４章　重積分

１節　重積分（p.126~143）

練習 1

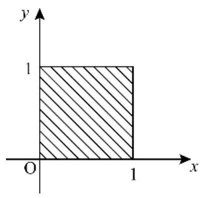
⑴

$$\iint_D xy\,dx\,dy = \int_0^1 \left(\int_0^1 xy\,dy \right) dx$$

$$= \int_0^1 x \left[\frac{1}{2} y^2 \right]_0^1 dx$$

$$= \frac{1}{2} \left[\frac{1}{2} x^2 \right]_0^1$$

$$= \frac{1}{4}$$



練習 2

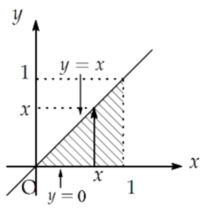
$$\iint_D xy\,dx\,dy = \int_0^1 \left(\int_0^x xy\,dy \right) dx$$

$$= \int_0^1 x \left[\frac{1}{2} y^2 \right]_0^x dx$$

$$= \frac{1}{2} \int_0^1 x^3\,dx$$

$$= \frac{1}{2} \left[\frac{1}{4} x^4 \right]_0^1$$

$$= \frac{1}{8}$$



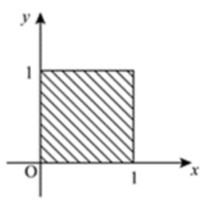
練習 3

$$\iint_D xy\,dx\,dy = \int_0^1 \left(\int_0^1 xy\,dx \right) dy$$

$$= \int_0^1 y \left[\frac{1}{2} x^2 \right]_0^1 dy$$

$$= \frac{1}{2} \left[\frac{1}{2} y^2 \right]_0^1$$

$$= \frac{1}{4}$$



練習 4

$$\iint_D xy\,dx\,dy=\int_0^1\left(\int_y^1 xy\,dx\right)dy$$

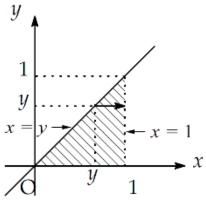
$$=\int_0^1 y\left[\frac{1}{2}x^2\right]_y^1 dy$$

$$=\frac{1}{2}\int_0^1 y\left(1-y^2\right)dy$$

$$=\frac{1}{2}\int_0^1\left(y-y^3\right)dy$$

$$=\frac{1}{2}\left[\frac{1}{2}y^2-\frac{1}{4}y^4\right]_0^1$$

$$=\frac{1}{8}$$



練習 5

(1)

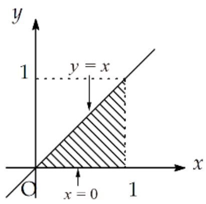
$$\int_0^1\left(\int_0^x(3-x-y)\,dy\right)dx=\int_0^1\left[(3-x)y-\frac{y^2}{2}\right]_0^x dx$$

$$=\int_0^1\left((3-x)x-\frac{y^2}{2}\right)dx$$

$$=\int_0^1\left((3-x)x-\frac{x^2}{2}\right)dx$$

$$=\int_0^1\left(3x-\frac{3x^2}{2}\right)dx$$

$$=\left[\frac{3x^2}{2}-\frac{x^3}{2}\right]_0^1=\frac{3}{2}-\frac{1}{2}=1$$



順序交換：

$$\int_0^1\left(\int_y^1(3-x-y)\,dx\right)dy=\int_0^1\left[3x-\frac{x^2}{2}-yx\right]_y^1 dy$$

$$=\int_0^1\left(3-\frac{1}{2}-y-3y+\frac{y^2}{2}+y^2\right)dy$$

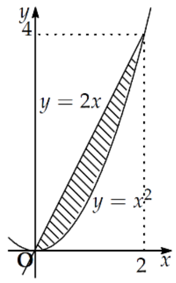
$$=\int_0^1\left(\frac{5}{2}-4y+\frac{3y^2}{2}\right)dy$$

$$=\left[\frac{5y}{2}-2y^2+\frac{y^3}{2}\right]_0^1=\frac{5}{2}-2+\frac{1}{2}=1$$

