２章　積分法

１節　定積分と不定積分

A

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(1) $\displaystyle \lim_{n\to\infty}\sum_{k=1}^{n}\left\{a+\frac{k(b-a)}{n}-1\right\}\frac{b-a}{n}\lim_{n\to\infty}=\left\{\sum_{k=1}^{n}\frac{(b-a)(a-1)}{n}+\frac{(b-a)^2}{n^2}\sum_{k=1}^{n}k\right\}$

$$=\lim_{n\to\infty}\left\{\frac{(b-a)(a-1)}{\cancel{n}}\cancel{n}+\frac{(b-a)^2}{n^2}\cdot\frac{1}{2}n(n+1)\right\}$$

$$=\lim_{n\to\infty}\left\{(b-a)(a-1)+\frac{1}{2}(b-a)^2\left(1+\frac{1}{n}\right)\right\}$$

$$=(b-a)(a-1)+\frac{1}{2}(b-a)^2=\frac{1}{2}(b-a)(b+a-2),$$

$$\int_a^b(x-1)\,dx=\left[\frac{1}{2}(x-1)^2\right]_a^b=\frac{1}{2}\left\{(b-1)^2-(a-1)^2\right\}$$

$$=\frac{1}{2}(b-a)(b+a-2)$$

(2) $\displaystyle \lim_{n\to\infty}\sum_{k=1}^{n}\left\{-2\left(a+\frac{k(b-a)}{n}\right)\cdot\frac{b-a}{n}\right\}=\lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{-2a(b-a)}{n}-\frac{2(b-a)^2}{n^2}\sum_{k=1}^{n}k\right\}$

$$=\lim_{n\to\infty}\left\{\frac{-2a(b-a)}{n}\cdot n-\frac{2(b-a)^2}{n^2}\cdot\frac{1}{\cancel{2}}n(n+1)\right\}$$

$$=\lim_{n\to\infty}\left\{-2a(b-a)-(b-a)^2\left(1+\frac{1}{n}\right)\right\}$$

$$=-2a(b-a)-(b-a)^2=a^2-b^2,$$

$$\int_a^b(-2x)\,dx=-2\left[\frac{1}{2}x^2\right]_a^b=-(b^2-a^2)=a^2-b^2$$

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(1) $\displaystyle \lim_{n\to0}\sum_{k=1}^{n}\left(\frac{2k}{n}\right)^2\frac{2}{n}=\lim_{n\to0}\frac{8}{n^3}\sum_{k=1}^{n}k^2=\lim_{n\to0}\frac{8}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)=\lim_{n\to0}\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)=\frac{8}{3},$

$$\int_0^2x^2\,dx=\left[\frac{1}{3}x^3\right]_0^2=\frac{8}{3}$$

(2) $\displaystyle \lim_{n\to0}\sum_{k=1}^{n}\left\{\left(\frac{2k}{n}\right)^2+1\right\}\frac{2}{n}=\lim_{n\to0}\left\{\frac{8}{n^3}\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}\frac{2}{n}\right\}=\lim_{n\to0}\left\{\frac{8}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{2}{\cancel{n}}\cdot\cancel{n}\right\}$

$$=\lim_{n\to0}\left\{\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+2\right\}=\frac{8}{3}+2=\frac{14}{3},$$

$$\int_0^2\left(x^2+1\right)dx=\left[\frac{1}{3}x^3+x\right]_0^2=\frac{8}{3}+2=\frac{14}{3}$$

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⑴ $$\lim_{n\to 0}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^2\frac{1}{n}=\lim_{n\to 0}\sum_{k=1}^{n}\left\{\frac{1}{n}+\frac{2k}{n^2}+\frac{k^2}{n^3}\right\}=\lim_{n\to 0}\left\{\sum_{k=1}^{n}\frac{1}{n}+\frac{2}{n^2}\sum_{k=1}^{n}k+\frac{1}{n^3}\sum_{k=1}^{n}k^2\right\}$$

$$=\lim_{n\to 0}\left\{\frac{1}{\cancel{n}}\cdot\cancel{n}+\frac{\cancel{2}}{n^2}\cdot\frac{1}{2}n(n+1)+\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)\right\}$$

$$=\lim_{n\to 0}\left\{1+\left(1+\frac{1}{n}\right)+\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right\}=1+1+\frac{1}{3}=\frac{7}{3},$$

$$\int_1^2 x^2\,dx=\frac{1}{3}\Big[\,x^3\,\Big]_1^2=\frac{1}{3}(8-1)=\frac{7}{3}$$

⑵ $$\lim_{n\to 0}\sum_{k=1}^{n}\left(\cancel{1}+\frac{k}{n}-\cancel{1}\right)^2\frac{1}{n}=\lim_{n\to 0}\sum_{k=1}^{n}\frac{k^2}{n^3}=\lim_{n\to 0}\frac{1}{n^3}\sum_{k=1}^{n}k^2=\lim_{n\to 0}\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)$$

$$=\lim_{n\to 0}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)=\frac{1}{3},$$

$$\int_1^2 (x-1)^2\,dx=\frac{1}{3}\Big[\,(x-1)\,\Big]_1^2=\frac{1}{3}(1-0)=\frac{1}{3}$$

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⑴ $2x^2-x$ を $x-1$ で割ると，商が $2x+1$，余りが 1 だから， $2x^2-x=(x-1)(2x+1)+1$

$$\therefore\ \int\frac{2x^2-x}{x-1}\,dx=\int\left(2x+1+\frac{1}{x-1}\right)dx=x^2+x+\log\left|x-1\right|+C$$ 公式 25

⑵ x^2-2x+1 を x^2+1 で割ると，商が 1，余りが $-2x$ だから， $x^2-2x+1=(x^2+1)\cdot 1-2x$

$$\therefore\ \int\frac{x^2-2x+1}{x^2+1}\,dx=\int\left(1-\frac{2x}{x^2+1}\right)dx=x-\log(x^2+1)+C$$ 公式 30

⑶ $3x^3+5x^2+2x+1$ を $x+1$ で割ると，商が $3x^2+2x$，余りが 1 だから，

$3x^3+5x^2+2x+1=(x+1)(3x^2+2x)+1$

$$\therefore\ \int\frac{3x^2+5x^2+2x+1}{x+1}\,dx=\int\left(3x^2+2x+\frac{1}{x+1}\right)dx=x^3+x^2+\log\left|x+1\right|+C$$

⑷ x^3+x+2 を x^2+1 で割ると，商が x ，余りが 2 だから， $x^3+x+2=(x^2+1)x+2$

$$\therefore\ \int\frac{x^3+x+2}{x^2+1}\,dx=\int\left(x+\frac{2}{x^2+1}\right)dx=\frac{1}{2}x^2+2\,\mathrm{Tan}^{-1}+x+C$$ 公式 28

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(1) $\displaystyle\int\frac{1}{x^2+3x+2}\,dx=\int\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx=\log|x+1|-\log|x+2|+C=\log\left|\frac{x+1}{x+2}\right|+C$　公式 $\boxed{1}$ $\boxed{6}$

(2) $P(x)=x^3-x^2+x-1$ とおくと，$P(1)=1^3-1^2+1-1=0$ だから，因数定理（公式 $\boxed{2}$ ）より，
$P(x)$ は $x-1$ で
割り切れ，$P(x)=(x-1)(x^2+1)$ となる。

$\dfrac{3x^2-x}{x^3-x^2+x-1}=\dfrac{a}{x-1}+\dfrac{bx+c}{x^2+1}$ とおき，分母を払うと，$3x^2-x=(a+b)x^2+(-b+c)x+(a-c)$ だから，

両辺を比較して，$a=1$，$b=2$，$c=1$ を得る。よって，$\dfrac{3x^2-x}{x^3-x^2+x-1}=\dfrac{1}{x-1}+\dfrac{2x}{x^2+1}+\dfrac{1}{x^2+1}$

$$\therefore\ \int\frac{3x^2-x}{x^3-x^2+x-1}\,dx=\int\left(\frac{1}{x-1}+\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx$$

$$=\log|x-1|+\log(x^2+1)+\mathrm{Tan}^{-1}+x+C$$ （公式 $\boxed{28}$ $\boxed{30}$ ）

$$=\log|x-1|(x^2+1)+\mathrm{Tan}^{-1}x+C$$

(3) $\dfrac{3x^2+7x+5}{(x+1)(x^2+2x+2)}=\dfrac{a}{x+1}+\dfrac{bx+c}{x^2+2x+2}$ とおき，分母を払うと，

$3x^2+7x+5=(a+b)x^2+(2a+b+c)x+(2a+c)$ だから，両辺を比較して，$a=1$，$b=2$，$c=3$ を得る。

よって，$\dfrac{1}{x+1}+\dfrac{2x+3}{x^2+2x+2}=\dfrac{1}{x+1}+\dfrac{2x+2}{x^2+2x+2}+\dfrac{1}{(x+1)^2+1}$

$$\therefore\ \int\frac{3x^2+7x+5}{(x+1)(x^2+2x+2)}\,dx=\int\left(\frac{1}{x+1}+\frac{2x+2}{x^2+2x+2}+\frac{1}{(x+1)^2+1}\right)dx$$

$$=\log|x+1|+\log(x^2+2x+2)+\mathrm{Tan}^{-1}(x+1)+C$$

$$=\log|x+1|(x^2+2x+2)+\mathrm{Tan}^{-1}(x+1)+C$$ （公式 $\boxed{28}$ $\boxed{30}$ ）

(4) $\dfrac{2x^2+7x+4}{x(x+2)^2}=\dfrac{a}{x}+\dfrac{b}{x+2}+\dfrac{c}{(x+2)^2}$ とおき，分母を払うと，

