \text{与式} &= \int \frac{-x+7}{(x-2)(x+3)} dx = \int \frac{(x+3)-2(x-2)}{(x-2)(x+3)} dx = \int \left(\frac{1}{x-2} - \frac{2}{x+3} \right) dx \\
 &= \log |x-2| - 2 \log |x+3| + C = \log \frac{|x-2|}{(x+3)^2} + C
 \end{aligned}$$

練習 18

$$(1) \quad \text{与式} = \int \frac{1 + \cos 2x}{2} dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

$$(2) \quad \text{与式} = \frac{1}{2} \int (\sin 4x - \sin 2x) dx = -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + C$$

$$(3) \quad \text{与式} = -\frac{1}{2} \int (\cos 3x - \cos x) dx = -\frac{1}{6}\sin 3x + \frac{1}{2}\sin x + C$$

練習 19

$$(1) \quad \cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x) \cos x \quad \text{より} \quad \sin x = t \quad \text{とおくと} \quad \cos x dx = dt$$

$$\text{よって} \quad \text{与式} = \int (1 - t^2) dt = -\frac{1}{3}t^3 + t + C = -\frac{1}{3}\sin^3 x + \sin x + C$$

$$(2) \quad e^{2x} - 1 = t \quad \text{とおくと} \quad 2e^{2x} dx = dt$$

$$\text{すなわち} \quad dx = \frac{1}{2e^{2x}} dt = \frac{1}{2(t+1)} dt$$

よって

$$\begin{aligned} \text{与式} &= \int \frac{1}{t} \cdot \frac{1}{2(t+1)} dt = \frac{1}{2} \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt \\ &= \frac{1}{2} (\log |t| - \log |t+1|) + C = \frac{1}{2} \log |e^{2x} - 1| - x + C \end{aligned}$$

練習 20

$$(1) \quad \int_0^2 x^3 dx = \left[\frac{1}{4}x^4 \right]_0^2 = 4$$

$$(2) \quad \int_{-2}^2 (x+2) dx = \left[\frac{1}{2}x^2 + 2x \right]_{-2}^2 = (2+4) - (2-4) = 8$$

$$(3) \quad \int_1^2 (3x^2 - 8x + 5) dx = \left[x^3 - 4x^2 + 5x \right]_1^2 = (8-16+10) - (1-4+5) = 2-2 = 0$$

$$(4) \quad \int_{-1}^1 x(x-2) dx = \int_{-1}^1 (x^2 - 2x) dx = \left[\frac{1}{3}x^3 - x^2 \right]_{-1}^1 = \left(\frac{1}{3} - 1 \right) - \left(-\frac{1}{3} - 1 \right) = \frac{2}{3}$$

$$\begin{aligned} (5) \quad \int_{-1}^3 (x+1)(x-3) dx &= \int_{-1}^3 (x^2 - 2x - 3) dx \\ &= \left[\frac{1}{3}x^3 - x^2 - 3x \right]_{-1}^3 = (9-9-9) - \left(-\frac{1}{3} - 1 + 3 \right) = -\frac{32}{3} \end{aligned}$$

$$(6) \quad \int_0^3 (t-1)^2 dt = \int_0^3 (t^2 - 2t + 1) dt = \left[\frac{1}{3}t^3 - t^2 + t \right]_0^3 = (9-9+3) - 0 = 3$$

別解

$$\int_0^3 (t-1)^2 dx = \left[\frac{1}{3}(t-1)^3 \right]_0^3 = \frac{8}{3} - \left(-\frac{1}{3} \right) = 3$$

$$(7) \quad \int_{-3}^1 (t+1)^3 dt = \left[\frac{1}{4}(t+1)^4 \right]_{-3}^1 = \frac{1}{4}(2^4 - 2^4) = 0$$

$$(8) \quad \int_{-2}^2 x(x+1)(x-1) dt = \int_{-2}^2 (x^3 - x) dt = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2 \right]_{-2}^2 = 2 - 2 = 0$$

