

2章 微分法 解答

2節 導関数

練習 1

$$(1) \quad \frac{f(5) - f(2)}{5 - 2} = \frac{5^2 - 2^2}{3} = 7$$

$$(2) \quad \frac{f(1) - f(-3)}{1 - (-3)} = \frac{1^2 - (-3)^2}{4} = -2$$

練習 2

$$(1) \quad f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ = \lim_{h \rightarrow 0} \frac{\{(2+h)^2 + 3(2+h)\} - (2^2 + 3 \cdot 2)}{h} = \lim_{h \rightarrow 0} \frac{7h + h^2}{h} = \lim_{h \rightarrow 0} (7 + h) = 7$$

$$(2) \quad f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ = \lim_{h \rightarrow 0} \frac{\{-2(2+h)^2 + 1\} - (-2 \cdot 2^2 + 1)}{h} = \lim_{h \rightarrow 0} \frac{-8h - 2h^2}{h} = \lim_{h \rightarrow 0} (-8 - 2h) = -8$$

練習 3

$$\lim_{h \rightarrow +0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow +0} \frac{|(2+h)^2 - 2(2+h)| - |2^2 - 2 \cdot 2|}{h} \\ = \lim_{h \rightarrow +0} \frac{|h^2 + 2h|}{h} = \lim_{h \rightarrow +0} \frac{h(h+2)}{h} = 2$$
$$\lim_{h \rightarrow -0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow -0} \frac{|(2+h)^2 - 2(2+h)| - |2^2 - 2 \cdot 2|}{h} \\ = \lim_{h \rightarrow -0} \frac{|h(h+2)|}{h} = \lim_{h \rightarrow -0} \frac{-h(h+2)}{h} = -2$$

よって、 $f'(2)$ は存在しないから、 $x=2$ で微分可能ではない。

練習 4

$$(1) \quad f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2$$

$$(2) \quad f'(x) = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{x(x+h)} = -\frac{1}{x^2}$$

練習 5

$$(1) \quad y' = (x^2 - 4x + 7)' = (x^2)' - 4(x)' + (7)' = 2x - 4 \cdot 1 = 2x - 4$$

$$(2) \quad y' = (-3x^2 + 2x + 3)' = -3(x^2)' + 2(x)' + (3)' = -3 \cdot 2x + 2 \cdot 1 = -6x + 2$$

$$(3) \quad \text{展開して整理すると } y = 4x^2 + 20x + 25 \text{ であるから}$$

$$y' = (4x^2 + 20x + 25)' = 4(x^2)' + 20(x)' + (25)' = 4 \cdot 2x + 20 \cdot 1 = 8x + 20$$

$$(4) \quad \text{展開して整理すると } y = 1 - 9x^2 \text{ であるから}$$

$$y' = (1 - 9x^2)' = (1)' - 9(x^2)' = -9 \cdot 2x = -18x$$

$$(5) \quad y' = \left(x^3 + \frac{1}{2}x^2 + \frac{2}{3} \right)' = (x^3)' + \left(\frac{1}{2}x^2 \right)' + \left(\frac{2}{3} \right)' = 3x^2 + x$$

$$(6) \quad y' = \left(-\frac{1}{6}x^3 + \frac{3}{4}x^2 + 5x \right)' = \left(-\frac{1}{6}x^3 \right)' + \left(\frac{3}{4}x^2 \right)' + (5x)'$$

$$= -\frac{1}{2}x^2 + \frac{3}{2}x + 5$$

$$(7) \quad \text{展開して整理すると } y = -2x^3 + 7x^2 - 3x \text{ であるから}$$

$$y' = (-2x^3 + 7x^2 - 3x)' = -2(x^3)' + 7(x^2)' - 3(x)' = -2 + 3x^2 \cdot 2 - 3 = -6x^2 + 14x - 3$$

$$(8) \quad \text{展開して整理すると } y = x^3 - 6x^2 + 12x - 8 \text{ であるから}$$

$$y' = (x^3 - 6x^2 + 12x - 8)' = (x^3)' - 6(x^2)' + 12(x)' - (8)' = 3x^2 - 6 \cdot 2x + 12 \cdot 1 = 3x^2 - 12x + 12$$

