

1 章 微分法 I

2 節 導関数

A 問題

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$$(1) \quad f'(-1) = \lim_{h \rightarrow 0} \frac{\{(-1+h)^2 - (-1+h)\} - \{(-1)^2 - (-1)\}}{h} = \lim_{h \rightarrow 0} \frac{-3h + h^2}{h} = \lim_{h \rightarrow 0} (-3 + h) = -3$$

$$(2) \quad f'(-1) = \lim_{h \rightarrow 0} \frac{\{2(-1+h)^2 - 1\} - \{2(-1)^2 - 1\}}{h} = \lim_{h \rightarrow 0} \frac{-4h + 2h^2}{h} = \lim_{h \rightarrow 0} (-4 + 2h) = -4$$

$$(3) \quad f'(-1) = \lim_{h \rightarrow 0} \frac{\{(-1+h)^3 - 1\} - \{(-1)^3 - 1\}}{h} = \lim_{h \rightarrow 0} \frac{3h - 3h^2 + h^3}{h} = \lim_{h \rightarrow 0} (3 - 3h + h^2) = 3$$

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$$\lim_{h \rightarrow +0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow +0} \frac{2(1+h) - 1 - (2 \cdot 1 - 1)}{h} = \lim_{h \rightarrow +0} \frac{2h}{h} = 2$$

$$\lim_{h \rightarrow -0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow -0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \rightarrow -0} \frac{\boxed{h}(2+h)}{\boxed{h}} = 2$$

よって、 $f'(1) = 2$ となり存在するから微分可能である。

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$$(1) \quad f'(x) = \lim_{h \rightarrow 0} \frac{\{2(x+h) + 1\} - \{2x + 1\}}{h} = \lim_{h \rightarrow 0} \frac{2h}{h} = \lim_{h \rightarrow 0} 2 = 2$$

$$(2) \quad f'(x) = \lim_{h \rightarrow 0} \frac{\{(x+h)^2 + (x+h)\} - \{x^2 + x\}}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2 + h}{h} = \lim_{h \rightarrow 0} (2x + h + 1) = 2x + 1$$

$$(3) \quad f'(x) = \lim_{h \rightarrow 0} \frac{\{(x+h)^3 - (x+h)\} - \{x^3 - x\}}{h} = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

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$$(1) \quad y' = 6x - 5$$

$$(2) \quad y' = -3x^2 + 4x + 1$$

$$(3) \quad y' = -6x^2 - 2x + 5$$

$$(4) \quad y' = \frac{1}{2}x^2 - \frac{1}{2}x - 1$$

$$(5) \quad y' = -2x^2 + 5x$$

$$(6) \quad y' = 4x^3 - 10x + \frac{2}{3}$$

$$33 \quad f'(x) = 3x^2 - 2x + 1$$

$$f'(1) = 3 - 2 + 1 = 2, \quad f'(0) = 1, \quad f'(-2) = 12 + 4 + 1 = 17$$

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$$(1) \quad y' = 1 \cdot (2x + 6) + (x - 1) \cdot 2 = 4x + 4$$

$$(2) \quad y' = 3 \cdot (x^2 - 2x - 1) + (3x + 2)(2x - 2) = 9x^2 - 8x - 7$$

$$(3) \quad y' = 2x(x^3 - 2) + (x^2 + 3) \cdot 3x^2 = 5x^4 + 9x^2 - 4x$$

$$(4) \quad y' = 1 \cdot (2x + 1)(3x - 1) + (x + 1) \cdot 2 \cdot (3x - 1) + (x + 1)(2x + 1) \cdot 3 = 18x^2 + 14x$$

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$$(1) \quad y' = -\frac{4}{(4x - 1)^2}$$

$$(2) \quad y' = \frac{2(x + 1) - (2x - 1) \cdot 1}{(x + 1)^2} = \frac{3}{(x + 1)^2}$$

$$(3) \quad y' = \frac{1 \cdot (x^2 + 2) - (x + 1) \cdot 2x}{(x^2 + 2)^2} = -\frac{x^2 + 2x - 2}{(x^2 + 2)^2}$$