練習 21

$$(1) \quad \text{与式} = \left[2\sqrt{x} \right]_1^4 = 4 - 2 = 2$$

$$(2) \quad \text{与式} = \left[\sin t \right]_0^\pi = 0 - 0 = 0$$

$$(3) \quad \text{与式} = \left[\log u \right]_1^e = 1 - 0 = 1$$

$$(4) \quad (\text{与式}) = \left[\frac{3}{5}x^{\frac{5}{3}} \right]_0^1 = \frac{3}{5}$$

$$(5) \quad \text{与式} = \left[\tan x \right]_0^{\frac{\pi}{4}} = 1$$

$$(6) \quad (\text{与式}) = \int_{-1}^0 e^{-x} dx = \left[-e^x \right]_{-1}^0 = -1 + e$$

$$(7) \quad (\text{与式}) = \int_0^1 \frac{1}{e} \cdot e^x dx = \frac{1}{e} \left[e^x \right]_0^1 = 1 - \frac{1}{e}$$

$$(8) \quad (\text{与式}) = \left[\frac{2^x}{\log 2} \right]_1^2 = \frac{4-2}{\log 2} = \frac{2}{\log 2}$$

$$(9) \quad (\text{与式}) = \int_0^2 \frac{1 - \sin 2x}{2} dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x \right]_0^2 = \frac{\pi}{4}$$

練習 22

$$(1) \quad \text{与式} = \int_0^\pi (\cos x - 3 \sin x) dx = \left[\sin x + 3 \cos x \right]_0^\pi = -3 - 3 = -6$$

$$(2) \quad \text{与式} = \int_1^2 \frac{4x-2}{x^2-x+1} dx = 2 \int_1^2 \frac{(x^2-x+1)'}{x^2-x+1} dx = 2 \left[\log(x^2-x+1) \right]_1^2 = 2 \log 3$$

練習 23

$$(1) \quad \text{与式} = \int_0^{\frac{\pi}{2}} \sin x dx = \left[-\cos x \right]_0^{\frac{\pi}{2}} = 1$$

$$(2) \quad \text{与式} = \int_1^2 (x^3 - x + 1 - x^3 + x) dx = \int_1^2 1 dx = \left[x \right]_1^2 = 1$$

$$\begin{aligned} (3) \quad \text{与式} &= \int_1^{2e} \left(x + \frac{1}{x} \right) dx = \left[\frac{1}{2}x^2 + \log x \right]_1^{2e} \\ &= 2e^2 + \log 2e - \frac{1}{2} = 2e^2 + \log e + \log 2 - \frac{1}{2} \\ &= 2e^2 + \log 2 + \frac{1}{2} \end{aligned}$$

練習 24

(1) $2x - 1 = t$ すなわち $x = \frac{1}{2}(t + 1)$ とおくと $dx = \frac{1}{2} dt$

よって

$$\text{与式} = \int_{-1}^1 \frac{1}{2}(t + 1) \cdot t^4 \cdot \frac{1}{2} dt$$

$$= \frac{1}{4} \int_{-1}^1 (t^5 + t^4) dt = \frac{1}{4} \left[\frac{1}{6} t^6 + \frac{1}{5} t^5 \right]_{-1}^1 = \frac{1}{10}$$

x	$0 \rightarrow 1$
t	$-1 \rightarrow 1$

(2) $t + 1 = s$ すなわち $t = s - 1$ とおくと $dt = ds$

よって

$$\text{与式} = \int_1^2 \frac{s-1}{s^3} ds = \int_1^2 \left(\frac{1}{s^2} - \frac{1}{s^3} \right) ds$$

$$= \left[-\frac{1}{s} + \frac{1}{2s^2} \right]_1^2 = \frac{1}{8}$$

t	$0 \rightarrow 1$
s	$1 \rightarrow 2$

練習 25

(1) $\sin x = t$ とおくと $\cos x dx = dt$

$$\text{与式} = \int_0^{\frac{\pi}{2}} \sin^4 x \cos x dx = \int_0^1 t^4 dt = \left[\frac{1}{5} t^5 \right]_0^1 = \frac{1}{5}$$

x	$0 \rightarrow \frac{\pi}{2}$
t	$0 \rightarrow 1$

(2) $\cos x = 1$ とおくと $-\sin x dx = dt$

$$\text{与式} = -\int_0^{\frac{\pi}{2}} e^{\cos x} (-\sin x) dx = -\int_1^0 e^t dt = \int_0^1 e^t dt$$

$$= \left[e^t \right]_0^1 = e - 1$$

x	$0 \rightarrow \frac{\pi}{2}$
t	$1 \rightarrow 0$

練習 26

(1) $x = \sin \theta$ とおくと $dx = \cos \theta d\theta$

よって

$$\begin{aligned} \text{与式} &= \int_0^{\frac{\pi}{6}} \sqrt{1 - \sin^2 \theta} \cdot \cos \theta d\theta = \int_0^{\frac{\pi}{6}} \cos^2 \theta d\theta = \int_0^{\frac{\pi}{6}} \frac{1 + \cos 2\theta}{2} d\theta \\ &= \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{6}} = \frac{1}{2} \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} \right) = \frac{\pi}{12} + \frac{\sqrt{3}}{8} \end{aligned}$$