練習 6

$$(1) \quad y' = (x + 2x)'(4x - 3) + (x + 2)(4x - 3)' = 1 \cdot (4x - 3) + (x + 2) \cdot 4 = 8x + 5$$

$$(2) \quad y' = (x^2 + 1)'(2x - 5) + (x^2 + 1)(2x - 5)' = 2x(2x - 5) + (x^2 + 1) \cdot 2 = 6x^2 - 10x + 2$$

$$(3) \quad y' = (x + 1)'(x^2 - x + 1) + (x + 1)(x^2 - x + 1)' = 1 \cdot (x^2 - x + 1) + (x + 1)(2x - 1) = 3x^2$$

$$(4) \quad y' = (x + 1)'(x + 2)(x + 3) + (x + 1)(x + 2)'(x + 3) + (x + 1)(x + 2)(x + 3)'$$

$$= (x^2 + 5x + 6) + (x^2 + 4x + 3) + (x^2 + 3x + 2) = 3x^2 + 12x + 11$$

練習 7

$$(1) \quad y' = -\frac{(3x + 2)'}{(3x + 2)^2} = -\frac{3}{(3x + 2)^2}$$

$$(2) \quad y' = \frac{(x - 1)'(x^2 - 3) - (x - 1)(x^2 - 3)'}{(x^2 - 3)^2} = \frac{x^2 - 3 - (x - 1) \cdot 2x}{(x^2 - 3)^2} = \frac{-x^2 + 2x - 3}{(x^2 - 3)^2}$$

$$(3) \quad y' = -\frac{(x^2 + 2)'(x^2 + 1) - (x^2 + 2)(x^2 + 1)'}{(x^2 + 1)^2} = \frac{2x(x^2 + 1) - (x^2 + 2) \cdot 2x}{(x^2 + 1)^2} = \frac{-2x}{(x^2 + 1)^2}$$

練習 8

$$(1) \quad y' = \left(\frac{1}{x} \right)' = (x^{-1})' = -x^{-2} = -\frac{1}{x^2}$$

$$(2) \quad y' = \left(\frac{3}{x^2} \right)' = (3x^{-2})' = -6x^{-3} = -\frac{6}{x^3}$$

$$(3) \quad y' = \left(-\frac{1}{2x^4} \right)' = \left(-\frac{1}{2} x^{-4} \right)' = 2x^{-5} = \frac{2}{x^5}$$

練習 9

$$(1) \quad y' = 2(3x-5)(3x-5)' = 6(3x-5)$$

$$(2) \quad y' = 3(1-2x)^2(1-2x)' = -6(1-2x)^2$$

$$(3) \quad y' = 4(x^2+x+1)^3(x^2+x+1)' = 4(2x+1)(x^2+x+1)^3$$

$$(4) \quad y' = -3(x+1)^{-4} \cdot (x+1)' = -\frac{3}{(x+1)^4}$$

$$(5) \quad y' = 3\left(x + \frac{1}{x}\right)^2 \left(x + \frac{1}{x}\right)' = 3\left(x + \frac{1}{x}\right)^2 \left(1 - \frac{1}{x^2}\right) = \frac{3(x^2+1)^2(x+1)(x-1)}{x^4}$$

$$(6) \quad y' = 4\left(\frac{x-2}{x}\right)^3 \left(\frac{x-2}{x}\right)' = 4\left(\frac{x-2}{x}\right)^3 \cdot \frac{2}{x^2} = \frac{8(x-2)^3}{x^5}$$

練習 10

$$(1) \quad y = f(ax+b), \quad u = ax+b \quad \text{とおくと} \quad y = f(u)$$

よって

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot a = af'(ax+b)$$

$$(2) \quad y = \{f(x)\}^n, \quad u = f(x) \quad \text{とおくと} \quad y = u^n$$

よって

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1} \cdot f'(x) = n\{f(x)\}^{n-1} f'(x)$$