⑵ $\int_0^2\left(\int_{x^2}^{2x}(2x+1)\,dy\right)dx=\int_0^2\Big[(2x+1)\,y\Big]_{x^2}^{2x}\,dx$

$=\int_0^2\left(-2x^3+3x^2+2x\right)dx$

$=\left[-\frac{x^4}{2}+x^3+x^2\right]_0^2=-\frac{16}{2}+8+4=4$



順序交換：

$\int_0^4\left(\int_{\frac{y}{2}}^{\sqrt{y}}(2x+1)\,dx\right)dy=\int_0^4\Big[x^2+x\Big]_{\frac{y}{2}}^{\sqrt{y}}\,dy$

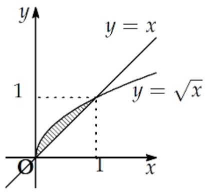
$=\int_0^4\left(-\frac{y^2}{4}+\frac{y}{2}+y^{\frac{1}{2}}\right)dy$

$=\left[-\frac{y^3}{12}+\frac{y^2}{4}+\frac{2}{3}y^{\frac{3}{2}}\right]_0^4=-\frac{16}{3}+4+\frac{16}{3}=4$

⑶ $\int_0^1\left(\int_x^{\sqrt{x}}2xy\,dy\right)dx=\int_0^1\Big[xy^2\Big]_x^{\sqrt{x}}\,dx$

$=\int_0^1\left(x^2-x^3\right)dx$

$=\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$



順序交換：

$\int_0^1\left(\int_{y^2}^{y}2xy\,dx\right)dy=\int_0^1\Big[x^2y\Big]_{y^2}^{y}\,dy$

$=\int_0^1\left(y^3-y^5\right)dy$

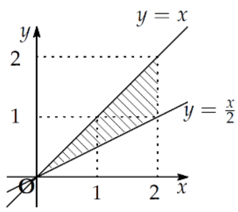
$=\left[\frac{y^4}{4}-\frac{y^6}{6}\right]_0^1=\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$

(4) $$\int_0^1\left(\int_y^{2y}2xy\,dx\right)dy+\int_1^2\left(\int_y^{2}2xy\,dx\right)dy=\int_0^1\left[x^2y\right]_y^{2y}dy+\int_1^2\left[x^2y\right]_y^{2}dy$$

$$=\int_0^1\left(4y^3-y^3\right)dy+\int_0^2\left(4y-y^3\right)dy$$

$$=\left[\frac{3y^4}{4}\right]_0^1+\left[2y^2-\frac{y^4}{4}\right]_1^2$$

$$=\frac{3}{4}+8-\frac{16}{4}-2+\frac{1}{4}=3$$



順序交換

$$\int_0^2\left(\int_{\frac{x}{2}}^{x}2xy\,dy\right)dx=\int_0^2\left[xy^2\right]_{\frac{x}{2}}^{x}dx$$

$$=\int_0^2\left(x^3-\frac{x^3}{4}\right)dx$$

$$=\int_0^2\frac{3x^3}{4}dx$$

$$=\left[\frac{3}{16}x^4\right]_0^2=3$$

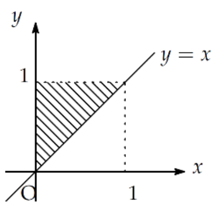
練習 6

(1) $$\int_0^1\left(\int_x^1 2e^{y^2}dy\right)dx=\int_0^1\left(\int_0^y 2e^{y^2}dx\right)dy$$

$$=\int_0^1 2e^{y^2}\left(\int_0^y dx\right)dy$$

$$=\int_0^1 2ye^{y^2}dy$$

$$=\left[e^{y^2}\right]_0^1=e-1$$



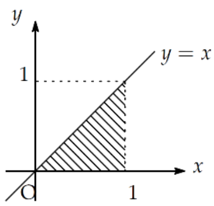
(2) $$\int_0^1\left(\int_y^1\frac{\sin x}{x}dx\right)dy=\int_0^1\left(\int_0^x\frac{\sin x}{x}dy\right)dx$$

$$=\int_0^1\frac{\sin x}{x}\left(\int_0^x dy\right)dx$$

$$=\int_0^1\frac{\sin x}{x}x\,dx$$

$$=\int_0^1\sin x\,dx$$

$$=\left[-\cos x\right]_0^1=1-\cos(1)$$



(3) $\displaystyle \int_0^2 \int_x^2 2y^2 \sin(xy)\,dy\,dx = \int_0^2 \int_0^y 2y^2 \sin(xy)\,dx\,dy$