x	$0 \rightarrow \frac{1}{2}$
θ	$0 \rightarrow \frac{\pi}{6}$

(2) $t = 2 \sin \theta$ とおくと $dt = 2 \cos \theta d\theta$

よって

$$\begin{aligned} \text{与式} &= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{4 - 4 \sin^2 \theta}} \cdot 2 \cos \theta d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{2 \cos \theta} \cdot 2 \cos \theta d\theta = \int_0^{\frac{\pi}{6}} d\theta \\ &= \left[\theta \right]_0^{\frac{\pi}{6}} = \frac{\pi}{6} \end{aligned}$$

t	$0 \rightarrow 1$
θ	$0 \rightarrow \frac{\pi}{6}$

(3) $x = 2 \sin \theta$ とおくと $dx = 2 \cos \theta d\theta$

よって

$$\begin{aligned} \text{与式} &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \sqrt{4 - 4 \sin^2 \theta} \cdot 2 \cos \theta d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} 4 \cos^2 \theta d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} 2(1 + \cos 2\theta) d\theta \\ &= \left[2\theta + \sin 2\theta \right]_{-\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{5}{6} \pi + \frac{2 + \sqrt{3}}{2} \end{aligned}$$

x	$-1 \rightarrow \sqrt{2}$
θ	$-\frac{\pi}{6} \rightarrow \frac{\pi}{4}$

練習 27

(1) $x = 3 \tan \theta$ とおくと $dx = \frac{3}{\cos^2 \theta} d\theta$

よって

$$\text{与式} = \int_0^{\frac{\pi}{4}} \frac{1}{9 + 9 \tan^2 \theta} \cdot \frac{3}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos^2 \theta}{9} \cdot \frac{3}{\cos^2 \theta} d\theta = \frac{1}{3} \left[\theta \right]_0^{\frac{\pi}{4}} = \frac{\pi}{12}$$

x	$0 \rightarrow 3$
θ	$0 \rightarrow \frac{\pi}{4}$

(2) $x = \sqrt{3} \tan \theta$ とおくと $dx = \frac{\sqrt{3}}{\cos^2 \theta} d\theta$

よって

$$\text{与式} = \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{3 \tan^2 \theta + 3} \cdot \frac{\sqrt{3}}{\cos^2 \theta} d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos^2 \theta}{3} \cdot \frac{\sqrt{3}}{\cos^2 \theta} d\theta = \frac{\sqrt{3}}{3} \left[\theta \right]_{-\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{5\sqrt{3}}{36} \pi$$

x	$-1 \rightarrow \sqrt{3}$
θ	$-\frac{\pi}{6} \rightarrow \frac{\pi}{4}$

練習 28

$$(1) \quad \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \tan \frac{x}{2} \cos^2 \frac{x}{2} = 2 \tan \frac{x}{2} \cdot \frac{1}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1 + t^2}$$

$$\cos x = \cos 2 \cdot \frac{x}{2} = 2 \cos^2 \frac{x}{2} - 1 = \frac{2}{1 + \tan^2 \frac{x}{2}} - 1 = \frac{2}{1 + t^2} - 1 = \frac{1 - t^2}{1 + t^2}$$

$$\frac{dt}{dx} = \frac{1}{2} \cdot \frac{1}{\cos^2 \frac{x}{2}} = \frac{1}{2} \left(1 + \tan^2 \frac{x}{2} \right) = \frac{1 + t^2}{2}$$

よって, $\frac{dx}{dt} = \frac{2}{1 + t^2}$ (終)

(別解)

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}} = \frac{2 \tan \frac{x}{2}}{\tan^2 \frac{x}{2} + 1} = \frac{2t}{1 + t^2}$$

また, 次のように求めることもできる。

$$\tan x = \tan 2 \cdot \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} = \frac{2}{1 + t^2} \quad \text{から} \quad \sin x = \cos x \tan x = \frac{1 - t^2}{1 + t^2} \times \frac{2t}{1 - t^2} = \frac{2t}{1 + t^2}$$