練習 11

$$(1) \quad y' = \left(-x^{\frac{2}{3}} \right)' = \frac{2}{3} x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$(2) \quad y' = \left(\sqrt{x^2+x+1} \right)' = \left\{ (x^2+x+1)^{\frac{1}{2}} \right\}' = \frac{1}{2} (x^2+x+1)^{-\frac{1}{2}} (x^2+x+1)' = \frac{2x+1}{2\sqrt{x^2+x+1}}$$

$$(3) \quad y' = \left(\sqrt[4]{2x+1} \right)' = \left\{ (2x+1)^{\frac{1}{4}} \right\}' = \frac{1}{4} (2x+1)^{-\frac{3}{4}} (2x+1)' = \frac{2}{4\sqrt[4]{(2x+1)^3}} = \frac{1}{2\sqrt[4]{(2x+1)^3}}$$

練習 12

$$y = \sqrt[4]{x} \quad \text{より} \quad x = y^4 \quad \text{であるから} \quad \frac{dx}{dy} = 4y^3$$

$$\text{よって} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{4y^3} = \frac{1}{4\sqrt[4]{x^3}}$$

練習 13

$$(1) \quad y' = 1 \cdot \cos x + x(-\sin x) - \cos x = -x \sin x$$

$$(2) \quad y' = (\cos 3x) \cdot (3x)' = 3 \cos 3x$$

$$(3) \quad y' = -\sin(1-x) \cdot (1-x)' = \sin(1-x)$$

$$(4) \quad y' = \frac{1}{\cos^2 2x} \cdot (2x)' = \frac{2}{\cos^2 2x}$$

$$(5) \quad y' = 3(\cos 2x)^2 \cdot (\cos 2x)' = 3(\cos 2x)^2 \cdot (-2 \sin 2x) = -6 \cos^2 2x \cdot \sin 2x$$

$$(6) \quad y' = 2 \tan 3x \cdot (\tan 3x)' = 2 \tan 3x \cdot \frac{1}{\cos^2 3x} (3x)' = \frac{6 \tan 3x}{\cos^2 3x} = \frac{6 \sin 3x}{\cos^3 3x}$$

$$(7) \quad y' = \left\{ (1 - \cos x)^{\frac{1}{2}} \right\}' = \frac{1}{2} (1 - \cos x)^{-\frac{1}{2}} (1 - \cos x)' = \frac{\sin x}{2\sqrt{1 - \cos x}}$$

$$(8) \quad y' = -\frac{(1 + \sin 2x)'}{(1 + \sin 2x)^2} = -\frac{2 \cos 2x}{(1 + \sin 2x)^2}$$

$$(9) \quad y' = \frac{(\sin x)'x - \sin x \cdot (x)'}{(x)^2} = \frac{x \cos x - \sin x}{x^2}$$

練習 14 $y = \cos^{-1} x$

$$(1) \quad x = \cos y \quad \text{であるから} \quad 1 = \frac{d}{dx}(\cos y) = -\sin y \frac{dy}{dx}$$

$$\text{したがって,} \quad \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}$$

$$(2) \quad x = \cos y \quad \text{の両辺を } y \text{ で微分して} \quad \frac{dx}{dy} = -\sin y$$

$$\text{したがって} \quad \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}$$

練習 15

$$(1) \quad \frac{1}{x} = u \quad \text{とおくと} \quad y = \cos^{-1} u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{\sqrt{1 - u^2}} \left(\frac{1}{x} \right)' = -\frac{1}{\sqrt{1 - \left(\frac{1}{x} \right)^2}} \left(-\frac{1}{x^2} \right) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$(2) \quad \frac{1}{x} = u \quad \text{とおくと} \quad y = \text{Tan}^{-1} u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \left(\frac{1}{x} \right)' = \frac{1}{1+u^2} \left(-\frac{1}{x^2} \right) = \frac{1}{1+\frac{1}{x^2}} \left(-\frac{1}{x^2} \right) = -\frac{1}{x^2+1}$$

$$(3) \quad \sqrt{x} = u \quad \text{とおくと} \quad y = \text{Sin}^{-1} u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} (\sqrt{x})' = \frac{1}{\sqrt{1-x}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x(1-x)}}$$