$2x^2+7x+4=(a+b)x^2+(4a+2b+c)x+4a$ だから，両辺を比較して，$a=b=c=1$ を得る。

よって，$\dfrac{2x^2+7x+4}{x(x+2)^2}=\dfrac{1}{x}+\dfrac{1}{x+2}+\dfrac{1}{(x+2)^2}$

$$\therefore\ \int\frac{2x^2+7x+4}{x(x+2)^2}\,dx=\int\left(\frac{1}{x}+\frac{1}{x+2}+\frac{1}{(x+2)^2}\right)dx$$

$$=\log|x|+\log|x+2|-\frac{1}{x+2}+C$$

$$=\log|x(x+2)|-\frac{1}{x+2}+C$$

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(1) $t=\tan\dfrac{x}{2}$ とおくと， $\cos x=\dfrac{1-t^2}{1+t^2}$ ， $dx=\dfrac{2}{1+t^2}$ だから

$$\int\frac{1}{1+\cos x}\,dx=\int\frac{1}{1+\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt=\int\frac{2}{t^2+1+1-t^2}\,dt$$

$$=\int 1\,dt=t+C$$

$$=\tan\frac{x}{2}+C$$

(2) $t=\tan\dfrac{x}{2}$ とおくと， $\sin x=\dfrac{2t}{1+t^2}\,dt$ ， $dx=\dfrac{2}{1+t^2}\,dt$ だから

$$\int\frac{1}{1-\sin x}\,dx=\int\frac{1}{1-\dfrac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt=\int\frac{2}{1+t^2-2t}\,dt$$

$$=\int\frac{2}{(t-1)^2}\,dt=-2(t-1)^{-1}+C$$

$$=-\frac{2}{\tan\dfrac{x}{2}-1}+C$$

(3) $t=\tan\dfrac{x}{2}$ ㋐ とおくと， $\cos x=\dfrac{1-t^2}{1+t^2}$ ， $dx=\dfrac{2}{1+t^2}\,dt$ だから

$$\int\frac{1-\cos x}{1+\cos x}\,dx=\int\frac{1-\dfrac{1-t^2}{1+t^2}}{1+\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt=\int\frac{(1+t^2)-(1-t^2)}{(1+t^2)+(1-t^2)}\cdot\frac{2}{1+t^2}\,dt$$

$$=\int\frac{2t^2}{\cancel{2}}\cdot\frac{\cancel{2}}{1+t^2}\,dt=2\int\frac{t^2+1-1}{1+t^2}\,dt=2\int\left(1-\frac{1}{1+t^2}\right)dt$$

$$=2\left(t-\mathrm{Tan}^{-1}\,t\right)+C=2\left(\tan\frac{x}{2}-\frac{x}{2}\right)+C$$

（㋐より $\mathrm{Tan}^{-1}\,t=\dfrac{x}{2}$ ）

(4) $t=\tan\dfrac{x}{2}$ とおくと， $\tan x=\dfrac{2t}{1-t^2}$ ， $dx=\dfrac{2}{1+t^2}\,dt$ だから

$$\int\left(\frac{1}{\tan x}+\tan\frac{x}{2}\right)dx=\int\left(\frac{1}{\dfrac{2t}{1-t^2}}+t\right)\frac{2}{1+t^2}\,dt=\int\left(\frac{1-t^2}{2t}+\frac{2t^2}{2t}\right)\frac{2}{1+t^2}\,dt$$

$$=\int\frac{\cancel{1+t^2}}{\cancel{2}t}\cdot\frac{\cancel{2}}{\cancel{1+t^2}}\,dt=\int\frac{1}{t}\,dt=\log|t|+C=\log\left|\tan\frac{x}{2}\right|+C$$

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(1) $$\int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x^2-4x+4)}}\,dx = \int \frac{1}{\sqrt{2^2-(x-2)^2}}\,dx = \mathrm{Sin}^{-1}\frac{x-2}{2} + C$$

(2) $$\int \frac{1}{\sqrt{8-6x-9x^2}}\,dx = \int \frac{1}{\sqrt{9-(9x^2+6x+1)}}\,dx = \int \frac{1}{\sqrt{3^2-(3x+1)^2}}\,dx = \frac{1}{3}\mathrm{Sin}^{-1}\frac{3x+1}{3} + C$$

(3) $$\int \sqrt{16-6x-x^2}\,dx = \int \sqrt{25-(x^2+6x+9)}\,dx = \int \sqrt{5^2-(x+3)^2}\,dx$$
$$= \frac{1}{2}\left\{(x+3)\sqrt{16-6x-x^2} + 25\,\mathrm{Sin}^{-1}\frac{x+3}{5}\right\} + C$$

(4) $$\int \sqrt{6x-9x^2}\,dx = \int \sqrt{1-(9x^2-6x+1)}\,dx = \int \sqrt{1-(3x-1)^2}\,dx$$
$$= \frac{1}{6}\left\{(3x-1)\sqrt{6x-9x^2} + \mathrm{Sin}^{-1}(3x-1)\right\} + C$$

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(1) $$\int \frac{1}{\sqrt{x^2+4x+5}}\,dx = \int \frac{1}{\sqrt{(x+2)^2+1}}\,dx = \log\left|x+2\sqrt{x^2+4x+5}\right| + C$$

(2) $$\int \frac{1}{\sqrt{4x^2+4x+3}}\,dx = \int \frac{1}{\sqrt{(2x+1)^2+2}}\,dx = \frac{1}{2}\log\left|2x+1+\sqrt{4x^2+4x+3}\right| + C$$

(3) $$\int \sqrt{x^2+6x+10}\,dx = \int \sqrt{(x+3)^2+1}\,dx$$
$$= \frac{1}{2}\left\{(x+3)\sqrt{x^2+6x+10} + \log\left|x+3+\sqrt{x^2+6x+10}\right|\right\} + C$$

(4) $$\int \sqrt{4x^2-4x+7}\,dx = \int \sqrt{(2x-1)^2+6}\,dx$$
$$= \frac{1}{2}\left\{(2x-1)\sqrt{4x^2-4x+7} + 6\log\left|(2x-1)+\sqrt{4x^2-4x+7}\right|\right\}\cdot\frac{1}{2} + C$$
$$= \frac{1}{4}\left\{(2x-1)\sqrt{4x^2-4x+7} + 6\log\left|2x-1+\sqrt{4x^2-4x+7}\right|\right\} + C$$

B

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(2) $$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n}k^3 = \lim_{n\to\infty}\frac{1}{n^4}\cdot\frac{1}{4}n^2(n+1)^2 = \frac{1}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = \frac{1}{4}$$

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⑴　$f(x)$ は偶関数，$\sin x$ は奇関数だから， $f(x)\sin x$ は奇関数である。よって $\displaystyle\int_{-\pi}^{\pi} f(x)\sin x\,dx = 0$

⑵　$f(x)\cos x$ はともに偶関数だから， $f(x)\cos x$ は偶関数である。
よって

$$\int_{-\pi}^{\pi} f(x)\cos x\,dx = 2\int_0^{\pi} (\pi - x)\cos x\,dx = 2\left\{\Big[(\pi - x)\sin x\Big]_0^{\pi} - \int_0^{\pi} (-1)\sin x\,dx\right\}$$

$$= 2\int_0^{\pi} \sin x\,dx = 2\cdot\Big[-\cos x\Big]_0^{\pi} = 4$$

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⑴　$$\int \frac{x^2+2x+2}{x^2+1}\,dx = \int\left(1 + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right)dx = x + \log(x^2+1) + \mathrm{Tan}^{-1}\,x + C$$

⑵　$$\int \frac{1}{\sqrt{2-x-x^2}}\,dx = \int \frac{1}{\sqrt{\frac{9}{4} - \left(x^2 + x + \frac{1}{4}\right)}}\,dx = \int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2}}\,dx$$

$$= \mathrm{Sin}^{-1}\,\frac{x+\frac{1}{2}}{\frac{3}{2}} + C = \mathrm{Sin}^{-1}\,\frac{2x+1}{3} + C$$

⑶　$t = \tan\dfrac{x}{2}$ とおくと， $\sin x = \dfrac{2t}{1+t^2}$ ， $dx = \dfrac{2}{1+t^2}$ であるから

$$\int \frac{1}{\sin x + \sin^2 x}\,dx = \int \frac{1}{\frac{2t}{1+t^2} + \frac{4t^2}{(1+t^2)^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{1}{t + \frac{2t^2}{1+t^2}}\,dt = \int \frac{t^2+1}{t^3+2t^2+t}\,dt$$

$$= \int \frac{t^2+1}{t(t+1)^2}\,dt = \int\left\{\frac{1}{t} - \frac{2}{(t+1)^2}\right\}dt = \log|t| + \frac{2}{t+1} + C$$

$$= \log\left|\tan\frac{x}{2}\right| + \frac{2}{\tan\frac{x}{2} + 1} + C$$

⑷　$$\int \sqrt{x^2+x+1}\,dx = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx$$ （P20 公式 （Ⅳ））

$$= \frac{1}{2}\left\{\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{3}{4}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{x^2+x+1}\right|\right\} + C$$

(5) $\displaystyle\int \frac{2x+1}{\sqrt{x^2+1}}\,dx = \int\left(\frac{2x}{\sqrt{x^2+1}}+\frac{1}{\sqrt{x^2+1}}\right)dx = 2\sqrt{x^2+1}+\log\left|x+\sqrt{x^2+1}\right|+C$

（第一項は $x^2+1=t$ とおく置換積分，第二項は P.20 公式（Ⅲ）を使用）

(6) $\displaystyle\int (2x+1)\sqrt{1-x^2}\,dx = \int\left(2x\sqrt{1-x^2}+\sqrt{1-x^2}\right)dx = -\frac{2}{3}\sqrt{(1-x^2)^3}+\frac{1}{2}\left\{x\sqrt{1-x^2}+\mathrm{Sin}^{-1}x\right\}+C$

（第一項は $1-x^2=t$ とおく置換積分，第二項は P.20 公式（Ⅰ）を使用）

(7) $t=\tan\dfrac{x}{2}$ とおくと，　$\sin x=\dfrac{2t}{1+t^2}$， $\cos x=\dfrac{1-t^2}{1+t^2}$， $dx=\dfrac{2}{1+t^2}\,dt$ だから

$$\int\frac{1}{\sin x+\cos x}\,dx = \int\frac{1}{\dfrac{2t}{1+t^2}+\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int\frac{2}{2t+1-t^2}\,dt$$

$$= -2\int\frac{1}{t^2-2t-1}\,dt = -2\int\frac{1}{(t-1)^2-2}\,dt$$

$$= -2\int\left\{\frac{1}{(t-1)-\sqrt{2}}+\frac{-1}{(t-1)+\sqrt{2}}\right\}\frac{1}{2\sqrt{2}}\,dt$$

（教 p.65公式 $\displaystyle\int\frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C$ を使うと簡単）

$$= -\frac{1}{\sqrt{2}}\left\{\log\left|t-1-\sqrt{2}\right|-\log\left|t-1+\sqrt{2}\right|\right\}+C$$

$$= -\frac{1}{\sqrt{2}}\log\left|\frac{\tan\dfrac{x}{2}-1-\sqrt{2}}{\tan\dfrac{x}{2}-1+\sqrt{2}}\right|+C$$