(2) (1)の結果から, $\tan \frac{x}{2} = t$ とおくと

$$\frac{1}{1 + \sin x} = \frac{1}{1 + \frac{2t}{1 + t^2}} = \frac{1 + t^2}{1 + t^2 + 2t} = \frac{1 + t^2}{(t + 1)^2}$$

x	$0 \rightarrow \frac{2}{3}\pi$
t	$0 \rightarrow \sqrt{3}$

$$\begin{aligned} \text{よって, } \int_0^{\frac{2}{3}\pi} \frac{dx}{1 + \sin x} &= \int_0^{\sqrt{3}} \frac{1 + t^2}{(t + 1)^2} \cdot \frac{2}{1 + t^2} dt = \int_0^{\sqrt{3}} \frac{2}{(t + 1)^2} dt \\ &= \left[-\frac{2}{t + 1} \right]_0^{\sqrt{3}} = -\left(\frac{2}{\sqrt{3} + 1} - 2 \right) = -(\sqrt{3} - 1) + 2 = 3 - \sqrt{3} \end{aligned}$$

練習 29

$$(1) \quad \text{与式} = \int_0^{\frac{\pi}{2}} x (\sin x)' dx = \left[x \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx = \frac{\pi}{2} + \left[\cos x \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

$$(2) \quad \text{与式} = \int_0^1 x (e^x)' dx = \left[x e^x \right]_0^1 - \int_0^1 e^x dx = e - \left[e^x \right]_0^1 = e - (e - 1) = 1$$

$$(3) \quad \begin{aligned} \text{与式} &= \int_1^e \left(\frac{1}{3} x^3 \right)' \log x dx \\ &= \left[\frac{1}{3} x^3 \log x \right]_1^e - \int_1^e \frac{1}{3} x^3 \cdot \frac{1}{x} dx \\ &= \frac{1}{3} e^3 - \frac{1}{3} \int_1^e x^2 dx = \frac{1}{3} e^3 - \frac{1}{3} \left[\frac{1}{3} x^3 \right]_1^e = \frac{1}{3} e^3 - \frac{1}{9} (e^3 - 1) = \frac{1}{9} (2e^3 + 1) \end{aligned}$$

$$(4) \quad \begin{aligned} \text{与式} &= \int_1^e \left(-\frac{1}{x} \right)' \log x dx \\ &= \left[-\frac{1}{x} \log x \right]_1^e + \int_1^e \frac{1}{x} \cdot \frac{1}{x} dx = -\frac{1}{e} + \left[-\frac{1}{x} \right]_1^e = -\frac{1}{e} + \left(-\frac{1}{e} + 1 \right) = 1 - \frac{2}{e} \end{aligned}$$

練習 30

$$(1) \quad \begin{aligned} \text{与式} &= \int_{\alpha}^{\beta} (x - \alpha) \left\{ \frac{1}{2} (x - \beta)^2 \right\}' dx \\ &= \left[\frac{1}{2} (x - \alpha) (x - \beta)^2 \right]_{\alpha}^{\beta} - \frac{1}{2} \int_{\alpha}^{\beta} (x - \beta)^2 dx \\ &= -\frac{1}{2} \left[\frac{1}{3} (x - \beta)^3 \right]_{\alpha}^{\beta} = -\frac{1}{6} \times (\alpha - \beta)^3 \quad (\text{終}) \end{aligned}$$

$$(2) \quad \begin{aligned} \text{与式} &= \int_{\alpha}^{\beta} (x - \alpha) \left\{ \frac{1}{3} (x - \beta)^3 \right\}' dx \\ &= \left[\frac{1}{3} (x - \alpha) (x - \beta)^3 \right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \frac{1}{3} (x - \beta)^3 dx \\ &= 0 - \frac{1}{3} \left[\frac{(x - \beta)^4}{4} \right]_{\alpha}^{\beta} = -\frac{1}{3} \times \left\{ -\frac{(\alpha - \beta)^4}{4} \right\} = \frac{1}{12} (\beta - \alpha)^4 \quad (\text{終}) \end{aligned}$$

練習 31

$$\frac{\pi}{2} - x = t \quad \text{とおくと} \quad x = \frac{\pi}{2} - t, \quad dx = -dt$$

x	$0 \rightarrow \frac{\pi}{2}$
t	$\frac{\pi}{2} \rightarrow 0$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^n x dx &= \int_{\frac{\pi}{2}}^0 \sin^n \left(\frac{\pi}{2} - t \right) (-dt) \\ &= \int_0^{\frac{\pi}{2}} \cos^n t dt = \int_0^{\frac{\pi}{2}} \cos^n x dx \quad (\text{終}) \end{aligned}$$

練習 32

$$(1) \int_0^{\frac{\pi}{2}} \sin^6 x \, dx$$

$$= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5}{32} \pi$$

$$(2) \int_0^{\frac{\pi}{2}} \sin^7 x \, dx$$

$$= \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = \frac{16}{35}$$

$$(3) \int_0^{\frac{\pi}{2}} \cos^8 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^8 x \, dx$$

$$= \frac{8}{7} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{35}{256} \pi$$