練習 16

$$(1) \quad y' = \frac{1}{3x} \cdot (3x)' = \frac{1}{x}$$

$$(2) \quad y' = \frac{1}{2-3x} \cdot (2-3x)' = \frac{-3}{2-3x} = \frac{3}{3x-2}$$

$$(3) \quad y' = \frac{1}{x \log 10}$$

別解 底を e に変換して

$$y = \frac{\log x}{\log 10} \text{ より } y' = \frac{1}{\log 10} \cdot \frac{1}{x} = \frac{1}{x \log 10}$$

$$(4) \quad y' = (x^2)' \log x + x^2 (\log x)' = 2x \log x + x^2 \cdot \frac{1}{x} = 2x \log x + x$$

$$(5) \quad y' = 2(\log x)(\log x)' = \frac{2}{x} \log x$$

$$(6) \quad y' = -\frac{(\log x)'}{(\log x)^2} = -\frac{1}{x(\log x)^2}$$

練習 17

$$(1) \quad y' = \frac{(3x+1)'}{3x+1} = \frac{3}{3x+1}$$

$$(2) \quad y' = \frac{(x^2-3x+2)'}{x^2-3x+2} = \frac{2x-3}{x^2-3x+2}$$

$$(3) \quad y' = \frac{(\sin x)'}{\sin x} = \frac{\cos x}{\sin x}$$

練習 18

両辺の絶対値の自然対数をとると

$$\log |y| = \log |x+1| + 2 \log |x-2| - 3 \log |x-1|$$

両辺を x で微分すると

$$\begin{aligned} \frac{y'}{y} &= \frac{1}{x+1} + \frac{2}{x-2} - \frac{3}{x-1} = \frac{(x-2)(x-1) + 2(x-1)(x-1) - 3(x+1)(x-2)}{(x+1)(x-2)(x-1)} \\ &= \frac{6}{(x+1)(x-2)(x-1)} \end{aligned}$$

$$\text{ゆえに} \quad y' = \frac{6}{(x+1)(x-2)(x-1)} \cdot \frac{(x+1)(x-2)^2}{(x-1)^3} = \frac{6(x-2)}{(x-1)^4}$$

練習 19

$y = x^\alpha$ において、両辺の自然対数をとると $\log y = \alpha \log x$

両辺を x で微分すると $\frac{y'}{y} = \frac{\alpha}{x}$

$$\text{よって} \quad y' = \frac{\alpha}{x} \cdot y = \frac{\alpha}{x} \cdot x^\alpha = \alpha x^{\alpha-1}$$

練習 20

$$(1) \quad y' = e^{2x} \cdot (2x)' = 2e^{2x}$$

$$(2) \quad y' = (x+1)'e^{-x} + (x+1)(e^{-x})' = 1 \cdot e^{-x} + (x+1)(-e^{-x}) = -xe^{-x}$$

$$(3) \quad y' = 3^{x+2} \log 3 \cdot (x+2)' = 3^{x+2} \log 3$$

$$(4) \quad y' = (x)'a^{3x} + x(a^{3x})' = 1 \cdot a^{3x} + x \cdot a^{3x} \log a \cdot (3x)' = (1 + 3x \log a)a^{3x}$$

練習 21

$$(1) \quad y' = e^x \log x + e^x \cdot \frac{1}{x} = \left(\log x + \frac{1}{x} \right) e^x$$

$$(2) \quad y' = -e^{-x} \sin x + e^{-x} \cos x = (\cos x - \sin x)e^{-x}$$

$$(3) \quad y' = e^{\sin x} (\sin x)' = e^{\sin x} \cos x$$

$$(4) \quad y' = e^{\frac{1}{x}} \cdot \left(\frac{1}{x} \right)' = -\frac{1}{x^2} e^{\frac{1}{x}}$$