(8) $t=\tan\dfrac{x}{2}$ とおくと，　$\sin x=\dfrac{2t}{1+t^2}$， $\cos x=\dfrac{1-t^2}{1+t^2}$， $dx=\dfrac{2}{1+t^2}\,dt$ だから

$$\int\frac{\sin x+1}{(\cos x+1)\sin x}\,dx = \int\frac{1}{\cos x+1}\cdot\left(1+\frac{1}{\sin x}\right)dx = \int\frac{1}{\dfrac{1-t^2}{1+t^2}+1}\cdot\left(1+\frac{1}{\dfrac{2t}{1+t^2}}\right)\cdot\frac{2}{1+t^2}\,dt$$

$$= \int\frac{\cancel{1+t^2}}{1\cancel{-t^2}+1+t^2}\cdot\left(1+\frac{1+t^2}{2t}\right)\cdot\frac{\cancel{2}}{\cancel{1+t^2}}\,dt = \int\left(1+\frac{1}{2t}+\frac{t}{2}\right)dt$$

$$= t+\frac{1}{2}\log|t|+\frac{1}{4}t^2+C$$

$$= \tan\frac{x}{2}+\frac{1}{2}\log\left|\tan\frac{x}{2}\right|+\frac{1}{4}\left(\tan\frac{x}{2}\right)^2+C$$

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⑴ $\displaystyle\int_{-1}^{0}\frac{3x^2+4x^2-4x+1}{x+2}\,dx=\int_{-1}^{0}\left(3x^2-2x+\frac{1}{x+2}\right)dt=\Big[\,x^3-x^2+\log\left|x+2\right|\Big]_{-1}^{0}=2+\log 2$

⑵ $\displaystyle\int_{1}^{2}\frac{1}{x^2(x+1)}\,dx=\int_{1}^{2}\left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx=\left[-\log\left|x\right|-\frac{1}{x}+\log\left|x+1\right|\right]_{1}^{2}$

$\displaystyle=-\log 2-\frac{1}{2}+\log 3+1-\log 2=\frac{1}{2}+\log\frac{3}{4}$

（第二式は $\displaystyle\frac{1}{x^2(x+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$ とおいて A，B，C を見つける）

⑶ $\displaystyle\int_{0}^{\frac{\pi}{3}}\frac{1}{\cos x}\,dx=\int_{0}^{\frac{1}{\sqrt{3}}}\frac{\cancel{1+t^2}}{1-t^2}\cdot\frac{2}{\cancel{1+t^2}}\,dt=\int_{0}^{\frac{1}{\sqrt{3}}}\frac{2}{(t-1)(t+1)}\,dt=-\int_{0}^{\frac{1}{\sqrt{3}}}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$

$\displaystyle=-\Big[\,\log\left|t-1\right|-\log\left|t+1\right|\Big]_{0}^{\frac{1}{\sqrt{3}}}=-\log\left|\frac{1}{\sqrt{3}}-1\right|+\log\left|\frac{1}{\sqrt{3}+1}\right|+\cancel{\log 1}-\cancel{\log 1}$

$\displaystyle=\log\left|\frac{\frac{1}{\sqrt{3}}+1}{\frac{1}{\sqrt{3}}-1}\right|=-\log\left|\frac{1+\sqrt{3}}{1-\sqrt{3}}\right|=\log\left|\frac{4+2\sqrt{3}}{-2}\right|=\log\left(2+\sqrt{3}\right)$

（59 ⑺の解答と同様，教 p.65 の公式を用いると簡単）

⑷ $\displaystyle\int_{0}^{1}\sqrt{2-x^2}\,dx=\left[\frac{1}{2}\left\{x\sqrt{2-x^2}+2\,\mathrm{Sin}^{-1}\frac{x}{\sqrt{2}}\right\}\right]_{0}^{1}$ （P.20 公式（Ⅱ））

$\displaystyle=\frac{1}{2}\left\{1+2\,\mathrm{Sin}^{-1}\frac{1}{\sqrt{2}}-0-2\,\cancel{\mathrm{Sin}^{-1}0}\right\}=\frac{1}{2}\left\{1+2\cdot\frac{\pi}{4}\right\}=\frac{2+\pi}{4}$

⑸ $\displaystyle\int_{0}^{4}\frac{1}{\sqrt{x^2+9}}\,dx=\Big[\,\log\left|x+\sqrt{x^2+9}\right|\Big]_{0}^{4}$ （P.20 公式（Ⅲ））

$\displaystyle=\log\left|4+\sqrt{25}\right|-\log\left|0+\sqrt{9}\right|=\log 9-\log 3=\log 3$

⑹ $\displaystyle\int_{0}^{3}\sqrt{x^2+16}\,dx=\left[\frac{1}{2}\left\{x\sqrt{x^2+16}+16\log\left|x+\sqrt{x^2+16}\right|\right\}\right]_{0}^{3}$ （P.20 公式（Ⅳ））

$\displaystyle=\frac{1}{2}\left\{3\sqrt{25}+16\log\left|3+\sqrt{25}\right|-0-16\log\sqrt{16}\right\}$

$\displaystyle=\frac{1}{2}\left\{15+16\log 8-16\log 4\right\}$

$\displaystyle=\frac{15}{2}+8\log 2$

61

(1) $t=\dfrac{\pi}{2}-x$ とおくと，$\dfrac{dt}{dx}=-1$ $\therefore$ $dx=-\,dt$

x	$0 \to \dfrac{\pi}{2}$
t	$\dfrac{\pi}{2} \to 0$

$$\int_0^{\frac{\pi}{2}} f(\sin x)\,dx=\int_0^{\frac{\pi}{2}} f\left(\cos\left(\frac{\pi}{2}-x\right)\right)dx=\int_{\frac{\pi}{2}}^{0} f(\cos t)\cdot(-1)\,dt=\int_0^{\frac{\pi}{2}} f(\cos t)\,dt=\int_0^{\frac{\pi}{2}} f(\cos x)\,dx$$

(2) $t=Lx$ とおくと，$\dfrac{dt}{dx}=L$ $\therefore$ $dx=\dfrac{1}{L}\,dt$

x	$-1 \to 1$
t	$-L \to L$

$$\therefore\ \int_{-1}^{1} f(Lx)\,dx=\int_{-L}^{L} f(t)\frac{1}{L}\,dt=\frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

(3) $s=xt$ とおくと，$\dfrac{ds}{dt}=x$ $\therefore$ $dt=\dfrac{1}{x}\,ds$

t	$0 \to 1$
s	$0 \to x$

$$\therefore\ \frac{d}{dx}\left\{x\int_0^1 f(xt)\,dt\right\}=\frac{d}{dx}\left\{\cancel{x}\int_0^x f(s)\cdot\frac{1}{\cancel{x}}\,ds\right\}=\frac{d}{dx}\left\{\int_0^x f(s)s)\,ds\right\}=f(x)$$

62

(1) $(\text{与式})=\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\left(\frac{k}{n}\right)^2}}\cdot\frac{1}{n}=\int_0^1\frac{1}{\sqrt{1+x^2}}\,dx$

（[教] P.51 公式で $f(x)=\dfrac{1}{\sqrt{1+x^2}}$，$x_k=\dfrac{k}{n}$，$\Delta x=\dfrac{1}{n}$ とする）

$=\left[\log\left|x+\sqrt{x^2+1}\right|\right]_0^1$ （P.20 公式（Ⅲ））

$=\log\left|1+\sqrt{2}\right|\cancel{-\log\left|0+\sqrt{1}\right|}=\log(1+\sqrt{2})$

(2) $(\text{与式})=\displaystyle\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\sum_{k=1}^{n}\sqrt{k}=\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\cdot\frac{1}{n}=\int_0^1\sqrt{x}\,dx=\left[\frac{2}{3}\sqrt{x^3}\right]_0^1=\frac{2}{3}$

（[教] P.51 公式で $f(x)=\sqrt{x}$，$x_k=\dfrac{k}{n}$，$\Delta x=\dfrac{1}{n}$ とする）

２章　積分法

２節　定積分の応用

A

63

(1) $$\int_0^{\frac{\pi}{2}} \left|2(1-\cos t)\cdot 2(1-\cos t)\right| dt = 4\int_0^{\frac{\pi}{2}} (1-2\cos t+\cos^2 t)\, dt = 4\int_0^{\frac{\pi}{2}} \left(1-2\cos t+\frac{1+\cos 2t}{2}\right) dt$$

$$= 4\left[t-2\sin t+\frac{1}{2}\left(t+\frac{1}{2}\sin 2t\right)\right]_0^{\frac{\pi}{2}}$$

$$= 4\left(\frac{\pi}{2}-2\sin\frac{\pi}{2}+\frac{\pi}{4}+\frac{1}{4}\cancel{\sin\pi}\right) = 3\pi-8$$

（ 公式 $\boxed{32}$ より $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \frac{1}{2}\cdot\frac{\pi}{2}$ を用いると簡単 ）

(2) $$\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \left|2\sin t\cdot(-2\sin t)\right| dt = 2\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} 2\sin^2 + dt$$

$$= 2\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} (1-\cos 2t)\, dt = 2\left[t-\frac{1}{2}\sin 2t\right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi}$$

$$= 2\left\{\left(\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right)-\left(-\frac{\pi}{4}-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)\right)\right\}$$

$$= 2\left\{\left(\frac{\pi}{4}-\frac{1}{2}\right)+\left(\frac{\pi}{4}-\frac{1}{2}\right)\right\} = \pi+2$$

(3) 曲線の $t=0$ から $t=1$ までの部分と x 軸および直線 $x=1$ で囲まれた図形の面積を求めて 2 倍すればよい。

$$2\int_0^1 \left|t^3\cdot 2t\right| dt = 4\int_0^1 t^4\, dt = 4\left[\frac{1}{5}t^5\right]_0^1 = \frac{4}{5}$$

64

(1) $$\frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\sin\theta)^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+2\sin\theta+\sin^2\theta)\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[\theta-2\cos\theta+\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left\{\left(\frac{\pi}{2}-\cancel{2\cos\frac{\pi}{2}}+\frac{1}{2}\left(\frac{\pi}{2}-\frac{1}{2}\cancel{\sin\pi}\right)\right)-\left(0-2\cos 0+\frac{1}{2}\cancel{(0-0)}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{\pi}{2}+\frac{\pi}{4}+2\right\} = \frac{3\pi}{8}+1$$

