２章　積分法

１節　定積分と不定積分（p.48～65）

練習 1

（証明）

任意の x について $f(x) \geqq g(x)$ より，任意の k について $f(\xi_k)\Delta x_k \geqq g(\xi_k)\Delta x_k$ だから，

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k \geqq \sum_{k=1}^{n} g(\xi_k)\Delta x_k \quad \therefore \quad S_f = \lim_{|\Delta|\to 0}\sum_{k=1}^{n} f(\xi_k)\Delta x_k \geqq \lim_{|\Delta|\to 0}\sum_{k=1}^{n} g(\xi_k)\Delta x_k = S_g$$

練習 2

⑴ $$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left\{\frac{k(b-a)}{n}+1\right\}\frac{b-a}{n} = \left\{\frac{b-a}{n}\right\}^2 \frac{1}{2}n(n+1) + \frac{b-a}{n}\cdot n$$

$$= \frac{1}{2}(b-a)^2\left(1+\frac{1}{n}\right)+(b-a)$$

$$\therefore \quad \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \frac{1}{2}(b-a)^2+(b-a)$$

⑵ $$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left\{2\left(a+\frac{k(b-a)}{n}\right)\cdot\frac{b-a}{n}\right\} = \sum_{k=1}^{n}\left\{\frac{2a(b-a)}{n}+\frac{2(b-a)^2}{n^2}\cdot k\right\}$$

$$= \frac{2a(b-a)}{n}\cdot n + \frac{2(b-a)^2}{n^2}\cdot\frac{1}{2}n(n+1) = 2a(b-a)+(b-a)^2\left(1+\frac{1}{n}\right)$$

$$\therefore \quad \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = 2a(b-a)+(b-a)^2 = b^2-a^2$$

練習 3

⑴ $$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left\{\left(\frac{k}{n}\right)^2+1\right\}\frac{1}{n} = \lim_{n\to\infty}\left\{\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{1}{n}\cdot n\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+1\right\} = \frac{4}{3}$$

⑵ $$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}+1\right)^2\frac{1}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k^2}{n^3}+\frac{2k}{n^2}+\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\left\{\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{2}{n^2}\cdot\frac{1}{2}n(n+1)+\frac{1}{n}\cdot n\right\}$$

$$= \lim_{n\to\infty}\left\{\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\left(1+\frac{1}{n}\right)+1\right\} = \frac{7}{3}$$

練習 4

⑴ $\frac{1}{4}$　　⑵ $\frac{5}{4}$

練習 5

⑴ $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k\right)\Delta x=\int_1^2 x^3\,dx=\left[\frac{1}{4}x^4\right]_1^2=\frac{15}{4}$

⑵ $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k\right)\Delta x=\int_1^2 \frac{1}{x^2}\,dx=\left[-\frac{1}{x}\right]_1^2=-\frac{1}{2}+1=\frac{1}{2}$

⑶ $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k\right)\Delta x=\int_1^2 \sqrt{x}\,dx=\left[\frac{2}{3}\sqrt{x^3}\right]_1^2=\frac{2}{3}\left(2\sqrt{2}-1\right)$

練習 6

⑴ $\frac{28}{3}$　　⑵ 2

練習 7

⑴ $\int\frac{x^3-4x-1}{x-2}\,dx=\int\left(x^2+2x-\frac{1}{x-2}\right)dx=\frac{1}{3}x^3+x^2-\log|x-2|+C$

⑵ $\int\frac{2x^3-x^2+2x}{x^2+1}\,dx=\int\left(2x-1+\frac{1}{x^2+1}\right)dx=x^2-x+\mathrm{Tan}^{-1}x+C$

練習 8

⑴ $\frac{3}{x^2+x-2}=\frac{a}{x-1}+\frac{b}{x+2}$ とおく。分母を払い，両辺を比較すると，$a=1$，$b=-1$，よって

$\int\frac{3}{x^2+x-2}\,dx=\int\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx=\log|x-1|-\log x|x+2|+C=\log\left|\frac{x-1}{x+2}\right|+C$

⑵ $P\left(x\right)=x^3-2x^2+x-2$ とおくと，$P\left(2\right)=0$ だから，因数定理より，$P\left(x\right)$ は $x-2$ で割り切れ，

$P\left(x\right)=\left(x-2\right)\left(x^2+1\right)$ となる。$\frac{x^2-4x-1}{x^3-2x^2+x-2}=\frac{a}{x-2}+\frac{bx+c}{x^2+1}$ とおく。分母を払い，両辺を

比較すると，$a=-1$，$b=2$，$c=0$，よって

$\int\frac{x^2-4x-1}{x^3-2x^2+x-2}\,dx=\int\left(-\frac{1}{x-2}+\frac{2x}{x^2+1}\right)dx=-\log|x-2|+\log|x^2+1|+C$