練習 22

$$(1) \quad y' = 4x^3 - 6x \quad \text{よ} \quad y'' = 12x^2 - 6$$

$$(2) \quad y' = \frac{1}{x} \quad \text{よ} \quad y'' = -\frac{1}{x^2}$$

$$(3) \quad y' = e^{2x} + 2xe^{2x} = (2x+1)e^{2x}$$

$$y'' = 2e^{2x} + (4x+2)e^{2x} = (4x+4)e^{2x}$$

練習 23

- (1) $y' = 3x^2$ $y'' = 6x$ より $y''' = 6$
 (2) $y' = e^x$ $y'' = e^x$ より $y''' = e^x$
 (3) $y' = -\sin x$ $y'' = -\cos x$ より $y''' = \sin x$

練習 24

$$y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}, \quad y^{(3)} = \frac{1 \cdot 2}{x^3} = \frac{2!}{x^3}$$

$$y^{(4)} = -\frac{3 \cdot 2!}{x^4} = -\frac{3!}{x^4}$$

$$y^{(5)} = \frac{4 \cdot 3!}{x^5} = \frac{4!}{x^5}$$

$$\text{これより } y^{(n)} = (-1)^{n-1} \frac{(n-1)!}{x^n}$$

練習 25

$$y' = e^x \cos x - e^x \sin x = e^x (\cos x - \sin x)$$

$$y'' = e^x (\cos x - \sin x) + e^x (-\sin x - \cos x) = -2e^x \sin x$$

$$\text{ゆえに } y'' - 2y' + 2y = -2e^x \sin x - 2e^x (\cos x - \sin x) + 2e^x \cos x = 0$$

節末問題

1. $f(x) = x^2 + x$

$$(1) \quad \frac{f(3) - f(1)}{3 - 1} = \frac{(9 + 3) - (1 + 1)}{3 - 1} = 5$$

$$(2) \quad \frac{f(2+h) - f(2)}{2+h-2} = \frac{\{(2+h)^2 + (2+h)\} - (2^2 + 2)}{h} = \frac{5h + h^2}{h} = 5 + h$$

$$(3) \quad f'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} (5 + h) = 5$$

2.

$$(1) \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\{(x+h)^3 - 2(x+h)\} - (x^3 - 2x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 2h}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 2) = 3x^2 - 2$$

$$(2) \quad f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

3.

$$f(a+h) - f(a-h) = \{f(a+h) - f(a)\} - \{f(a-h) - f(a)\} \text{ より}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(a+h) - f(a-h)}{h} &= \lim_{h \rightarrow 0} \left\{ \frac{f(a+h) - f(a)}{h} - \frac{f(a-h) - f(a)}{h} \right\} \\ &= \lim_{h \rightarrow 0} \left\{ \frac{f(a+h) - f(a)}{h} + \frac{f(a-h) - f(a)}{h} \right\} \\ &= f'(a) + f'(a) = 2f'(a) \end{aligned}$$

$$f'(x) = e^x \text{ より } f'(a) = e^a \text{ であるから } \lim_{h \rightarrow 0} \frac{f(a+h) - f(a-h)}{h} = 2e^a$$

4. $f(x) = ax^2 + bx + c$ とおくと $f'(x) = 2ax + b$ であるから

$$\begin{cases} f(1) = a + b + c = 0 & \dots \textcircled{1} \\ f'(1) = 2a + b = 1 & \dots \textcircled{2} \\ f'(2) = 4a + b = 3 & \dots \textcircled{3} \end{cases}$$

①, ②, ③を解いて $a=1, b=-1, c=0$

よって, $f(x) = x^2 - x$

5.

$$(1) \quad \frac{dV}{dr} = \left(\frac{4}{3} \pi r^3 \right)' = 4\pi r^2$$

$$(2) \quad \frac{ds}{dt} = (t^3 + at^2 + b)' = 3t^2 + 2at$$

6.