（ 公式 $\boxed{32}$ より $\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{1}{2}\cdot\frac{\pi}{2}$ を用いると簡単 ）

⑵　レムニスケートの　$\theta = 0$　から　$\theta = \dfrac{\pi}{4}$　までの部分と x 軸で囲まれた図形の面積を 4 倍すればよい。

（ 教 P.18 に図示）

$$4 \cdot \frac{1}{2}\int_0^{\frac{\pi}{4}} a^2 \cos 2\theta \, d\theta = \cancel{2}a^2\left[\frac{1}{\cancel{2}}\sin 2\theta\right]_0^{\frac{\pi}{4}} = a^2\left(\sin\frac{\pi}{2} - \cancel{\sin 0}\right) = a^2$$

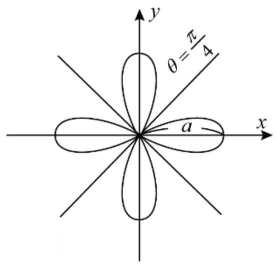
⑶　四葉線の　$\theta = 0$　から　$\theta = \dfrac{\pi}{4}$　までの部分と x 軸で囲まれた図形の面積を 8 倍すればよい。

$$8 \cdot \frac{1}{2}\int_0^{\frac{\pi}{4}} (a\cos 2\theta)^2 \, d\theta = 4a^2\int_0^{\frac{\pi}{4}} \cos^2 2\theta \, d\theta \quad \left(t = 2\theta \text{ とおくと } \frac{dt}{d\theta} = 2 \quad \begin{array}{c|ccc} \theta & 0 & \to & \frac{\pi}{4} \\ \hline t & 0 & \to & \frac{\pi}{2} \end{array} \right)$$

$$= 4a^2 \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2 t \, dt$$

$$= 2a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad （公式 \boxed{32}）$$

$$= \frac{1}{2}\pi a^2$$



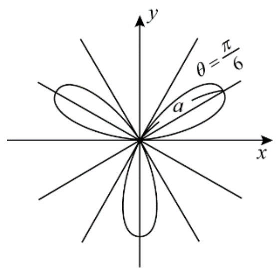
⑷　三葉線の　$\theta = 0$　から　$\theta = \dfrac{\pi}{6}$　までの部分と x 軸で囲まれた図形の面積を 6 倍すればよい。

$$6 \cdot \frac{1}{2}\int_0^{\frac{\pi}{6}} (a\sin 3\theta)^2 \, d\theta = 3a^2\int_0^{\frac{\pi}{6}} \sin^2 3\theta \, d\theta \quad \left(t = 3\theta \text{ とおくと } \frac{dt}{d\theta} = 3 \quad \begin{array}{c|ccc} \theta & 0 & \to & \frac{\pi}{6} \\ \hline t & \theta & \to & \frac{\pi}{2} \end{array} \right)$$

$$= 3a^2 \cdot \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^2 t \, dt$$

$$= a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad （公式 \boxed{32}）$$

$$= \frac{1}{4}\pi a^2$$



65

(1)　$y=\frac{1}{4}x^2$　より　$y'=\frac{1}{2}x$　だから

$$\int_0^1\sqrt{1+\frac{x^2}{4}}\,dx=\frac{1}{2}\int_0^1\sqrt{x^2+4}\,dx=\frac{1}{2}\left[\frac{1}{2}\left\{x\sqrt{x^2+4}+4\log\left|x+\sqrt{x^2+4}\right|\right\}\right]_0^1$$

$$=\frac{1}{4}\left\{\sqrt{5}+4\log\left|1+\sqrt{5}\right|-0-4\log 2\right\}=\frac{\sqrt{5}}{4}+\log\frac{1+\sqrt{5}}{2}$$

(2)　$y=\log|\cos x|$　より　$y'=-\frac{\sin x}{\cos x}$　だから，求める長さを L とすると

$$L=\int_0^{\frac{\pi}{3}}\sqrt{1+\frac{\sin^2 x}{\cos^2 x}}\,dx=\int_0^{\frac{\pi}{3}}\sqrt{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}\,dx=\int_0^{\frac{\pi}{3}}\frac{1}{\cos x}\,dx$$

ここで，$t=\tan\frac{x}{2}$　とおくと，$\cos x=\frac{1-t^2}{1+t^2}$，$dx=\frac{2}{1+t^2}\,dt$，
$\begin{array}{c|ccc} x & 0 & \to & \frac{\pi}{3} \\ \hline t & 0 & \to & \frac{\sqrt{3}}{3} \end{array}$
だから

$$L=\int_0^{\frac{\sqrt{3}}{3}}\frac{\cancel{1+t^2}}{1-t^2}\cdot\frac{2}{\cancel{1+t^2}}\,dt=-\int_0^{\frac{\sqrt{3}}{3}}\frac{2}{t^2-1}\,dt=-\int_0^{\frac{\sqrt{3}}{3}}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

(59(7)の解答同様 教 P.65公式を用いると簡単)

$$=-\Big[\log|t-1|-\log|t+1|\Big]_0^{\frac{\sqrt{3}}{3}}=\left[\log\left|\frac{t+1}{t-1}\right|\right]_0^{\frac{\sqrt{3}}{3}}=\log\left|\frac{\frac{\sqrt{3}}{3}+1}{\frac{\sqrt{3}}{3}-1}\right|\cancel{-\log\left|\frac{1}{1}\right|}=\log\left|\frac{1+\sqrt{3}}{1-\sqrt{3}}\right|$$

$$=\log\left|\frac{4+2\sqrt{3}}{2}\right|=\log(2+\sqrt{3})$$

66

(1)　$x=2t^2$，$y=t^3$　より　$\frac{dx}{dt}=4t$，$\frac{dy}{dt}=3t^2$　だから，求める曲線の長さを L とすると

$$\int_0^1\sqrt{16t^2+9t^4}\,dt=\int_0^1 t\sqrt{16+9t^2}\,dt$$

ここで，$s=16+9t^2$　とおくと，$\frac{ds}{dt}=18t$　より，$tdt=\frac{1}{18}ds$，
$\begin{array}{c|ccc} t & 0 & \to & 1 \\ \hline s & 16 & \to & 25 \end{array}$
だから

$$L=\int_{16}^{25}\sqrt{s}\cdot\frac{1}{18}\,dx=\frac{1}{18}\left[\frac{2}{3}\sqrt{s^3}\right]_{16}^{25}=\frac{1}{27}(5^3-4^3)=\frac{61}{27}$$

(2) $x=a(t-\sin t)$， $y=a(1-\cos t)$ より $\dfrac{dx}{dt}=a(1-\cos t)$， $\dfrac{dy}{dt}=a\sin t$ だから

$$\int_0^{\frac{\pi}{2}}\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\;dt=a\int_0^{\frac{\pi}{2}}\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\;dt=\int_0^{\frac{\pi}{2}}\sqrt{2(1-\cos t)}\;dt$$

$$=a\int_0^{\frac{\pi}{2}}\sqrt{4\sin^2\frac{t}{2}}\;dt\ (\text{公式}\boxed{10})=2a\int_0^{\frac{\pi}{2}}\sin\frac{t}{2}\,dt=2a\left[-2\cos\frac{t}{2}\right]_0^{\frac{\pi}{2}}$$

$$=-4a\left(\cos\frac{\pi}{4}-\cos 0\right)=-4a\left(\frac{\sqrt{2}}{2}-1\right)=2a\left(2-\sqrt{2}\right)$$

(3) $r=e^{\theta}$ より $\dfrac{dr}{d\theta}=e^{\theta}$ だから

$$\int_0^{\pi}\sqrt{e^{2\theta}+e^{2\theta}}\;d\theta=\int_0^{\pi}\sqrt{2e^{2\theta}}\;d\theta=\sqrt{2}\int_0^{\pi}e^{\pi}d\theta=\sqrt{2}\left[\,e^{\theta}\,\right]_0^{\pi}=\sqrt{2}(e^{\pi}-1)$$

(4) $r=a(1+\cos\theta)$ より $\dfrac{dr}{d\theta}=-a\sin\theta$ だから

$$\int_{\frac{\pi}{2}}^{\pi}\sqrt{a^2(1+\cos\theta)^2+a^2\sin^1\theta}\;d\theta=a\int_{\frac{\pi}{2}}^{\pi}\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\;d\theta$$

$$=a\int_{\frac{\pi}{2}}^{\pi}\sqrt{2(1+\cos\theta)}\;d\theta=a\int_{\frac{\pi}{2}}^{\pi}\sqrt{4\cos^2\frac{\theta}{2}}\;d\theta\ (\text{公式}\ \boxed{10}\ )$$

$$=2a\int_{\frac{\pi}{2}}^{\pi}\cos\frac{\theta}{2}\,d\theta=2a\left[\,2\sin\frac{\theta}{2}\right]_{\frac{\pi}{2}}^{\pi}=2a\left(2\sin\frac{\pi}{2}-2\sin\frac{\pi}{4}\right)$$

$$=2a\left(2-\sqrt{2}\right)$$

67

(1) $$\pi\int_0^{\frac{\pi}{4}}\tan^2 x\,dx=\pi\int_0^{\frac{\pi}{4}}\left(\frac{1}{\cos^2 x}-1\right)dx\ (\text{公式}\boxed{5})=\pi\left[\,\tan x-x\,\right]_0^{\frac{\pi}{4}}=\pi\left(1-\frac{\pi}{4}-0+0\right)=\frac{\pi(4-\pi)}{4}$$