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$$(1) \quad y' = (x^{-4})' = -4x^{-5} = -\frac{4}{x^5}$$

$$(2) \quad y' = (2x^{-2})' = -4x^{-3} = -\frac{4}{x^3}$$

$$(3) \quad y' = \left(-\frac{1}{2}x^{-6}\right)' = 3x^{-7} = \frac{3}{x^7}$$

$$(1) \quad u = 5x + 4 \text{ とおく と } y = u^2 \text{ であり, } \frac{dy}{du} = 2u, \quad \frac{du}{dx} = 5 \text{ だから } y' = 2u \cdot 5 = 10(5x + 4)$$

$$(2) \quad u = 4x - 1 \text{ とおく と } y = u^3 \text{ であり, } \frac{dy}{du} = 3u^2, \quad \frac{du}{dx} = 4 \text{ だから } y' = 3u^2 \cdot 4 = 12(4x - 1)^2$$

$$(3) \quad u = 2x^2 + 1 \text{ とおく と } y = u^4 \text{ であり, } \frac{dy}{du} = 4u^3, \quad \frac{du}{dx} = 4x \text{ だから } y' = 4u^3 \cdot 4x = 16x(2x^2 + 1)^3$$

$$(4) \quad u = 3x^2 - x + 1 \text{ とおく と } y = u^3 \text{ であり, } \frac{dy}{du} = 3u^2, \quad \frac{du}{dx} = 6x - 1 \text{ だから}$$

$$y' = 3u^2 \cdot (6x - 1) = 3(6x - 1)(3x^2 - x + 1)^2$$

$$(5) \quad u = x - 1 \text{ とおく と } y = u^{-2} \text{ であり, } \frac{dy}{du} = -2u^{-3}, \quad \frac{du}{dx} = 1 \text{ だから}$$

$$y' = -2u^{-3} \cdot 1 = -\frac{2}{(x-1)^3}$$

$$(6) \quad u = x^2 + 3 \text{ とおく と } y = u^{-4} \text{ であり, } \frac{dy}{du} = -4u^{-5}, \quad \frac{du}{dx} = 2x \text{ だから}$$

$$y' = -4u^{-5} \cdot 2x = -\frac{8x}{(x^2 + 3)^5}$$

$$(1) \quad y' = \left(x^{\frac{3}{2}} \right)' = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2} \sqrt{x}$$

$$(2) \quad y' = \left\{ (x^2 + 1)^{\frac{1}{2}} \right\}' = \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}$$

$$(3) \quad y' = \left\{ (3x^2 + 1)^{\frac{1}{3}} \right\}' = \frac{1}{3} (3x^2 + 1)^{-\frac{2}{3}} \cdot 6x = \frac{2x}{\sqrt[3]{(3x^2 + 1)^2}}$$

$$(1) \quad \text{両辺を 2 乗して } y^2 = x + 1 \quad \therefore x = y^2 - 1 \quad \therefore \frac{dx}{dy} = 2y \quad \therefore y' = \frac{1}{2y} = \frac{1}{2\sqrt{x+1}}$$

$$(2) \quad \text{両辺を 3 乗して } y^3 = \frac{27}{x} \quad \therefore x = \frac{27}{y^3} \quad \therefore \frac{dx}{dy} = -\frac{81}{y^4}$$

$$y' = \frac{1}{-\frac{81}{y^4}} = -\frac{1}{81} y^4 = -\frac{1}{81} \cdot \frac{81}{\sqrt[3]{x^4}} = -\frac{1}{\sqrt[3]{x^4}}$$