$$\begin{aligned} (1) \quad y' &= 3x^2(x+4)^3 + x^3 \cdot 3(x+4)^2 \\ &= 3x^2(x+4)^2(2x+4) = 6x^2(x+2)(x+4)^2 \end{aligned}$$

$$\begin{aligned} (2) \quad y' &= 2(2x^3 - 6x + 3) \cdot (2x^3 - 6x + 3)' \\ &= 2(2x^3 - 6x + 3)(6x^2 - 6) = 12(x+1)(x-1)(2x^3 - 6x + 3) \end{aligned}$$

$$(3) \quad y' = -\frac{(x+3)'}{(x+3)^2} = -\frac{1}{(x+3)^2}$$

$$(4) \quad y' = \frac{(1+x^2)'(1-x^2) - (1+x^2)(1-x^2)'}{(1-x^2)^2} = \frac{2x(1-x^2) - (1+x^2)(-2x)}{(1-x^2)^2} = \frac{4x}{(x^2-1)^2}$$

$$(5) \quad y' = \left\{ (4x+3)^{\frac{1}{2}} \right\}' = \frac{1}{2}(4x+3)^{-\frac{1}{2}} \cdot 4 = \frac{2}{\sqrt{4x+3}}$$

$$(6) \quad y' = \left\{ (3x-2)^{\frac{2}{3}} \right\}' = \frac{2}{3}(3x-2)^{-\frac{1}{3}} \cdot 3 = \frac{2}{\sqrt[3]{3x-2}}$$

7.

$$(1) \quad y = \sqrt{\frac{x^2+1}{2x+1}} \quad \text{の両辺の自然対数をとると} \quad \log y = \frac{1}{2} \left\{ \log |x^2+1| - \log |2x+1| \right\}$$

$$\text{両辺を } x \text{ で微分すると} \quad \frac{y'}{y} = \frac{1}{2} \left(\frac{2x}{x^2+1} - \frac{2}{2x+1} \right) = \frac{x^2+x-1}{(x^2+1)(2x+1)}$$

$$\text{したがって} \quad y' = \frac{x^2+x-1}{(x^2+1)(2x+1)} \sqrt{\frac{x^2+1}{2x+1}} = \frac{x^2+x-1}{\sqrt{(2x+1)^3(x^2+1)}}$$

$$(2) \quad y = x^x \text{ の両辺の自然対数をとると} \quad \log y = x \log x$$

$$\text{両辺を微分すると} \quad \frac{y'}{y} = \log x + x \cdot \frac{1}{x} = \log x + 1$$

$$\text{したがって} \quad y' = (\log x + 1)y = x^x(\log x + 1)$$

8.

$$(1) \quad y' = \frac{-\sin x(1+\sin x) - \cos x \cdot \cos x}{(1+\sin x)^2} = \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= -\frac{1+\sin x}{(1+\sin x)^2} = -\frac{1}{1+\sin x}$$

$$(2) \quad y' = \frac{-\frac{1}{\cos^2 x}(1+\tan x) - (1-\tan x) \cdot \frac{1}{\cos^2 x}}{(1+\tan x)^2} = \frac{-(1+\tan x) - (1-\tan x)}{\cos^2 x(1+\tan x)^2}$$

$$= \frac{-2}{\{\cos x(1+\tan x)\}^2} = -\frac{2}{(\cos x + \sin x)^2}$$

$$(3) \quad y = e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) = e^{\frac{1}{x}} \left(1 - \frac{1}{x}\right)$$

$$(4) \quad y' = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$

$$(5) \quad y' = (\log 2) \cdot 2^{\log x} \cdot (\log x)' = \frac{(\log 2) \cdot 2^{\log x}}{x}$$