(2) $$\pi\int_1^{e}(\log x)^2\,dx=\pi\int_1^{e}(x)'(\log x)^2\,dx=\pi\left\{\left[\,x(\log x)^2\,\right]_1^{e}-\int_1^{e}x\left\{(\log x)^2\right\}'\,dx\right\}\ (\text{公式}\ \boxed{31}\ )$$

$$=\pi\left\{\left[\,x(\log x)^2\,\right]_1^{e}-\int_1^{e}\cancel{x}\cdot 2\log x\cdot\frac{1}{\cancel{x}}\,dx\right\}=\pi\left\{e-0-2\int_1^{e}\log x\,dx\right\}$$

$$=\pi\left\{e-2\int_1^{e}(x)'\log x\,dx\right\}=\pi\left\{e-2\left(\left[\,x\log x\,\right]_1^{e}-\int_1^{e}x(\log x)'\,dx\right)\right\}\ (\text{公式}\ \boxed{31}\ )$$

$$=\pi\left\{e-2\left(\left[\,x\log x\,\right]_1^{e}-\int_1^{e}\cancel{x}\cdot\frac{1}{\cancel{x}}\,dx\right)\right\}=\pi\left\{e-2\left(e-0-\left[\,x\,\right]_1^{e}\right)\right\}$$

$$=\pi\left\{e-2(\cancel{e}-(\cancel{e}-1))\right\}=\pi(e-2)$$

⑴ $x=\log(t^2+1)$, $y=6t^3$ より, $\dfrac{dx}{dt}=\dfrac{2t}{t^2+1}$ だから，求める面積を S とすると

$$S=\int_0^1\left|6t^3\cdot\frac{2t}{t^2+1}\right|dt=12\int_0^1\frac{t^4}{t^2+1}\,dt$$

t^4 を t^2+1 で割ると，商が t^2-1，余りが 1 だから，$t^4=(t^2+1)(t^2-1)+1$

$$\therefore\quad \frac{t^4}{t^2+1}=\frac{(t^2+1)(t^2-1)+1}{t^2+1}=t^2-1+\frac{1}{t^2+1}$$

$$\therefore\quad S=12\int_0^1\left(t^2-1+\frac{1}{t^2+1}\right)dt=12\left[\frac{1}{3}t^3-t+\mathrm{Tan}^{-1}\,t\right]_0^1 \quad (\text{公式 } \boxed{28})$$

$$=12\left\{\left(\frac{1}{3}-1+\frac{\pi}{4}-(0-0+0)\right)\right\}=3\pi-8$$

⑵ $x=\tan t$, $y=\sin t$ より, $\dfrac{dx}{dt}=\dfrac{1}{\cos^2 t}$ （公式 $\boxed{26}$ ） だから，求める面積を S とすると

$$S=\int_0^{\frac{\pi}{3}}\left|\sin t\cdot\frac{1}{\cos^2 t}\right|dt$$

ここで，$u=\cos t$ とおくと，$du=-\sin t\,dt$，

t	$0\ \to\ \frac{\pi}{3}$
u	$1\ \to\ \frac{1}{2}$

だから

$$S=\int_1^{\frac{1}{2}}\frac{1}{u^2}(-1)\,du=\int_{\frac{1}{2}}^1\frac{1}{u^2}\,du=\left[-\frac{1}{u}\right]_{\frac{1}{2}}^1=-1+\frac{1}{\frac{1}{2}}=-1+2=1$$

⑶ $x=\tan t$, $y=\cos t$ より, $\dfrac{dx}{dt}=\dfrac{1}{\cos^2 t}$ （公式 $\boxed{26}$ ） だから，求める面積を S とすると

$$S=\int_0^{\frac{\pi}{3}}\left|\cos t\cdot\frac{1}{\cos^2 t}\right|dt=\int_0^{\frac{\pi}{3}}\frac{1}{\cos t}\,dt$$

ここで，$u=\tan\dfrac{t}{2}$ とおくと，$\cos t=\dfrac{1-u^2}{1+u^2}$，$dt=\dfrac{2}{1+u^2}\,du$

t	$0\ \to\ \frac{\pi}{3}$
u	$0\ \to\ \frac{1}{\sqrt{3}}$

だから，

$$S=\int_0^{\frac{1}{\sqrt{3}}}\frac{\cancel{1+u^2}}{1-u^2}\cdot\frac{2}{\cancel{1+u^2}}\,du=-\int_0^{\frac{1}{\sqrt{3}}}\frac{2}{(u-1)(u+1)}\,du=-\int_0^{\frac{1}{\sqrt{3}}}\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

（59 ⑺の解答と同様 $\boxed{\text{教}}$ P.65 の公式を用いると簡単）

$$= -\left[\log|u-1| - \log|u+1|\right]_0^{\frac{1}{\sqrt{3}}} = \left[\log\left|\frac{u+1}{u-1}\right|\right]_0^{\frac{1}{\sqrt{3}}} = \log\left|\frac{\frac{1}{\sqrt{3}}+1}{\frac{1}{\sqrt{3}}-1}\right| \log\left|\frac{1}{-1}\right|$$

$$= \log\left|\frac{1+\sqrt{3}}{1-\sqrt{3}}\right| - \cancel{\log 1} = \log\frac{\sqrt{3}+1}{\sqrt{3}-1} = \log\frac{4+2\sqrt{3}}{3-1} = \log\left(2+\sqrt{3}\right)$$

69

(1) $\displaystyle\int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{\varepsilon\to+0}\int_{1+\varepsilon}^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{\varepsilon\to+0}\left[2\sqrt{x-1}\right]_{1+\varepsilon}^2 = \lim_{\varepsilon\to+0}\left(2-2\sqrt{\varepsilon}\right) = 2$

(2) $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{\varepsilon\to+0}\int_0^{2-\varepsilon} \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{\varepsilon\to+0}\left[\mathrm{Sin}^{-1}\frac{x}{2}\right]_0^{2-\varepsilon}$ （P.20公式(I)）

$$= \lim_{\varepsilon\to+0}\left(\mathrm{Sin}^{-1}\frac{2-\varepsilon}{2} - \cancel{\mathrm{Sin}^{-1}0}\right) = \mathrm{Sin}^{-1}1 = \frac{\pi}{2}$$

(3) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\substack{\varepsilon\to+0\\ \varepsilon'\to+0}}\int_{-1+\varepsilon}^{1-\varepsilon'} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\substack{\varepsilon\to+0\\ \varepsilon'\to+0}}\left[\mathrm{Sin}^{-1}x\right]_{-1+\varepsilon}^{1-\varepsilon'}$ （P.20公式(I)）

$$= \lim_{\substack{\varepsilon\to+0\\ \varepsilon'\to+0}}\left\{\mathrm{Sin}^{-1}(1-\varepsilon') - \mathrm{Sin}^{-1}(-1+\varepsilon)\right\} = \mathrm{Sin}^{-1}1 - \mathrm{Sin}^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

70

(1) $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx = \lim_{K\to\infty}\int_1^K \frac{1}{x^4}\,dx = \lim_{K\to\infty}\left[-\frac{1}{3x^3}\right]_1^K = -\frac{1}{3}\lim_{K\to\infty}\left(\frac{1}{K^3}-1\right) = \frac{1}{3}$

(2) $\displaystyle\int_{-\infty}^0 e^x\,dx = \lim_{K\to\infty}\int_{-K}^0 e^x\,dx = \lim_{K\to\infty}\left[e^x\right]_{-K}^0 = \lim_{K\to\infty}\left(1-e^{-K}\right) = 1$

(3) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \lim_{\substack{K\to\infty\\ K'\to\infty}}\int_{-K'}^K \frac{1}{1+x^2}\,dx = \lim_{\substack{K\to\infty\\ K'\to\infty}}\left[\mathrm{Tan}^{-1}x\right]_{-K}^K$ （公式 28 ）

$$= \lim_{\substack{K\to\infty\\ K'\to\infty}}\left(\mathrm{Tan}^{-1}K - \mathrm{Tan}^{-1}K'\right) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

B

71

(1) $r\cos\theta = \cos 2\theta$ より， $r = \dfrac{\cos 2\theta}{\cos\theta}$ だから，

$$r^2 = \left(\frac{\cos 2\theta}{\cos\theta}\right)^2 = \frac{(2\cos^2\theta-1)^2}{\cos^2\theta} = \frac{4\cos^4\theta - 4\cos^2\theta + 1}{\cos^2\theta} = 4\cos^2\theta - 4 + \frac{1}{\cos^2\theta}$$ （公式 10 ）

$$= 4\cdot\frac{1+\cos 2\theta}{2} - 4 + \frac{1}{\cos^2\theta} = 2\cos 2\theta - 2 + \frac{1}{\cos^2\theta}$$

よって，求める面積は

$$\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{\cos 2\theta}{\cos\theta}\right)^2 d\theta=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(2\cos 2\theta-2+\frac{1}{\cos^2\theta}\right)d\theta=\frac{1}{2}\Big[\sin\theta-2\theta+\tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$ （公式 $\boxed{26}$ ）

$$=\frac{1}{2}\left\{\left(\sin\frac{\pi}{2}-\frac{\pi}{2}+\tan\frac{\pi}{4}\right)-\left(\sin\left(-\frac{\pi}{2}\right)+\frac{\pi}{2}+\tan\left(-\frac{\pi}{4}\right)\right)\right\}$$

$$=\frac{1}{2}\left\{1-\frac{\pi}{2}+1+1-\frac{\pi}{2}+1\right\}=2-\frac{\pi}{2}$$

⑵　$r=1+2\cos\theta$　より

$$r^2=(1+2\cos\theta)^2=1+4\cos\theta+4\cos^2\theta=1+4\cos\theta+4\cdot\frac{1+\cos 2\theta}{2}=3+4\cos\theta+2\cos 2\theta$$ （公式 $\boxed{10}$ ）

よって，求める面積は

$$\frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}(1+2\cos\theta)^2\,d\theta=\frac{1}{2}\int_{-\frac{2\pi}{4}}^{\frac{2\pi}{3}}(3+4\cos\theta+2\cos 2\theta)\,d\theta=\frac{1}{2}\Big[3\theta+4\sin\theta+\sin 2\theta\Big]_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}$$

$$=\frac{1}{2}\left\{\left(2\pi+4\sin\frac{2\pi}{3}+\sin\frac{4\pi}{3}\right)-\left(-2\pi+4\sin\left(-\frac{2\pi}{3}\right)+\sin\left(-\frac{4\pi}{3}\right)\right)\right\}$$

$$=\frac{1}{2}\left\{2\pi+2\sqrt{3}-\frac{\sqrt{3}}{2}+2\pi+2\sqrt{3}-\frac{\sqrt{3}}{2}\right\}=2\pi+\frac{3\sqrt{3}}{2}$$

（　$(1+2\cos\theta)^2$ が偶関数なので $\frac{1}{2}\int_{-\frac{4}{3}\pi}^{\frac{2}{3}\pi}=\frac{1}{2}\times 2\int_{0}^{\frac{2}{3}x}$　とすると簡単　）

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⑴　$y=\log|\sin x|$　より $y'=\frac{\cos x}{\sin x}$，　だから

$$\sqrt{1+(y')^2}=\sqrt{1+\frac{\cos^2 x}{\sin^2 x}}=\sqrt{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}}=\frac{1}{\sin x}$$