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$$(1) \quad y' = -\sin 2x \cdot 2 = -2 \sin 2x$$

$$(3) \quad y' = \frac{1}{\cos^2 3x} \cdot 3 = \frac{3}{\cos^2 3x}$$

$$(5) \quad y' = 3 \cos^2 x \cdot (-\sin x) = -3 \cos^2 x \sin x$$

$$(7) \quad y' = -\frac{\cos x}{\sin^2 x}$$

$$(9) \quad y' = \frac{-\sin x \cdot x - \cos x \cdot 1}{x^2} = -\frac{x \sin x + \cos x}{x^2}$$

$$(2) \quad y' = \cos(1-x) \cdot (-1) = -\cos(1-x)$$

$$(4) \quad y' = 2 \sin x \cdot \cos x = 2 \sin x \cos x$$

$$(6) \quad y' = 2 \tan x \cdot \frac{1}{\cos^2 x} = \frac{2 \tan x}{\cos^2 x}$$

$$(8) \quad y' = -\frac{-\sin^2 x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

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$$(1) \quad y' = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

$$(3) \quad y' = \frac{1}{1+(2x)^2} \cdot 2 = \frac{2}{1+4x^2}$$

$$(5) \quad y' = \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2} = \frac{2}{4+x^2}$$

$$(2) \quad y' = \frac{1}{\sqrt{1-(3x)^2}} \cdot 3 = -\frac{3}{\sqrt{1-9x^2}}$$

$$(4) \quad y' = \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \cdot \frac{1}{3} = \frac{1}{\sqrt{9-x^2}}$$

$$(6) \quad y' = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2(x+1)\sqrt{2}}$$

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$$(1) \quad y' = \frac{1}{2x} \cdot 2 = \frac{1}{x}$$

$$(2) \quad y' = \frac{1}{3x+1} \cdot 3 = \frac{3}{3x+1}$$

$$(3) \quad y' = \frac{1}{x \log 3}$$

$$(4) \quad y' = 3x^2 \log x + x^3 \cdot \frac{1}{x} = x^2 (3 \log x + 1)$$

$$(5) \quad y' = 1 \cdot \log_2 x + x \cdot \frac{1}{x \log 2} = \frac{\log x}{\log 2} + \frac{1}{\log 2} = \frac{1}{\log 2} (\log x + 1)$$

$$(6) \quad y' = \frac{\frac{1}{x} \cdot x - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2}$$

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$$(1) \quad y' = \frac{1}{2x-1} \cdot 2 = \frac{2}{2x-1}$$

$$(2) \quad y' = \frac{1}{x^2 - x} \cdot (2x - 1) = \frac{2x - 1}{x^2 - x}$$

$$(3) \quad y' = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x$$

$$44 \quad \text{両辺の対数をとると} \quad \log y = \log \frac{(x-2)^3}{(x-1)^2} = \log (x-2)^3 - \log (x-1)^2 = 3 \log (x-2) - 2 \log (x-1)$$

$$\text{この両辺を } x \text{ で微分すると} \quad \frac{y'}{y} = \frac{3}{x-2} - \frac{2}{x-1} = \frac{x+1}{(x-2)(x-1)}$$

$$\therefore y' = \frac{x+1}{(x-2)(x-1)} \cdot \frac{(x-2)^3}{(x-1)^2} = \frac{(x+1)(x-2)^2}{(x-1)^3}$$

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$$(1) \quad y' = e^{3x+1} \cdot 3 = 3e^{3x+1}$$

$$(2) \quad y' = 1 \cdot e^x + x e^x = (x+1) e^x$$

$$(3) \quad y' = 2^{1-x} \log 2 \cdot (-1) = -2^{1-x} \log 2$$

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$$(1) \quad y' = e^x \cos x + e^x (-\sin x) = e^x (\cos x - \sin x)$$

$$(2) \quad y' = \frac{e^x (x+1) - e^x \cdot 1}{(x+1)^2} = \frac{x e^x}{(x+1)^2}$$

$$(3) \quad y' = e^{-x^2} \cdot (-2x) = -2x e^{-x^2}$$

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$$(1) \quad y' = 6x^2 - 6x + 4 \quad y'' = 12x - 6$$

$$(2) \quad y' = \frac{1}{1+x^2} \quad y'' = -\frac{2x}{(1+x^2)^2}$$

$$(3) \quad y' = \sin x + x \cos x \quad y'' = \cos x + \cos x + x(\sin x) = 2 \cos x - x \sin x$$