$=\log\left|\frac{x^2+1}{x-2}\right|+C$

練習 9

⑴ $\dfrac{-x^2+3x+3}{(x-1)(x^2+2x+2)}=\dfrac{a}{x-1}+\dfrac{bx+c}{x^2+2x+2}$ とおく。分母を払い，両辺を比較すると，

$a=1$， $b=-2$， $c=-1$， よって

$$\frac{-x^2+3x+3}{(x-1)(x^2+2x+2)}=\frac{1}{x-1}-\frac{2x+1}{x^2+2x+2}=\frac{1}{x-1}-\frac{2x+2-1}{x^2+2x+2}$$

$$=\frac{1}{x-1}-\frac{2x+2}{x^2+2x+2}+\frac{1}{x^2+2x+2}$$

$$\therefore \int\frac{-x^2+3x+3}{(x-1)(x^2+2x+2)}\,dx=\log|x-1|-\log|x^2+2x+2|+\mathrm{Tan}^{-1}(x+1)+C$$

$$=\log\left|\frac{x-1}{x^2+2x+2}\right|+\mathrm{Tan}^{-1}(x+1)+C$$

⑵ $\dfrac{-3x+1}{(x+1)(x-1)^2}=\dfrac{a}{x+1}+\dfrac{b}{x-1}+\dfrac{c}{(x-1)^2}$ とおく。分母を払い，両辺を比較すると，

$a=1$， $b=-1$， $c=-1$， よって， $\dfrac{-3x+1}{(x+1)(x-1)^2}=\dfrac{1}{x+1}-\dfrac{1}{x-1}-\dfrac{1}{(x-1)^2}$

$$\therefore \int\frac{-3x+1}{(x+1)(x-1)^2}\,dx=\log|x+1|-\log|x-1|+\frac{1}{x-1}+C=\log\left|\frac{x+1}{x-1}\right|+\frac{1}{x-1}+C$$

⑶ $\dfrac{1}{2}\log|x|-3\log|x-1|+\dfrac{5}{2}\log|x-2|+C$

練習 10

$\displaystyle\int\frac{1}{x^2+a^2}\,dx$ （分母分子を $\div a$）

$$=\int\frac{\frac{1}{a}}{\left(\frac{x}{a}\right)^2+1}\,dx$$

$\left(\begin{array}{l}\dfrac{x}{a}=t \text{ とおくと} \\ \dfrac{1}{a}=\dfrac{dt}{dx}\ (dx=a\,dt)\end{array}\right.$

$$=\int\frac{\frac{1}{a}}{t^2+1}\cdot a\,dt$$

$$=\int\frac{1}{t^2+1}\,dt$$

$=\mathrm{Tan}^{-1}t+C$ （公式$\boxed{28}$より）

$=\mathrm{Tan}^{-1}\dfrac{x}{a}+C$

（Cは積分定数）

練習 11

(1) $$\int \frac{1}{\sin x}\,dx = \int \frac{1+t^2}{2t}\cdot\frac{2}{1+t^2}\,dt = \int \frac{1}{t}\,dt = \log|t| + C = \log\left|\tan\frac{x}{2}\right| + C$$

(2) $$\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\dfrac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{2}{1+t^2+2t}\,dt = 2\int \frac{1}{(1+t)^2}\,dt$$

$$= -\frac{2}{1+t} + C = -\frac{2}{1+\tan\dfrac{x}{2}} + C$$

練習 12

(1) $$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x^2-2x+1)}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \mathrm{Sin}^{-1}(x-1) + C$$

(2) $$\int \frac{1}{\sqrt{3x-2x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x^2+2x+1)}}\,dx = \int \frac{1}{\sqrt{4-(x+1)^2}}\,dx = \mathrm{Sin}^{-1}\frac{x+1}{2} + C$$

練習 13

(1) $$\int \sqrt{2x-x^2}\,dx = \int \sqrt{1-(x^2-2x+1)}\,dx = \int \sqrt{1-(x-1)^2}\,dx$$

$$= \frac{1}{2}\left\{(x-1)\sqrt{2x-x^2} + \mathrm{Sin}^{-1}(x-1)\right\} + C$$

(2) $$\int \sqrt{8-2x-x^2}\,dx = \int \sqrt{9-(x^2+2x+1)}\,dx = \int \sqrt{9-(x+1)^2}\,dx$$

$$= \frac{1}{2}\left\{(x-1)\sqrt{8-2x-x^2} + \mathrm{Sin}^{-1}\frac{x+1}{3}\right\} + C$$

練習 14

(1) $$\int \frac{1}{\sqrt{x^2-4x+5}}\,dx = \int \frac{1}{\sqrt{(x-2)^2+1}}\,dx = \log\left|x-2+\sqrt{x^2-4x+5}\right| + C$$