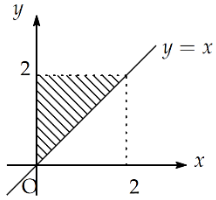
$$= \int_0^2 2y \int_0^y y \sin(xy)\,dx\,dy$$

$$= \int_0^2 2y \Big[-\cos(xy) \Big]_0^y\,dy$$

$$= \int_0^2 2y\left(-\cos\left(y^2\right)+1\right)dy$$

$$= \int_0^2 \left(-2y\cos\left(y^2\right)+2y\right)dy$$

$$= \Big[-\sin\left(y^2\right)+y^2 \Big]_0^2 = 4-\sin 4$$

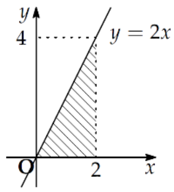


(4) $\displaystyle \int_0^4 \left(\int_{\frac{y}{2}}^2 \cos\left(x^2\right)dx \right) dy = \int_0^2 \left(\int_0^{2x} \cos\left(x^2\right)dy \right) dx$

$$= \int_0^2 \cos\left(x^2\right)\left(\int_0^{2x} dy \right) dx$$

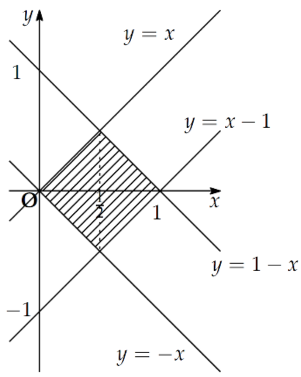
$$= \int_0^2 \cos\left(x^2\right) 2x\,dx$$

$$= \Big[\sin\left(x^2\right) \Big]_0^2 = \sin 4$$



練習 7

$\begin{cases} u = x+y \\ v = x-y \end{cases}$ とおくと $\begin{cases} x = \dfrac{1}{2}(u+v) \\ y = \dfrac{1}{2}(u-v) \end{cases}$ であるから,



$$J = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\therefore \quad |J| = \frac{1}{2}$$

$$\int_0^1 \int_0^1 \frac{u-v}{2} \cdot \frac{1}{2}\,du\,dv = \frac{1}{4} \int_0^1 \int_0^1 (u-v)\,du\,dv$$

$$= \frac{1}{4} \int_0^1 \left[\frac{u^2}{2} - vu \right]_0^1 dv$$

$$= \frac{1}{4} \int_0^1 \left(\frac{1}{2} - v \right) dv$$

$$= \frac{1}{4} \left[\frac{1}{2}v - \frac{1}{2}v^2 \right]_0^1$$

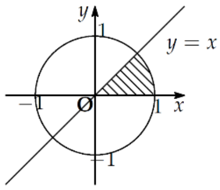
$$= \frac{1}{4} \left(\frac{1}{2} - \frac{1}{2} \right) = 0$$

練習 8

(1) $$\iint_D (x^2+y^2)\,dx\,dy = \int_0^1 \int_0^{\frac{\pi}{4}} r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta \cdot \int_0^1 r^3\,dr$$

$$= \frac{\pi}{4}\cdot\left[\frac{r^4}{4}\right]_0^1 = \frac{\pi}{16}$$



(2) $$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi}\int_0^{\sin\theta}\sqrt{r^2}\cdot r\,dr\,d\theta$$