$$(6) \quad y = e^{-x} \cos x$$

$$y' = -e^{-x} \cos x + e^{-x}(-\sin x) = -e^{-x}(\sin x + \cos x)$$

$$(7) \quad y = \log(x + \sqrt{x^2 + a})$$

$$y' = \frac{(x + \sqrt{x^2 + a})'}{x + \sqrt{x^2 + a}} = \frac{1 + \frac{1}{2}(x^2 + a)^{-\frac{1}{2}} \cdot 2x}{x + \sqrt{x^2 + a}} = \frac{\sqrt{x^2 + a} + x}{(x + \sqrt{x^2 + a})\sqrt{x^2 + a}} = \frac{1}{\sqrt{x^2 + a}}$$

$$(8) \quad y = \log \left| \frac{x-a}{x+a} \right| \log(x-a) - \log(x+a)$$

$$y' = \frac{(x-a)'}{x-a} - \frac{(x+a)'}{x+a} = \frac{1}{x-a} - \frac{1}{x+a} = \frac{2a}{x^2 - a^2}$$

9.

(1) 両辺を x で微分すると

$$2x - y - x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$(x - 2y) \frac{dy}{dx} = 2x - y$$

よって, $x - 2y \neq 0$ すなわち

$$x \neq 2y \text{ のとき } \frac{dy}{dx} = \frac{2x - y}{x - 2y}$$

(2) 両辺を x で微分すると

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

ゆえに, $x \neq 0, y \neq 0$ のとき

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{y}{x}}$$

10.

$$y = A \cos kx + B \sin kx$$

$$y' = -Ak \sin kx + Bk \cos kx$$

$$y'' = -Ak^2 \cos kx - Bk^2 \sin kx \quad \text{だから}$$

$$y'' + k^2 y = -Ak^2 \cos kx - Bk^2 \sin kx + k^2 (A \cos kx + B \sin kx) = 0$$

11.

$$(1) \quad y' = e^x - e^{-x}$$

$$y'' = e^x + e^{-x}$$

$$y''' = e^x - e^{-x} \text{ より}$$

$\vdots$

$$y^{(n)} = e^x + (-1)^n e^{-x}$$

$$(2) \quad y' = -1 \cdot x^{-2}$$

$$y'' = (-1)(-2)x^{-3} = 2!x^{-3}$$

$$y''' = (-1)(-2)(-3)x^{-4} = -3!x^{-4}$$

$$y^{(4)} = (-1)(-2)(-3)(-4)x^{-5} = 4!x^{-5}$$

これより

$$y^{(n)} = (-1)^n \cdot n! x^{-(n+1)} = \frac{(-1)^n \cdot n!}{x^{n+1}}$$

12.

$$(I) \quad n = 1 \text{ のとき } y' = \cos x$$

$$\text{一方 } y^{(1)} = \sin\left(x + \frac{\pi}{2}\right) = \cos x$$

よって, $n = 1$ のとき与えられた等式が成り立つ。

$$(II) \quad n = k \text{ のとき } y^{(k)} = \sin\left(x + \frac{k\pi}{2}\right) \text{ であると仮定すると,}$$

$$n = k + 1 \text{ のとき } y^{(k+1)} = \left\{ y^{(k)} \right\}' = \cos\left(x + \frac{k\pi}{2}\right)$$

$$= \sin\left\{\left(x + \frac{k\pi}{2}\right) + \frac{\pi}{2}\right\} \quad \left(\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta \text{ より}\right)$$

$$= \sin\left(x + \frac{k+1}{2}\pi\right)$$

よって, $n = k + 1$ のときも与えられた等式が成り立つ。

したがって, (I), (II) より, すべての自然数 n に対して与えられた等式が成り立つ。

(終)

13.

- (1) 初項 1, 公比 $x (\neq 1)$, 項数 $n+1$ の等比数列の和であるから

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

- (2) (1)の両辺を x で微分すると

$$\begin{aligned} 1 + 2x + 3x^2 + \cdots + nx^{n-1} &= \frac{(n+1)x^n(x-1) - (x^{n+1} - 1)}{(x-1)^2} \\ &= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} \end{aligned}$$