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$$(1) \quad y''' = (5x^4 + 8x^3 - 9x^2)'' = (20x^3 + 24x^2 - 18x)' = 60x^2 + 48x - 18$$

$$(2) \quad y''' = (2 \cos 2x)'' = (-4 \sin 2x)' = -8 \cos 2x$$

$$(3) \quad y''' = \left(\frac{3}{2} x^{\frac{1}{2}} \right)'' = \left(\frac{3}{4} x^{-\frac{1}{2}} \right)' = -\frac{3}{8} x^{-\frac{3}{2}} = -\frac{3}{8\sqrt{x^3}}$$

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$$(1) \quad y' = -e^{-x}, \quad y'' = e^{-x}, \quad y''' = -e^{-x}, \quad y^{(4)} = e^{-x} \quad \dots \therefore y^{(n)} = (-1)^n e^{-x}$$

$$(2) \quad y' = e^{2x} + x \cdot 2e^{2x} = (2x+1)e^{2x}, \quad y'' = 2e^{2x} + (2x+1) \cdot 2e^{2x} = 2(2x+2)e^{2x}, \\ y''' = 2^2 e^{2x} + 2(2x+2) \cdot 2e^{2x} = 2^2(2x+3)e^{2x}, \quad \dots \quad y^{(4)} = \dots \therefore y^{(n)} = 2^{n-1}(2x+n)e^{2x}$$

$$(3) \quad y' = -\frac{1}{(x-1)^2}, \quad y'' = \frac{2 \cdot 1}{(x-1)^3}, \quad y''' = -\frac{3 \cdot 2 \cdot 1}{(x-1)^4}, \quad y^{(4)} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(x-1)^5} \quad \dots$$

$$\therefore y^{(n)} = \frac{(-1)^n n!}{(x-1)^{n+1}}$$

$$50 \quad y' = e^{-x} + x \cdot (-e^{-x}) = (1-x)e^{-x}, \quad y'' = -1 \cdot e^{-x} + (1-x) \cdot (-e^{-x}) = (x-2)e^{-x}$$

$$\therefore y'' + 2y' + y = (x-2)e^{-x} + 2 \cdot (1-x)e^{-x} + xe^{-x} = 0$$

B 問題

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$$(1) \quad y' = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \rightarrow 0} \frac{\frac{x - (x+h)}{(x+h)x}}{h} = \lim_{h \rightarrow 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$$

$$\begin{aligned} (2) \quad y' &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{-\cos x(1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right) = \lim_{h \rightarrow 0} \left(\frac{-\cos x(1 - \cos h)(1 + \cos h)}{h(1 + \cos h)} - \frac{\sin h}{h} \cdot \sin x \right) \\ &= \lim_{h \rightarrow 0} \left(-\frac{\cos x \sin^2 h}{h(1 + \cos h)} - \frac{\sin h}{h} \cdot \sin x \right) = \lim_{h \rightarrow 0} \left(-\frac{h \sin^2 h \cos x}{h^2(1 + \cos h)} - \frac{\sin h}{h} \cdot \sin x \right) \\ &= -\sin x \end{aligned}$$

別解

$$\begin{aligned} y' &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{-2 \sin\left(x + \frac{h}{2} \sin \frac{h}{2}\right)}{h} \\ &= \lim_{h \rightarrow 0} -2 \sin\left(x + \frac{h}{2}\right) \cdot \frac{1}{2} \cdot \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right) \\ &= -\sin x \end{aligned}$$