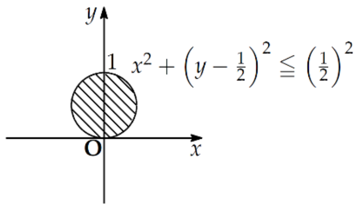
$$= \int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{\sin\theta} d\theta$$

$$= \int_0^{\pi}\frac{1}{3}\sin^3\theta\,d\theta$$

$$= \frac{2}{3}\int_0^{\frac{\pi}{2}}\sin^3\theta\,d\theta$$

$$= \frac{2}{3}\cdot\frac{2}{3}$$ (Wallis の公式から)

$$= \frac{4}{9}$$



練習 9

$$\iint_{D'} dx\,dy = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 r\,dr$$

$$= 2\pi\left[\frac{r^2}{2}\right]_0^1 = 2\pi\cdot\frac{1}{2} = \pi$$

練習 10　$x^2+4xy+13y^2=16 \Longrightarrow (x+2y)^2+(3y)^2=4^2$

$\begin{cases} u=x+2y \\ v=3y \end{cases}$ とおくと $\begin{cases} x=u-\dfrac{2}{3}v \\ y=\dfrac{v}{3} \end{cases}$ であるから，

$$J=\begin{vmatrix} 1 & -\dfrac{2}{3} \\ 0 & \dfrac{1}{3} \end{vmatrix}=\frac{1}{3}$$

よって

$$\iint_D dx\,dy=\iint_{D'}\frac{1}{3}du\,dv$$

$$=\frac{1}{3}\iint_{D'}du\,dv$$

ここで　$D'=\left\{(u,\ \ v)\,|\,u^2+v^2=4^2\right\}$ であるから

$\iint_{D'}dudv=\pi\cdot 4^2=16\pi$　となるので　$\iint_D dxdy=\dfrac{16}{3}\pi$

節末問題（p.144）

1

⑴

$$\int_{-1}^{1}\int_{0}^{2}(2x-3y^2)\,dy\,dx=\int_{-1}^{1}\Big[\,2xy-y^3\,\Big]_0^2\,dx$$

$$=\int_{-1}^{1}(4x-8)\,dx=2\int_{0}^{1}(-8)\,dx=-16$$

⑵

$$\int_0^1\int_\pi^{2\pi}x\sin(xy)\,dy\,dx=\int_0^1\Big[-\cos(xy)\Big]_\pi^{2\pi}\,dx$$

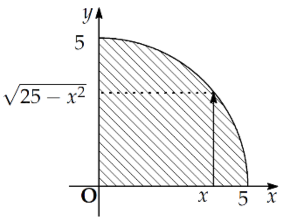
$$=\int_0^1(-\cos(2\pi x)+\cos(\pi x))\,dx$$

$$=\left[-\frac{1}{2\pi}\sin(2\pi x)+\frac{1}{\pi}\sin(\pi x)\right]_0^1=0$$

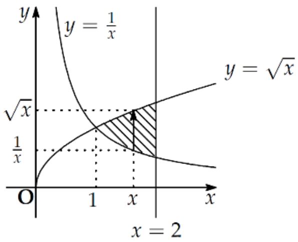
2

(1)

$$\int_0^5 \int_0^{\sqrt{25-x^2}} f(x,\ y)\,dy\,dx$$



(2) $$\int_1^2 \int_{\frac{1}{x}}^{\sqrt{x}} f(x,\ y)\,dy\,dx$$

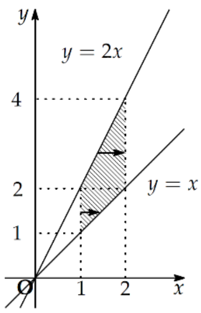


3

(1)

$$\int_1^2 \int_x^{2x} f\ dy\ dx$$

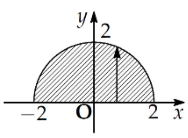
$$= \int_1^2 \int_1^y f\ dx\ dy + \int_2^4 \int_{\frac{y}{2}}^2 f\ dx\ dy$$



(2)