(2) $$\int \frac{1}{\sqrt{x^2+2x-1}}\,dx = \int \frac{1}{\sqrt{(x+1)^2-2}}\,dx = \log\left|x+1+\sqrt{x^2+2x-1}\right| + C$$

練習 15

(1) $$\int \sqrt{x^2+2x+4}\,dx = \int \sqrt{(x+1)^2+3}\,dx$$

$$= \frac{1}{2}\left\{(x+1)\sqrt{x^2+2x+4} + 3\log\left|x+1+\sqrt{x^2+2x+4}\right|\right\} + C$$

(2) $$\int \sqrt{4x(x-2)}\,dx = 2\int \sqrt{(x^2-2x+1)-1}\,dx = 2\int \sqrt{(x-1)^2-1}\,dx$$

$$= (x-1)\sqrt{x^2-2x} - \log\left|x-1+\sqrt{x^2-2x}\right| + C$$

節末問題（p.65）

1

⑴ $\displaystyle \lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k\right)\Delta x=\int_1^2 \frac{1}{x}\,dx=\Big[\log|x|\Big]_1^2=\log 2$

⑵ $\displaystyle \lim_{n\to\infty}\sum_{k=1}^{n} f\left(x_k\right)\Delta x=\int_0^1 e^x\,dx=\Big[e^x\Big]_0^1=e-1$

2

⑴ $\displaystyle \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan x\,dx=0$

⑵ $\displaystyle \int_{-\pi}^{\pi}|\sin x|\,dx=2\int_0^{\pi}|\sin x|\,dx=2\int_0^{\pi}\sin x\,dx=2\Big[-\cos x\Big]_0^{\pi}=4$

⑶ 0

3

⑴ $\displaystyle \int\frac{x^4+5x^2+5}{x^2+4}\,dx=\int\left(x^2+1+\frac{1}{x^2+4}\right)dx=\frac{1}{3}x^3+x+\frac{1}{2}\mathrm{Tan}^{-1}\frac{x}{2}+C$

⑵ $\displaystyle \int\frac{2x^3+x^2-2x}{x^2-1}\,dx=\int\left\{2x+1+\frac{1}{x^2-1}\right\}dx=\int\left\{2x+1+\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)\right\}dx$

$\displaystyle =x^2+x+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C$

⑶ $\displaystyle \int\frac{1}{1+\cos x}\,dx=\int\frac{1}{1+\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt=\int\frac{2}{1+\cancel{t^2}+1-\cancel{t^2}}\,dt=\int dt=t+C=\tan\frac{x}{2}+C$

⑷ $\displaystyle \int\frac{1}{4\sin x+3\cos x}\,dx=\int\frac{1}{4\cdot\dfrac{2t}{1+t^2}+3\cdot\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt=\int\frac{2}{8t+3-3t^2}\,dt$

$\displaystyle =-2\int\frac{1}{\left(3t+1\right)\left(t-3\right)}\,dt=-2\int\left\{-\frac{1}{10}\left(\frac{3}{3t+1}-\frac{1}{t-3}\right)\right\}dt$

$\displaystyle =\frac{1}{5}\Big\{\log|3t+1|-\log|t-3|\Big\}+C=\frac{1}{5}\log\left|\frac{3t+1}{t-3}\right|+C=\frac{1}{5}\log\left|\frac{3\tan\dfrac{x}{2}+1}{\tan\dfrac{x}{2}-3}\right|+C$

(5) $\displaystyle\int \frac{10-x^2}{\sqrt{9-x^2}}\,dx = \int \frac{\left(9-x^2\right)+1}{\sqrt{9-x^2}}\,dx = \int \left\{\sqrt{9-x^2} + \frac{1}{\sqrt{9-x^2}}\right\} dx$

$$= \frac{1}{2}\int\left\{x\sqrt{9-x^2} + 9\,\mathrm{Sin}^{-1}\frac{x}{3}\right\} + \mathrm{Sin}^{-1}\frac{x}{3} + C = \frac{1}{2}x\sqrt{9-x^2} + \frac{11}{2}\mathrm{Sin}^{-1}\frac{x}{3} + C$$

(6) $\displaystyle\int \frac{x^2+3}{\sqrt{x^2+2}}\,dx = \int \frac{\left(x^2+2\right)+1}{\sqrt{x^2+2}}\,dx = \int \left\{\sqrt{x^2+2} + \frac{1}{\sqrt{x^2+2}}\right\} xd$

$$= \frac{1}{2}\left\{x\sqrt{x^2+2} + 2\log\left|x+\sqrt{x^2+2}\right|\right\} + \log\left|x+\sqrt{x^2+2}\right| + C$$

$$= \frac{1}{2}\sqrt{x^2+2} + 2\log\left|x+\sqrt{x^2+2}\right| + C$$