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$$(1) \quad x^2 + y^2 = 1$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$(2) \quad x^2 + xy + 2y^2 = 1$$

$$2x + 1 \cdot y + x \frac{dy}{dx} + 4y \frac{dy}{dx} = 0$$

$$(x + 4y) \frac{dy}{dx} = -2x - y$$

$$\therefore \frac{dy}{dx} = -\frac{2x + y}{x + 4y}$$

$$(3) \quad x^{\frac{1}{3}} + y^{\frac{1}{3}} = 1$$

$$\frac{1}{3} x^{-\frac{2}{3}} + \frac{1}{3} y^{\frac{2}{3}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x^{-\frac{2}{3}}}{y^{\frac{2}{3}}} = -\left(\frac{y}{x}\right)^{\frac{2}{3}}$$

$$(1) \quad y' = 3x^2(x^2+1)^3 + x^3 \cdot 3(x^2+1)^2 \cdot 2x = 3x^2(x^2+1)^2(3x^2+1)$$

$$(2) \quad y' = 2(x^4+2x^2+3) \cdot (4x^3+4x) = 8x(x^4+2x^2+3)(x^2+1)$$

$$(3) \quad y' = \frac{2(1-x) - (2x+1)(-1)}{(1-x)^2} = \frac{3}{(1-x)^2}$$

$$(4) \quad y' = \frac{1 \cdot \sqrt{x+1} - (x-1) \cdot \frac{1}{2\sqrt{x+1}}}{x+1} = \frac{x+3}{2(x+1)\sqrt{x+1}}$$

$$(5) \quad y' = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

$$(6) \quad y' = \frac{3}{\cancel{4}}(2x^2+1)^{-\frac{1}{4}} \cdot \cancel{4}x = \frac{3x}{\sqrt[4]{2x^2+1}}$$

$$(1) \quad \text{両辺の対数をとると} \quad \log y = \log \sqrt[3]{\frac{x-1}{x+1}} = \frac{1}{3}(\log(x-1) - \log(x+1))$$

$$\text{この両辺を } x \text{ で微分すると} \quad \frac{y'}{y} = \frac{1}{3} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) = \frac{2}{3(x-1)(x+1)}$$

$$\therefore y' = \frac{2}{3(x-1)(x+1)} \sqrt[3]{\frac{x-1}{x+1}}$$

$$(2) \quad \text{両辺の対数をとると} \quad \log y = \log x^{\frac{1}{x}} = \frac{1}{x} \log x$$

$$\text{この両辺を } x \text{ で微分すると} \quad \frac{y'}{y} = -\frac{1}{x^2} \log x + \frac{1}{x} \cdot \frac{1}{x} = \frac{1}{x^2} (1 - \log x)$$

$$\therefore y' = x^{\frac{1}{x}} \cdot \frac{1}{x^2} (1 - \log x) = x^{\frac{1}{x}-2} (1 - \log x)$$

$$(1) \quad y' = \cos x \cdot \cos^2 x + \sin x \cdot 2 \cos x (-\sin x) = \cos^3 x - 2 \sin^2 x \cos x = 3 \cos^3 x - 2 \cos x$$

$$(2) \quad y' = \left(\log \frac{x^2+1}{x} \right)' = \left(\log(x^2+1) - \log x \right)' = \frac{2x}{x^2+1} - \frac{1}{x} = \frac{2x^2 - (x^2+1)}{x(x^2+1)} = \frac{x^2-1}{x(x^2+1)}$$

$$(3) \quad y' = \frac{(\cos x + \sin x)^2 + (\sin x - \cos x)^2}{(\sin x - \cos x)^2} = \frac{\cos^2 x + \cancel{2 \cos x \sin x} + \sin^2 x + \sin^2 x - \cancel{2 \sin x \cos x} + \cos^2 x}{(\sin x - \cos x)^2}$$

$$= \frac{2}{(\sin x + \cos x)^2}$$

$$(4) \quad y' = \cos x e^{\sin x}$$

$$(5) \quad y' = 2e^{2x} \sin^2 x + e^{2x} 2 \sin x \cos x = 2e^x \sin x (\sin x + \cos x)$$