練習 33

$$(1) F(x) = \int_{-1}^x 2t^2 \, dt - x \int_{-1}^x t \, dt \quad \text{より}$$

$$F'(x) = 2x^2 - \left(\int_{-1}^x t \, dt + x \cdot x \right)$$

$$= 2x^2 - \left[\frac{1}{2} t^2 \right]_{-1}^x - x^2 = \frac{1}{2} x^2 + \frac{1}{2}$$

$$(2) F'(x) = \left(e^x \int_{-3}^x e^t \, dt \right)' = e^x \int_{-3}^x e^t \, dt + e^x \cdot e^x$$

$$= e^x \left[e^t \right]_{-3}^x + e^{2x} = e^x (e^x - e^{-3}) + e^{2x}$$

$$= 2e^{2x} - e^{x-3}$$

節末問題

1.

$$(1) \quad \text{与式} = \int \frac{x - \sqrt{x} - 2}{x} dx = \int \left(1 - \frac{1}{\sqrt{x}} - \frac{2}{x} \right) dx = x - 2\sqrt{x} - 2 \log x + C$$

$$(2) \quad 2x + 3 = t \quad \text{とおくと} \quad 2dx = dt$$

$$\text{与式} = \int 10^t \cdot \frac{1}{2} dt = \frac{1}{2} \cdot \frac{10^t}{\log 10} + C = \frac{10^{2x+3}}{2 \log 10} + C$$

$$(3) \quad \sqrt{x-1} = t \quad \text{とおくと} \quad x = t^2 + 1 \quad \text{から} \quad dx = 2t dt$$

よって

$$\begin{aligned} \text{与式} &= \int (t^2 + 1) \cdot t \cdot 2t dt = \int (2t^4 + 2t^2) dt = \frac{2}{5} t^5 + \frac{2}{3} t^3 + C \\ &= \frac{2}{15} t^3 (3t^2 + 5) + C = \frac{2}{15} (3x + 2)(x - 1)\sqrt{x-1} + C \end{aligned}$$

$$(4) \quad \sqrt{1-x} = t \quad \text{とおくと} \quad 1-x = t^2$$

$$x = 1 - t^2 \quad \text{から} \quad dx = -2t dt$$

$$\begin{aligned} \text{与式} &= \int \frac{(1-t^2)^2}{t} \cdot (-2t) dt = -2 \int (1-t^2)^2 dt \\ &= -2 \int (t^4 - 2t^2 + 1) dt = -2 \left(\frac{1}{5} t^5 - \frac{2}{3} t^3 + t \right) + C \\ &= -\frac{2}{15} t (3t^4 - 10t^2 + 15) + C = -\frac{2}{15} \sqrt{1-x} \{ 3(1-x)^2 - 10(1-x) + 15 \} + C \\ &= -\frac{2}{15} (3x^2 + 4x + 8) \sqrt{1-x} + C \end{aligned}$$

$$(5) \quad \log x = t \quad \text{とおくと} \quad \frac{1}{x} dx = dt$$

よって

$$\text{与式} = \int t dt = \frac{1}{2} t^2 + C = \frac{(\log x)^2}{2} + C$$

$$(6) \quad \sin x = t \quad \text{とおくと} \quad \cos x dx = dt$$

よって

$$\text{与式} = \int \frac{1}{1+t} dt = \log |1+t| + C = \log (1 + \sin x) + C$$

2.

$$(1) \quad \text{与式} = \int x \left(-\frac{1}{3} \cos 3x \right)' dx = -\frac{1}{3} x \cos 3x + \int \frac{1}{3} \cos 3x dx = -\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + C$$

$$(2) \quad \begin{aligned} \text{与式} &= \int x \left(-\frac{1}{3} e^{-3x} \right)' dx = -\frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} dx \\ &= -\frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} + C = -\frac{1}{9} (3x + 1) e^{-3x} + C \end{aligned}$$

$$(3) \quad \begin{aligned} \text{与式} &= \int \left(\frac{1}{3} x^3 \right)' \log x dx = \frac{1}{3} x^3 \log x - \int \frac{1}{3} x^3 \cdot \frac{1}{x} dx \\ &= \frac{1}{3} x^3 \log x - \int \frac{1}{3} x^2 dx = \frac{1}{3} x^3 \log x - \frac{x^3}{9} + C \end{aligned}$$

$$(4) \quad \begin{aligned} \text{与式} &= \int \left(\frac{x^2}{2} \right)' \log(x+1) dx = \frac{x^2}{2} \log(x+1) - \int \frac{x^2}{2} \cdot \frac{1}{x+1} dx \\ &= \frac{x^2}{2} \log(x+1) - \frac{1}{2} \int \left(x - 1 + \frac{1}{x+1} \right) dx \\ &= \frac{x^2}{2} \log(x+1) - \frac{1}{4} x^2 + \frac{1}{2} x - \frac{1}{2} \log(x+1) + C \\ &= \frac{x^2 - 1}{2} \log(x+1) - \frac{1}{4} x^2 + \frac{1}{2} x + C \end{aligned}$$