$t=\tan\frac{x}{2}$　とおくと，$\sin x=\frac{2t}{1+t^2}$，　$dx=\frac{2}{1+t^2}\,dt$，

x	$\frac{\pi}{3}$	→	$\frac{\pi}{2}$
t	$\frac{1}{\sqrt{3}}$	→	1

だから，求める長さは

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sqrt{1+(y')^2}\,dx=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{\sin x}\,dx=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{\cancel{1+t^2}}{2t}\cdot\frac{2}{\cancel{1+t^2}}\,dt=\int_{\frac{1}{\sqrt{3}}}^{1}\frac{1}{t}\,dt=\Big[\log t\Big]_{\frac{1}{\sqrt{3}}}^{1}$$

$$=\cancel{\log 1}-\log\frac{1}{\sqrt{3}}=-\log 3^{-\frac{1}{2}}=\frac{1}{2}\log 3$$

⑵　$x=e^t\cos t$，　$y=e^t\sin t$　より，$\frac{dx}{dt}=e^t\cos t-e^t\sin t=e^t(\cos t-\sin t)$，

$\frac{dy}{dx}=e^t\sin t+e^t\cos t=e^t(\cos t+\sin t)$ だから，

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=e^{2t}\left(\cos^2 t-\cancel{2\cos t\sin t}+\sin^2 t\right)+e^{2t}\left(\cos^2 t+\cancel{2\cos t\sin t}+\sin^2 t\right)$$

$$=2e^{2t}\left(\cos^2 t+\sin^2 t\right)=2e^{2t}$$

よって，求める長さは

$$\int_0^1\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\sqrt{2}\int_0^1 e^t\,dt=\sqrt{2}\left[e^t\right]_0^1=\sqrt{2}\left(e-1\right)$$

⑶ $x=t\cos t$， $y=t\sin t$ より， $\frac{dx}{dt}=\cos t-t\sin t$， $\frac{dy}{dt}=\sin t+t\cos t$ だから，

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\cos^2 t-\cancel{2t\cos t\sin t}+t^2\sin^2 t\right)+\left(\sin^2 t+\cancel{2t\sin t\cos t}+t^2\cos^2 t\right)$$

$$=\left(\cos^2 t+\sin^2 t\right)+t^2\left(\sin^2 t+\cos^2 t\right)=1+t^2$$

よって，求める長さは

$$\int_0^1\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^1\sqrt{1+t^2}\,dt=\left[\frac{1}{2}\left\{t\sqrt{t^2+1}+\log\left|t+\sqrt{t^2+1}\right|\right\}\right]_0^1$$

$$=\frac{1}{2}\left\{\left(\sqrt{2}+\log\left|1+\sqrt{2}\right|\right)-\left(0+\cancel{\log 1}\right)\right\}=\frac{1}{2}\left(\sqrt{2}+\log\left|1+\sqrt{2}\right|\right)$$

⑷ $x=\sin^2 t$， $y=\sin t\cos t$ より， $\frac{dx}{dt}=2\sin t\cos t=\sin 2t$， $\frac{dy}{dt}=\cos^2 t-\sin^2 t=\cos 2t$ だから，

求める長さは

$$\int_0^{\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^{\pi}\sqrt{\sin^2 2t+\cos^2 2t}\,dt=\int_0^{\pi}dt=\left[t\right]_0^{\pi}=\pi$$

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⑴ $x=\sin t$， $y=\sqrt{t}$ より， $\frac{dx}{dt}=\cos t$ だから，求める体積は

$$\pi\int_0^{\frac{\pi}{2}}\left(\sqrt{t}\right)^2\cos t\,dt=\pi\int_0^{\frac{\pi}{2}}t\cos t\,dt=\pi\int_0^{\frac{\pi}{2}}t\left(\sin t\right)'\,dt$$

$$=\pi\left\{\left[t\sin t\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\left(t\right)'\sin t\,dt\right\}$$ （公式 $\boxed{31}$ ）

$$=\pi\left\{\left[t\sin t\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin t\,dt\right\}=\pi\left\{\frac{\pi}{2}\sin\frac{\pi}{2}-0-\left[-\cos t\right]_0^{\frac{\pi}{2}}\right\}$$

$$=\pi\left\{\frac{\pi}{2}+\cancel{\cos\frac{\pi}{2}}-\cos 0\right\}=\frac{\pi\left(\pi-2\right)}{2}$$

(2)　$x = t^2$，　$y = -\cos t$　より，　$\dfrac{dx}{dt} = 2t$　だから，求める体積は

$$\pi\int_0^{\pi} \cos^2 t \cdot 2t\, dt = \pi\int_0^{\pi} \frac{1+\cos 2t}{\cancel{2}} \cdot \cancel{2}t\, dt \ (\text{公式 } \boxed{10}\ ) = \pi\left\{\int_0^{\pi} t\, dt + \int_0^{\pi} t\cos 2t\, dt\right\}$$

$$\left(\int_0^{\pi} t\cos 2t\, dt = \int_0^{\pi} t\cdot\left(\frac{1}{2}\sin 2t\right)' dt = \left[t\cdot\frac{1}{2}\sin 2t\right]_0^{\pi} - \int_0^{\pi}\left(t\right)'\cdot\frac{1}{2}\sin 2t\, dt\right) \ (\text{公式}\boxed{31}\ )$$

$$= \pi\left\{\left[\frac{1}{2}t^2\right]_0^{\pi} + \left[t\cdot\frac{1}{2}\sin 2t\right]_0^{\pi} - \int_0^{\pi}\frac{1}{2}\sin 2t\, dt\right\}$$

$$= \pi\left\{\left(\frac{\pi^2}{2} - 0\right) + \left(0 - 0\right) - \frac{1}{2}\left[-\frac{1}{2}\cos 2t\right]_0^{\pi}\right\}$$

$$= \pi\left\{\frac{\pi^2}{2} + \frac{1}{4}\left(\cancel{\cos 2\pi - \cos 0}\right)\right\} = \frac{\pi^3}{2}$$

(3)　$x = \cos^2 t$，　$y = \sin t\cos t$　より，　$\dfrac{dx}{dt} = 2\cos t\sin t$　だから，求める体積を V とすると

$$\pi\int_0^{\frac{\pi}{2}}\left(\sin t\cos t\right)^2 \cdot 2\cos t\sin t\, dt = 2\pi\int_0^{\frac{\pi}{2}}\sin^3 t\cos^3 t\, dt = 2\pi\int_0^{\frac{\pi}{2}}\sin^3 t\left(1-\sin^2 t\right)\cos t\, dt$$

$$= 2\pi\int_0^{\frac{\pi}{2}}\left(\sin^3 t - \sin^5 t\right)\cos t\, dt$$

ここで，$u = \sin t$ とおくと，$\dfrac{du}{dt} = \cos t$　$\therefore$　$du = \cos t\, dt$　$\begin{array}{c|ccc} t & 0 & \to & \frac{\pi}{2} \\ \hline u & 0 & \to & 1 \end{array}$　だから

$$V = 2\pi\int_0^1\left(u^3 - u^5\right) du = 2\pi\left[\frac{1}{4}u^4 - \frac{1}{6}u^6\right]_0^1 = 2\pi\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{\pi}{6}$$

(4)　$x=\cos t$，　$y=e^t$　より，$\dfrac{dx}{dt}=-\sin t$　だから，求める体積を V とすると

$$V=\pi\int_0^{\pi}\left(e^t\right)^2\left|-\sin t\right|dt=\pi\int_0^{\pi}e^{2t}\sin t\,dt$$

$$=\int_0^{\pi}e^{2t}\left(-\cos t\right)'\,dt=\pi\left\{\left[e^{2t}\left(-\cos t\right)\right]_0^{\pi}-\int_0^{\pi}\left(e^{2t}\right)'\left(-\cos t\right)dt\right\}$$ （公式 31 ）

$$=\pi\left\{\left[e^{2t}\left(-\cos t\right)\right]_0^{\pi}-\int_0^{\pi}2e^{2t}\left(-\cos t\right)dt\right\}$$

$$=\pi\left\{-e^{2\pi}\cos\pi+e^0\cos 0+2\int_0^{\pi}e^{2t}\cos t\,dt\right\}=\pi\left\{e^{2\pi}+1+2\left(\left[e^{2t}\sin t\right]_0^{\pi}-\int_0^{\pi}2e^{2t}\sin t\,dt\right)\right\}$$

$$\left(\int_0^{\pi}e^{2t}\cos t\,dt=\int_0^{\pi}e^{2t}\left(\sin t\right)'\,dt=\left[e^{2t}\sin t\right]_0^{\pi}-\int_0^{\pi}\left(e^{2t}\right)'\sin t\,dt\right)$$ （公式 31 ）

$$=\pi\left\{e^{2\pi}+1+2\left(\left(e^{2\pi}\cancel{\sin\pi}-e^0\cancel{\sin 0}\right)-2\int_0^{\pi}e^{2t}\sin t\,dt\right)\right\}=\pi\left(e^{2\pi}+1\right)-4V$$

$$\therefore\ \ 5V=\pi\left(e^{2\pi}+1\right)\quad\therefore V=\frac{\pi\left(e^{2^{\pi}}+1\right)}{5}$$

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(1)　$$\int_1^2\frac{1}{\sqrt{x^2-1}}\,dx=\lim_{\varepsilon\to+0}\int_{1+\varepsilon}^2\frac{1}{\sqrt{x^2-1}}\,dx=\lim_{\varepsilon\to+0}\left[\log\left|x+\sqrt{x^2-1}\right|\right]_{1+\varepsilon}^2$$ （P.20公式(Ⅲ)）

$$=\lim_{\varepsilon\to+0}\left\{;og\left|2+\sqrt{3}\right|-\log\left|1+\varepsilon+\sqrt{\left(1+\varepsilon\right)^2-1}\right|\right\}=\log\left|2+\sqrt{3}\right|-\cancel{\log\left(1+0\right)}=\log\left(2+\sqrt{3}\right)$$

(2)　$$\int_0^{\infty}xe^{-x^2}\,dx=\lim_{K\to\infty}\int_0^{K}xe^{-x^2}\,dx=\lim_{K\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^{K}=-\frac{1}{2}\lim_{K\to\infty}\left(e^{-K^2-e^0}\right)=-\frac{1}{2}(0-1)=\frac{1}{2}$$