$$(6) \quad y' = \left(\log |1 - \cos x| - \log |1 + \cos x| \right)' = \frac{\sin x}{1 - \cos x} - \frac{-\sin x}{1 + \cos x}$$

$$= \frac{\sin x + \cancel{\sin x \cos x} + \sin x - \cancel{\sin x \cos x}}{1 - \cos^2 x} = \frac{2 \sin x}{\sin^2 x} = \frac{2}{\sin x}$$

$$(1) \quad y' = \frac{1}{2} \left(\sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{1 + \frac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}} \right)$$

$$= \frac{1}{2} \left(\sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}} + \frac{\cancel{\sqrt{x^2+1}} + x}{\left(\cancel{x + \sqrt{x^2+1}} \right) \sqrt{x^2+1}} \right) = \frac{1}{2} \left(\sqrt{x^2+1} + \frac{x^2+1}{\sqrt{x^2+1}} \right)$$

$$= \frac{1}{2} \left(\sqrt{x^2+1} + \sqrt{x^2+1} \right) = \sqrt{x^2+1}$$

$$(2) \quad y' = \frac{1}{2} \left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) = \frac{1}{2} \left(\sqrt{1-x^2} + \frac{1-x^2}{\sqrt{1-x^2}} \right)$$

$$= \frac{1}{2} \left(\sqrt{1-x^2} + \sqrt{1-x^2} \right) = \sqrt{1-x^2}$$

(1) 与式を①とする。

(I) $n=1$ のとき

$$y' = -\sin x = \cos\left(x + \frac{\pi}{2}\right) \quad \text{だから, } n=1 \text{ のとき①は成り立つ。}$$

(II) $n=k$ のとき①が成り立つと仮定すると

$n=k+1$ のとき

$$\begin{aligned} y^{(k+1)} &= \left\{ y^{(k)} \right\}' = \left\{ \cos\left(x + \frac{k\pi}{2}\right) \right\}' = -\sin\left(x + \frac{k\pi}{2}\right) = \cos\left\{\left(x + \frac{k\pi}{2}\right) + \frac{\pi}{2}\right\} \\ &= \cos\left(x + \frac{(k+1)\pi}{2}\right) \quad \text{だから, } n=k+1 \text{ のときも成り立つ。} \end{aligned}$$

(I), (II) より, 任意の自然数 n について, ①は成り立つ。

(2) 与式を①とする。

(I) $n=1$ のとき

$$y' = 2\cos(2x+1) = 2\sin\left(2x+1 + \frac{\pi}{2}\right) \quad \text{だから, } n=1 \text{ のとき①は成り立つ。}$$

(II) $n=k$ のとき①が成り立つとすると

$n=k+1$ のとき

$$\begin{aligned} y^{(k+1)} &= \left\{ y^{(k)} \right\}' = \left\{ 2^k \sin\left(2x+1 + \frac{k\pi}{2}\right) \right\}' = 2^k \cdot \cos\left(2x+1 + \frac{k\pi}{2}\right) \\ &= 2^{k+1} \sin\left\{\left(2x+1 + \frac{k\pi}{2}\right) + \frac{\pi}{2}\right\} = 2^{k+1} \sin\left(2x+1 + \frac{(k+1)\pi}{2}\right) \end{aligned}$$

だから, $n=k+1$ のときも成り立つ。

(I), (II) より, 任意の自然数 n について, ①は成り立つ。

発展問題

$$(1) \quad (\text{左辺}) = (\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x = (\text{右辺})$$

$$(2) \quad (\text{左辺}) = (\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \sinh x = (\text{右辺})$$

$$(3) \quad (\text{左辺}) = \cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{(\cancel{e^{2x}} + 2 + \cancel{e^{-x}}) - (\cancel{e^{2x}} - 2 + \cancel{e^{-x}})}{4} = 1 = (\text{右辺})$$

$$(4) \quad (\text{左辺}) = (\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = (\text{右辺})$$