(別解)

$$\begin{aligned} \text{与式} &= \int \{ (x+1) \log(x+1) - \log(x+1) \} dx \\ &= \frac{(x+1)^2}{2} \log(x+1) - \int \frac{(x+1)^2}{2} \cdot \frac{1}{x+1} dx - \{ (x+1) \log(x+1) - x \} + C \\ &= \frac{x^2 - 1}{2} \log(x+1) - \frac{1}{4} x^2 + \frac{1}{2} x + C \end{aligned}$$

3.

$$(1) \text{ 右の割り算より 与式} = \int \left(x - 3 - \frac{1}{x+2} \right) dx$$

$$= \frac{1}{2} x^2 - 3x - \log |x+2| + C$$

$$\begin{array}{r} x-3 \\ x+2 \overline{) x^2 - x - 7} \\ \underline{x^2 + 2x} \\ -3x - 7 \\ \underline{-3x - 6} \\ 5 \end{array}$$

$$(2) \frac{3x+3}{x^2-9} = \frac{3x+3}{(x+3)(x-3)} = \frac{a}{x+3} + \frac{b}{x-3} \text{ とおくと}$$

両辺に $(x+3)(x-3)$ を掛けて

$$3x+3 = a(x-3) + b(x+3)$$

$$= (a+b)x - 3a + 3b$$

両辺の係数を比較して

$$a+b=3, -3a+3b=3$$

これより $a=1, b=2$

よって,

$$\text{与式} = \int \left(\frac{2}{x-3} + \frac{1}{x+3} \right) dx$$

$$= 2 \log |x-3| + \log |x+3| + C = \log (x-3)^2 |x+3| + C$$

4.

$$(1) \text{ 与式} = \frac{1}{2} \int (\cos 6x + \cos 4x) dx = \frac{1}{12} \sin 6x + \frac{1}{8} \sin 4x + C$$

$$(2) \text{ 与式} = \int x \cdot \frac{1+\cos 2x}{2} dx = \int \left(\frac{1}{2} x + \frac{1}{2} x \cos 2x \right) dx = \frac{1}{4} x^2 + \frac{1}{2} \int x \left(\frac{1}{2} \sin 2x \right)' dx$$

$$= \frac{1}{4} x^2 + \frac{1}{2} \left(\frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x dx \right) = \frac{1}{4} x^2 + \frac{1}{4} x \sin 2x - \frac{1}{4} \int \sin 2x dx$$

$$= \frac{1}{4} x^2 + \frac{1}{4} x \sin 2x + \frac{1}{8} \cos 2x + C$$

$$(3) \cos^4 x \sin^3 x = \cos^4 x (1 - \cos^2 x) \sin x \text{ から } \cos x = t \text{ とおくと}$$

$$-\sin x dx = dt \text{ よって}$$

$$\text{与式} = \int t^4 (1-t^2) \cdot (-1) dt = \int (t^6 - t^4) dt = \frac{1}{7} t^7 - \frac{1}{5} t^5 + C = \frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + C$$

$$(4) e^x + 1 = t \text{ とおくと } e^x dx = dt \text{ よって } dx = \frac{1}{t} dt$$

$$\text{与式} = \int \frac{t^3}{t+1} \cdot \frac{1}{t} dt = \int \frac{t^2}{t+1} dt = \int \left(t - 1 + \frac{1}{t+1} \right) dt$$

$$= \frac{1}{2} t^2 - t + \log(t+1) + C = \frac{1}{2} e^{2x} - e^x + \log(e^x + 1) + C$$

(別解)

$$e^x = t \text{ とおくと } e^x dx = dt \text{ よって } dx = \frac{1}{t} dt$$

$$\begin{aligned}\text{与式} &= \int \frac{t^3}{t+1} \cdot \frac{1}{t} dt = \int \frac{t^2}{t+1} dt = \int \left(t-1 + \frac{1}{t+1} \right) dt \\ &= \frac{1}{2} t^2 - t + \log(t+1) + C \\ &= \frac{1}{2} e^{2x} - e^x + \log(e^x + 1) + C\end{aligned}$$