(3) $I=\int_0^K e^{-x}\cos x\,dx=\int_0^K e^{-x}\left(\sin x\right)'\,dx$ とおくと

$$I=\left[e^{-x}\sin x\right]_0^K-\int_0^K\left(-e^{-x}\right)\sin x\,dx\text{（公式 }\boxed{31}\text{）}=e^{-K}\sin K-e^{0}\sin 0+\int_0^K e^{-x}\sin x\,dx$$

$$\left(\int_0^K e^{-x}\sin x\,dx=\int_0^K e^{-x}(-\cos x)'\,dx=\left[e^{-x}(-\cos x)\right]_0^K-\int_0^K(e^{-x})'\,(-\cos x)\,dx\text{ より}\right)$$

$$=e^{-K}\sin K+\left[e^{-x}\left(-\cos x\right)\right]_0^K-\int_0^K\left(-e^{-x}\right)\left(-\cos x\right)'\,dx$$

$$=e^{-K}\sin K-e^{-K}\cos K+e^{0}\cos 0-\int_0^K\left(-e^{-x}\right)\left(-\cos x\right)dx=e^{-K}\cos K-e^{-K}\cos K$$

$\therefore\quad 2I=e^{-K}\sin K-e^{-K}\cos K+1\qquad\therefore\quad I=\frac{1}{2}e^{-K}\sin K-\frac{1}{2}e^{-K}\cos K+\frac{1}{2}$

ここで，$-1\leqq\sin K\leqq 1$ より，$-e^{-K}\leqq e^{-K}\sin K\leqq e^{-K}$　だから

$0=\lim_{K\to\infty}\left(-e^{-K}\right)\leqq\lim_{K\to\infty}e^{-K}\sin K\leqq\lim_{K\to\infty}e^{-K}=0\quad\therefore\quad\lim_{K\to\infty}e^{-K}\sin K=0$

同様にして，$\lim_{K\to\infty}e^{-K}\cos K=0$

$$\therefore\quad\int_0^\infty e^{-x}\cos x\,dx=\lim_{K\to\infty}\int_0^K e^{-x}\cos x\,dx=\lim_{K\to\infty}\left\{\frac{1}{2}e^{-K}\sin K-\frac{1}{2}e^{-K}\cos K+\frac{1}{2}\right\}$$

$$=0+\frac{1}{2}=\frac{1}{2}$$

(4) $2\lim_{K\to\infty}\int_0^K\frac{1}{x^2+2^2}\,dx=2\lim_{K\to\infty}\frac{1}{2}\left[\mathrm{Tan}^{-1}\frac{x}{2}\right]_0^K=2\lim_{K\to\infty}\frac{1}{2}\left(\mathrm{Tan}^{-1}\frac{K}{2}-\mathrm{Tan}^{-1}0\right)=2\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{2}$

(5)

$$\int_{-\infty}^{\infty}\frac{1}{\left(x+1\right)^2+1}\,dx=\int_{-\infty}^{\infty}\frac{1}{t^2+1}\,dt=2\lim_{K\to\infty}\int_0^K\frac{1}{t^2+1}\,dt=2\lim_{K\to\infty}\left[\mathrm{Tan}^{-1}t\right]_0^K=2\cdot\frac{\pi}{2}\cdot\pi$$

（$x+1=t$ とおく）

(6) $n\neq 1$ のとき

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1\frac{1}{x^n}\,dx=\lim_{\varepsilon\to 0}\left[\frac{1}{-n+1}x^{-n+1}\right]_\varepsilon^1=\frac{1}{-n+1}\lim_{\varepsilon\to 0}\left(1-\varepsilon^{-n+1}\right)=\begin{cases}\dfrac{1}{-n+1} & (0<n<1)\\ \infty & (1<n)\end{cases}$$

$n=1$ のとき

$$\lim_{\varepsilon\to 0}\int_0^1\frac{1}{x}\,dx=\lim_{\varepsilon\to 0}\left[\log\left|x\right|\right]_\varepsilon^1=\lim_{\varepsilon\to 0}\left(0-\log^{\varepsilon}\right)=\infty$$

２章　積分法

２章の問題

1

⑴　②　　⑵　③

2　　③

3

⑴　$\displaystyle\int\frac{1-x^2}{1+x^2}\,dx=-\int\frac{x^2-1}{1+x^2}\,dx=-\int\frac{x^2+1-2}{1+x^2}\,dx$

$\displaystyle=-\int\left(1-2\cdot\frac{1}{1+x^2}\right)dx=\int\left(2\cdot\frac{1}{1+x^2}-1\right)dx$

$=2\,\mathrm{Tan}^{-1}x-x+C$　　（公式 $\boxed{28}$ ）

⑵　$\displaystyle\frac{2x-9}{(x-2)(x+3)}=\frac{a}{x-2}+\frac{b}{x+3}$ とおき，分母を払うと，$2x-9=(a+b)x+(3a-2b)$

辺を比較して，$a=-1$，$b=3$ を得る。よって

$\displaystyle\int\frac{2x-9}{(x-2)(x+3)}\,dx=\int\left(-\frac{1}{x-2}+\frac{3}{x+3}\right)dx=-\log|x-2|+3\log|x+3|+C$

$\displaystyle=\log\left|\frac{(x+3)^3}{x-2}\right|+C$

⑶　$\displaystyle\int\frac{2x}{x^2+x+1}\,dx=\int\frac{(x^2+x+1)'-1}{x^2+x+1}\,dx=\int\left\{\frac{(x^2+x+1)'}{(x^2+x+1)}-\frac{1}{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1}\right\}dx$ ▷

$\displaystyle=\int\left\{\frac{(x^2+x+1)'}{(x^2+x+1)}-\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}dx$　　（公式 $\boxed{30}$ と $\boxed{教}$ p.59

公式　$\displaystyle\int\frac{1}{x^2+a^2}\,dx=\frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a}+C$ ）

$\displaystyle=\log|x^2+x+1|-\frac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$

$\displaystyle=\log|x^2+x+1|-\frac{2}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

⑷ $\dfrac{1}{x^2(x-2)}=\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x-1}$ (とおくと) $=\dfrac{ax(x-1)+b(x-1)+cx^2}{x^2(x-1)}$

$=\dfrac{(a+c)x^2+(-a+b)x-b}{x^2(x-1)}$ より $a+c=0$, $-a+b=0$, $-b=1$

よって　$a=-1$, $b=-1$, $c=1$　従って

与式 $=\displaystyle\int\left(\frac{-1}{x}+\frac{-1}{x^2}+\frac{1}{x-1}\right)dx=-\log|x|+x^{-1}+\log|x-1|+C$

$=\log\left|\dfrac{x-1}{x}\right|+\dfrac{1}{x}+C$

4

⑴ $\displaystyle\int\frac{x^2}{\sqrt{x^2+1}}dx=\int\frac{x^2+1-1}{\sqrt{x^2+1}}dx=\int\left(\sqrt{x^2+1}-\frac{1}{\sqrt{x^2+1}}\right)dx$

$=\dfrac{1}{2}\left\{x\sqrt{x^2+1}+\log\left|x+\sqrt{x^2+1}\right|\right\}-\log\left|x+\sqrt{x^2+1}\right|+C$

$=\dfrac{1}{2}\left\{x\sqrt{x^2+1}-\log\left|x+\sqrt{x^2+1}\right|\right\}+C$　(P.20公式(Ⅲ)(Ⅳ))

⑵ $\displaystyle\int\frac{3-2x^2}{\sqrt{1-x^2}}dx=\int\frac{1+2(1-x^2)}{\sqrt{1-x^2}}dx=\int\left(\frac{1}{\sqrt{1-x^2}}+2\sqrt{1-x^2}\right)dx$　(P.20公式(Ⅰ)(Ⅱ))

$=\mathrm{Sin}^{-1}x+\dfrac{1}{2}\left\{x\sqrt{1-x^2}+\mathrm{Sin}^{-1}x\right\}+C=x\sqrt{1-x^2}+2\,\mathrm{Sin}^{-1}+C$

⑶ $\displaystyle\int\frac{1-\sin x}{1+\sin x}dx=-\int\frac{\sin x-1}{\sin x+1}dx=-\int\frac{\sin x+1-2}{\sin x+1}dx$

$=-\displaystyle\int\left(1-\frac{2}{\sin x+1}\right)dx$　$\left(t=\tan\dfrac{x}{2}\text{とおくと}\sin x=\dfrac{2t}{1+t^2},\quad dx=\dfrac{2dt}{1+t^2}\right)$

$=-x+2\displaystyle\int\frac{1}{\frac{2t}{1+t^2}+1}\cdot\frac{2}{1+t^2}dt=-x+4\int\frac{1}{2t+1+t^2}dt$

$=-x+4\displaystyle\int\frac{1}{(t+1)^2}dt=-x+4\left(-(t+1)^{-1}\right)+C$

$=-x-4\cdot\dfrac{1}{\tan\frac{x}{2}+1}+C$

⑷ $t=\tan\dfrac{x}{2}$ とおくと $\cos x=\dfrac{1-t^2}{1+t^2}$, $dx=\dfrac{2}{1+t^2}dt$ より

与式$=\displaystyle\int\frac{1}{5+3\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt=\int\frac{1}{5(1+t^2)+3(1-t^2)}\cdot 2dt$

$=\displaystyle\int\frac{2}{2t^2+8}dt=\int\frac{1}{t^2+2^2}dt=\frac{1}{2}\mathrm{Tan}^{-1}\frac{t}{2}+C$　　3 ⑶の解答と同じ 教 P.59 公式

$=\dfrac{1}{2}\mathrm{Tan}^{-1}\left(\dfrac{1}{2}\tan\dfrac{x}{2}\right)+C$

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(1) $\sin(\pi - x) = \sin x$ より， $\int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx = \int_{\frac{\pi}{2}}^{\pi} f(\sin(\pi - x))\,dx$

ここで， $t = \pi - x$ とおくと， $\frac{dt}{dx} = -1$ $\therefore$ $dx = -dt$ ，

x	$\frac{\pi}{2}$	$\to$	π
t	$\frac{\pi}{2}$	$\to$	0

だから

$$\int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx = \int_{\frac{\pi}{2}}^{0} f(\sin t)\cdot(-1)\,dt = \int_{0}^{\frac{\pi}{2}} f(\sin t)\,dt = \int_{0}^{\frac{\pi}{2}} f(\sin x)\,dx$$

$$\therefore \int_{0}^{\pi} f(\sin x)\,dx = \int_{0}^{\frac{\pi}{2}} f(\sin x)\,dx + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx$$

$$= \int_{0}^{\frac{\pi}{2}} f(\sin x)\,dx + \int_{0}^{\frac{\pi}{2}} f(\sin x)\,dx = 2\int_{0}^{\frac{\pi}{2}} f(\sin x)\,dx$$