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f\ dx\ dy$$

$$= \int_{-2}^2 \int_0^{\sqrt{4-x^2}} f\ dy\ dx$$



4

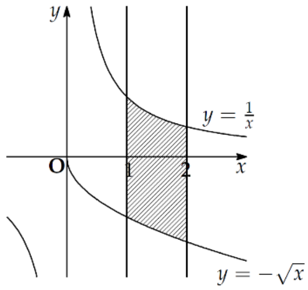
(1) $$\iint_D x^2y\,dx\,dy=\int_1^2\left(\int_{-\sqrt{x}}^{\frac{1}{x}}x^2y\,dy\right)dx$$

$$=\int_1^2\left[\frac{x^2y^2}{2}\right]_{-\sqrt{x}}^{\frac{1}{x}}dx$$

$$=\int_1^2\left(\frac{1}{2}-\frac{x^3}{2}\right)dx$$

$$=\left[\frac{x}{2}-\frac{x^4}{8}\right]_1^2$$

$$=1-2-\frac{1}{2}+\frac{1}{8}=-\frac{11}{8}$$



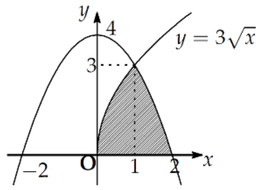
(2) $$\int_0^1\int_0^{3\sqrt{x}}xy\,dy\,dx+\int_1^2\int_0^{4-x^2}xy\,dy\,dx=\int_0^1\left[x\frac{y^2}{2}\right]_0^{3\sqrt{x}}dx+\int_1^2\left[x\frac{y^2}{2}\right]_0^{4-x^2}dx$$

$$=\int_0^1\frac{9}{2}x^2\,dx+\int_1^2\frac{x}{2}\left(4-x^2\right)^2dx$$

$$=\int_0^1\frac{9}{2}x^2\,dx-\frac{1}{4}\int_1^2(-2x)\left(4-x^2\right)^2dx$$

$$=\left[\frac{3x^3}{2}\right]_0^1-\frac{1}{4}\left[\frac{\left(4-x^2\right)^3}{3}\right]_1^2$$

$$=\frac{3}{2}-\frac{1}{12}(-3^3)=\frac{3}{2}+\frac{9}{4}=\frac{15}{4}$$



5

(1) $$\iint_D(x^2+y^2)\,dx\,dy=\int_0^{2\pi}\int_0^2 r^2\cdot r\,dr\,d\theta$$

$$=2\pi\left[\frac{r^4}{4}\right]_0^2=2\pi\cdot\frac{16}{4}=8\pi$$

(2) $$\iint_D\sqrt{x^2+y^2-9}\,dx\,dy=\int_0^{2\pi}\int_3^5\sqrt{r^2-9}\cdot r\,dr\,d\theta$$

$$=\frac{2\pi}{3}\left[(r^2-9)^{\frac{3}{2}}\right]_3^5=\frac{2\pi}{3}(25-9)^{\frac{3}{2}}$$

$$=\frac{2\pi}{3}(4^2)^{\frac{3}{2}}=\frac{128}{3}\pi$$

6

$\begin{cases} u = x - y \\ v = x + y \end{cases}$ とおくと $\begin{cases} x = \dfrac{1}{2}(u + v) \\ y = \dfrac{1}{2}(u - v) \end{cases}$ であるから,

$$J = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$\begin{aligned}
\iint_D \left\{ (x-y)^2 + (x+y)^2 \right\} dx\, dy &= \int_{-1}^{1}\int_{-1}^{1} (u^2 + v^2) \cdot \frac{1}{2}\, du\, dv \\
&= 4\int_0^1\int_0^1 (u^2 + v^2) \cdot \frac{1}{2}\, du\, dv \\
&= 2\int_0^1\int_0^1 (u^2 + v^2)\, du\, dv \\
&= 2\int_0^1 \left[\frac{u^3}{3} + v^2 u \right]_0^1 dv \\
&= 2\int_0^1 \left(\frac{1}{3} + v^2 \right) dv \\
&= 2\left[\frac{v}{3} + \frac{v^3}{3} \right]_0^1 = \frac{4}{3}
\end{aligned}$$

7　2π