5.

$$(1) \quad \frac{1}{\sin x} = \frac{\sin x}{\sin^2 x} = \frac{\sin x}{1 - \cos^2 x} \text{ だから } \int \frac{1}{\sin x} dx = \int \frac{\sin x}{1 - \cos^2 x} dx$$

$$\text{ここで, } \cos x = t \text{ とおくと } -\sin x dx = dt$$

よって

$$\begin{aligned}\text{与式} &= \int \frac{1}{1-t^2} \cdot (-1) dt = \int \frac{1}{(t-1)(t+1)} dt = \frac{1}{2} \int \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt \\ &= \frac{1}{2} (\log|t-1| - \log|t+1|) + C = \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + C \\ &= \frac{1}{2} \log \left| \frac{\cos x - 1}{\cos x + 1} \right| + C = \frac{1}{2} \log \frac{1 - \cos x}{1 + \cos x} + C\end{aligned}$$

$$(2) \quad \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x} \text{ だから } \int \frac{1}{\cos x} dx = \int \frac{\cos x}{1 - \sin^2 x} dx$$

$$\text{ここで, } \sin x = t \text{ とおくと } \cos x dx = dt$$

よって

$$\begin{aligned}\text{与式} &= \int \frac{1}{1-t^2} dt = \int \frac{-1}{(t-1)(t+1)} dt = \frac{1}{2} \int \left(\frac{1}{t+1} - \frac{1}{t-1} \right) dt \\ &= \frac{1}{2} (\log|t+1| - \log|t-1|) + C = \frac{1}{2} \log \left| \frac{t+1}{t-1} \right| + C \\ &= \frac{1}{2} \log \frac{1 + \sin x}{1 - \sin x} + C\end{aligned}$$

6.

(1) $\log x = t$ とおくと $\frac{1}{x} dx = dt$ よって

x	$1 \rightarrow e$
t	$0 \rightarrow 1$

$$\text{与式} = \int_0^1 t \, dt = \left[\frac{1}{2} t^2 \right]_0^1 = \frac{1}{2}$$

(2) $x = \sqrt{3} \sin \theta$ とおくと $dx = \sqrt{3} \cos \theta \, d\theta$

$$\sqrt{3-x^2} = \sqrt{3(1-\sin^2 \theta)} = \sqrt{3 \cos^2 \theta} = \sqrt{3} |\cos \theta|$$

$$-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3} \text{ のとき } \cos \theta > 0 \text{ だから}$$

x	$-\frac{\sqrt{3}}{2} \rightarrow \frac{3}{2}$
θ	$-\frac{\pi}{6} \rightarrow \frac{\pi}{3}$

$$\text{よって 与式} = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sqrt{3} \cos \theta} \cdot \sqrt{3} \cos \theta \, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta = \left[\theta \right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{2}$$

(3) $x = \tan \theta$ とおくと $dx = \frac{1}{\cos^2 \theta} d\theta$

ここで

$$(x^2 + 1)^{\frac{3}{2}} = (\tan^2 \theta + 1)^{\frac{3}{2}} = \left(\frac{1}{\cos^2 \theta} \right)^{\frac{3}{2}} = \frac{1}{\cos^3 \theta}$$

よって

$$\text{与式} = \int_0^{\frac{\pi}{4}} \cos^3 \theta \cdot \frac{1}{\cos^2 \theta} \, d\theta = \int_0^{\frac{\pi}{4}} \cos \theta \, d\theta = \left[\sin \theta \right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}$$

x	$0 \rightarrow 1$
θ	$0 \rightarrow \frac{\pi}{4}$

(4) $\text{与式} = \int_0^{\frac{\pi}{4}} x (\tan x)' \, dx = \left[x \tan x \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx = \frac{\pi}{4} + \int_0^{\frac{\pi}{4}} \frac{(\cos x)'}{(\cos x)} \, dx$

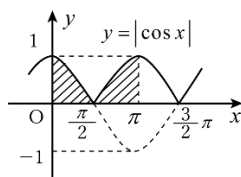
$$= \frac{\pi}{4} + \left[\log (\cos x) \right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \log \frac{1}{\sqrt{2}} = \frac{\pi}{4} - \log \sqrt{2}$$

7.