(2) $\sin(x - 2\pi) = \sin x$ より， $\int_{2\pi-c}^{2\pi} f\left(\sin x\right) dx = \int_{2\pi-c}^{2\pi} f\left(\sin\left(x - 2\pi\right)\right) dx$ より，

ここで， $t = x - 2\pi$ とおくと， $\frac{dt}{dx} = 1$ $\therefore$ $dt = dx$ ，

x	$2\pi - c$	$\to$	2π
t	$-c$	$\to$	0

だから

$$\int_{2\pi-c}^{2\pi} f\left(\sin x\right) dx = \int_{-c}^{0} f\left(\sin t\right) dt = \int_{-c}^{0} f\left(\sin x\right) dx$$

$$\therefore \int_{0}^{2\pi} f\left(\sin x\right) dx = \int_{0}^{2\pi-c} f\left(\sin x\right) dx + \int_{2\pi-c}^{2\pi} f\left(\sin x\right) dx$$

$$= \int_{0}^{2\pi-c} f\left(\sin x\right) dx + \int_{-c}^{0} f\left(\sin x\right) dx = \int_{-c}^{2\pi-c} f\left(\sin x\right) dx$$

⑶ 公式 10 より，$a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\alpha)$。ただし，$\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$，

$\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$ である。ここで，$t = x+\alpha$ とおくと，$dt = dx$，

x	$0 \to 2\pi$
t	$\alpha \to 2\pi+\alpha$

だから

$$\int_0^{2\pi} f(a\cos x + b\sin x)\,dx = \int_0^{2\pi} f\left(\sqrt{a^2+b^2}\sin(x+\alpha)\right)dx = \int_\alpha^{2\pi+\alpha} f\left(\sqrt{a^2+b^2}\sin t\right)dt$$

$$= \int_0^{2\pi} f\left(\sqrt{a^2+b^2}\sin t\right)dt \quad \text{（⑵で } c \text{ を特に} -\alpha \text{とした）}$$

$$= \int_{-\pi}^{\pi} f\left(\sqrt{a^2+b^2}\sin t\right)dt$$

$$= \int_{-\pi}^{0} f\left(\sqrt{a^2+b^2}\sin t\right)dt + \int_0^{\pi} f\left(\sqrt{a^2+b^2}\sin t\right)dt$$

（⑵で c を特にπとした）

第 2 項は $\displaystyle\int_0^{\pi} f\left(\sqrt{a^2+b^2}\sin t\right)dt = 2\int_0^{\frac{\pi}{2}} f\left(\sqrt{a^2+b^2}\sin t\right)dt$ （⑴より） ㋐

次に第 1 項を $2\displaystyle\int_{-\frac{\pi}{2}}^{0}$ の形に変形する。

$u = -t$ とおくと，$du = dt$

t	$-\pi \to 0$
u	$\pi \to 0$

だから，⑴より

$$\int_{-\pi}^{0} f\left(\sqrt{a^2+b^2}\sin t\right)dt = \int_{\pi}^{0} f\left(\sqrt{a^2+b^2}\sin(-u)\right)du = \int_{\pi}^{0} f\left(-\sqrt{a^2+b^2}\sin u\right)du$$

$$= 2\int_0^{\frac{\pi}{2}} f\left(-\sqrt{a^2+b^2}\sin u\right)du = 2\int_{\frac{\pi}{2}}^{0} f\left(\sqrt{a^2+b^2}\sin(-u)\right)du = 2\int_{-\frac{\pi}{2}}^{0} f\left(\sqrt{a^2+b^2}\sin t\right)dt \quad ㋑$$

よって ㋐ ㋑より

$$\int_0^{2\pi} f(a\cos x + b\sin x)\,dx = 2\int_{-\frac{\pi}{2}}^{0} f\left(\sqrt{a^2+b^2}\sin t\right)dt + 2\int_0^{\frac{\pi}{2}} f\left(\sqrt{a^2+b^2}\sin t\right)dt$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\left(\sqrt{a^2+b^2}\sin t\right)dt = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\left(\sqrt{a^2+b^2}\sin x\right)dx$$

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⑴ $\left(\int t\cos 2t\,dt=\int t\left(\frac{1}{2}\sin 2t\right)'dt=t\cdot\frac{1}{2}\sin 2t-\int (t)'\sin 2t\,dt\right)$ （公式 31 ）

$$\int t\sin^2 t\,dt=\int t\cdot\frac{1-\cos 2t}{2}\,dt=\frac{1}{2}\left\{\int t\,dt-\int t\cos 2t\,dt\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{2}t^2-\left(\frac{1}{2}t\sin 2t-\frac{1}{2}\int\sin 2t\,dt\right)\right\}=\frac{1}{4}\left\{t^2-t\sin 2t-\frac{1}{2}\cos 2t\right\}+C$$

$$=\frac{1}{8}\left(2t^2-2t\sin 2t-\cos 2t\right)+C$$

⑵ $x(t)=-\frac{t}{\pi}\cos t$, $y(t)=\sin t$ より，$\frac{dx}{dt}=-\frac{1}{\pi}\cos t+\frac{t}{\pi}\sin t$ だから，求める面積は

$u=\sin t$ とおくと $\frac{du}{dt}=\cos t$ 　 $\left(\int\sin t\cos t\,dt=\int u\,du=\frac{1}{2}u^2+C=\frac{1}{2}\sin^2 t+C\right)$

$\left(\int t\sin^2 t\,dt\right.$ の方は(1)を利用$\left.\right)$

$$\int_{\frac{\pi}{2}}^{\pi}\left|\sin t\cdot\left(-\frac{1}{\pi}\cos t+\frac{t}{\pi}\sin t\right)\right|dt=\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi}\left(-\sin t\cos t+t\sin^2 t\right)dt$$

$$=\frac{1}{\pi}\left[-\frac{1}{2}\sin^2 t+\frac{1}{8}\left(2t^2-2t\sin 2t-\cos 2t\right)\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-\frac{1}{8\pi}\left[4\sin^2 t-2t^2+2t\sin 2t+\cos 2t\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-\frac{1}{8\pi}\left\{\left(0-2\pi^2+0+1\right)-\left(4-\frac{\pi^2}{2}+0-1\right)\right\}$$

$$=-\frac{1}{8\pi}\left\{-\frac{3\pi^2}{2}-2\right\}=\frac{3\pi^2+4}{16\pi}$$

7

⑴ $S=\int_t^{2t}\left(3-e^x\right)dx=\left[\,3x-e^x\,\right]_t^{2t}=\left(6t-e^{2t}\right)-\left(2t-e^t\right)=3t-e^{2t}+e^t$

⑵ $\dfrac{dS}{st}=3-2e^{2t}+e^t=-\left(2e^t-3\right)\left(e^t+1\right)$ より，$\dfrac{dS}{dt}=0$ となるのは，$e^t=\dfrac{3}{2}$，すなわち，

$t=\log\dfrac{3}{2}$ のときである。

（または

$S''(t)=-4e^{2t}+e^t$ より

$S''\left(\log\dfrac{3}{2}\right)=-4e^{2\log\frac{3}{2}}+e^{\log\frac{3}{2}}$

$=-4e^{\log\frac{9}{4}}+e^{\log\frac{3}{2}}$

$=-4\cdot\dfrac{9}{4}+\dfrac{3}{2}<0$ より

S は $t=\log\dfrac{3}{2}$ で極大値をとる。すなわち最大値をとる。）

t	0		$\log\frac{3}{2}$		$\frac{1}{2}\log 3$
$\frac{dS}{dt}$	$+$	$+$	0	$-$	$-$
S	0	↗	$3\log\frac{3}{2}-\frac{3}{4}$	↘	$\frac{3}{2}\log 3-3+\sqrt{3}$

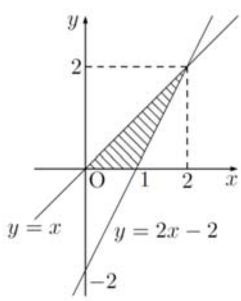
増減表より，$t=\log\dfrac{3}{2}$ のとき最大値 $3\log\dfrac{3}{2}-\dfrac{3}{4}$

8

⑴ $\pi\int_0^1\left(\sqrt{x}\right)^2dx=\pi\int_0^1 x\,dx=\pi\left[\dfrac{1}{2}x^2\right]_0^1=\dfrac{\pi}{2}$

⑵

直線 $y=x$ と $y=2x-2$ の交点は（2，2）であり，右図の斜線部を回転して得られる回転体の体積を求めればよいから



$\pi\int_0^2 x^2\,dx-\pi\int_1^2\left(2x-2\right)^2dx=\pi\int_0^2 x^2\,dx-4\pi\int_1^2\left(x^2-2x+1\right)dx$

$=\pi\left\{\left[\dfrac{1}{3}x^3\right]_0^2-4\left[\dfrac{1}{3}x^3-x^2+x\right]_1^2\right\}$

$=\pi\left\{\dfrac{8}{3}-4\left(\dfrac{8}{3}-4+2-\dfrac{1}{3}+1-1\right)\right\}=\pi\left\{\dfrac{8}{3}-4\cdot\dfrac{1}{3}\right\}=\dfrac{4\pi}{3}$

9

$x=\mathrm{Tan}^{-1}t$， $y=\dfrac{1}{t^2}$ より， $\dfrac{dx}{dt}=\dfrac{1}{1+t^2}$，

x	$\dfrac{\pi}{4}$	$\to$	$\dfrac{\pi}{2}$
t	1	$\to$	∞

だから，求める面積は

$$\int_1^{\infty}\left|\frac{1}{t^2}\cdot\frac{1}{1+t^2}\right|dt=\int_1^{\infty}\left(\frac{1}{t^2}-\frac{1}{1+t^2}\right)dt=\lim_{K\to\infty}\int_1^{K}\left(\frac{1}{t^2}-\frac{1}{1+t^2}\right)dt$$ （公式 $\boxed{28}$ ）

$$=\lim_{K\to\infty}\left[-\frac{1}{t}-\mathrm{Tan}^{-1}t\right]_1^K=\lim_{K\to\infty}\left\{-\frac{1}{K}-\mathrm{Tan}^{-1}K+\frac{1}{1}+\mathrm{Tan}^{-1}1\right\}=0-\frac{\pi}{2}+1+\frac{\pi}{4}$$

$$=1-\frac{\pi}{4}$$