(1) $\text{与式} = \int_0^{\frac{\pi}{2}} \cos x \, dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) \, dx$

$$= \left[\sin x \right]_0^{\frac{\pi}{2}} - \left[\sin x \right]_{\frac{\pi}{2}}^{\pi}$$

$$= 1 - (-1) = 2$$

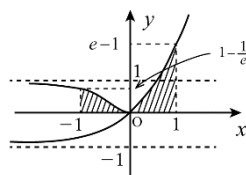


(2) $\text{与式} = \int_{-1}^0 (-e^x + 1) \, dx + \int_0^1 (e^x - 1) \, dx$

$$= \left[-e^x + x \right]_{-1}^0 - \left[e^x - x \right]_0^1$$

$$= -1 - \left(-\frac{1}{e} - 1 \right) + (e - 1) - 1$$

$$= e + \frac{1}{e} - 2$$



8.

$$(1) \quad x = \frac{e^t - e^{-t}}{2} \quad \text{とおくと} \quad dx = \frac{e^t + e^{-t}}{2} dt$$

$$\text{また} \quad \sqrt{x^2 + 1} = \sqrt{\left(\frac{e^t - e^{-t}}{2}\right)^2 + 1} = \sqrt{\frac{e^{2t} + 2 + e^{-2t}}{4}} = \sqrt{\left(\frac{e^t + e^{-t}}{2}\right)^2} = \frac{e^t + e^{-t}}{2}$$

したがって

$$A = \int \frac{2}{e^t + e^{-t}} \cdot \frac{e^t + e^{-t}}{2} dt = \int dt = t + C$$

$$(2) \quad e^t - \frac{1}{e^t} = 2x \quad \text{から} \quad (e^t)^2 - 2xe^t - 1 = 0$$

$$e^t = x \pm \sqrt{x^2 + 1}$$

$$e^t > 0 \quad \text{より適するのは} \quad e^t = x + \sqrt{x^2 + 1}$$

$$\text{よって} \quad t = \log(x + \sqrt{x^2 + 1})$$

$$\text{ゆえに} \quad \int \frac{1}{\sqrt{x^2 + 1}} dx = \log(x + \sqrt{x^2 + 1}) + C \quad \text{が成り立つ。 (終)}$$

9.

$$\int_a^x (x-t) f'(t) dt = \int_a^x x f'(t) dt - \int_a^x t f'(t) dt \quad \text{だから}$$

$$\begin{aligned} \frac{d}{dx} \int_a^x (x-t) f'(t) dt &= \left(x \int_a^x f'(t) dt \right)' - \left(\int_a^x t f'(t) dt \right) \\ &= \int_a^x f'(t) dt + x f'(x) - x f'(x) \\ &= \int_a^x f'(t) dt = [f(t)]_a^x = f(x) - f(a) \quad (\text{終り}) \end{aligned}$$

10.

与式の両辺を x で微分すると

$$f(x) = 2(\log x) \cdot \frac{1}{x} - \frac{1}{x} = \frac{2 \log x - 1}{x}$$

$$\text{与式に } x=a \text{ を代入すると} \quad \int_0^a f(x) dt = 0 \quad \text{だから}$$

$$(\log a)^2 - \log a - 6 = 0$$

$$(\log a - 3)(\log a + 2) = 0 \quad \therefore \log a = 3, -2$$

$$\text{したがって, } 3 = \log e^3, -2 = \log \frac{1}{e^2} \quad \text{より} \quad a = e^3, \frac{1}{e^2}$$

11.

$$I_n = \int_0^e (x)' (\log x)^n dx = \left[x (\log x)^n \right]_0^e - \int_0^e x \cdot n (\log x)^{n-1} \cdot \frac{1}{x} dx = e - n \int_0^{\frac{\pi}{2}} (\log x)^{n-1} dx$$

よって, $I_n = e - nI_{n-1}$ (終)

12. $\cos \alpha \cos \beta = \frac{1}{2} \{ \cos (\alpha + \beta) + \cos (\alpha - \beta) \}$ より

$$I = \int_0^{2\pi} \cos mx \cos nx dx \text{ とおくと}$$

$$I = \frac{1}{2} \int_0^{2\pi} \{ \cos (m+n)x + \cos (m-n)x \} dx$$

(1) $m \neq n$ のとき, $m-n \neq 0$ であるから

$$I = \frac{1}{2} \left[\frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{m-n} \right]_0^{2\pi} = 0$$

(2) $m = n$ のとき, $\cos (m-n)x = \cos 0 = 1$ であるから

$$I = \frac{1}{2} \int_0^{2\pi} (\cos 2mx + 1) dx = \frac{1}{2} \left[\frac{1}{2m} \sin 2mx + x \right]_0^{2\pi} = \pi$$

以上から

$$\int_0^{2\pi} \cos mx \cos nx dx = \begin{cases} 0 & (m \neq n) \\ \pi & (m = n) \end{cases}$$