５章　微分方程式

２節　１階微分方程式（p.164～171）

練習１　C は任意定数

⑴　$yy' = x$

$$\int y\,dx = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$\therefore\ y^2 = x^2 + C$$

⑵　$xy' = y$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\log|\,y\,| = \log|\,x\,| + C$$

$$\log\left|\frac{y}{x}\right| = C$$

$$\frac{y}{x} = \pm e^c = C \qquad \therefore\ y = Cx$$

⑶　$\sqrt{1 - x^2}\,y' = 1$

$$\int dy = \int \frac{1}{\sqrt{1 - x^2}}\,dx$$

$$\therefore\ y = \sin^{-1} x + C$$

⑷　$y' \cos y - \sin x = 0$

$$\int \cos y\,dy = \int \sin x\,dx$$

$$\sin y = -\cos x + C$$

練習２

⑴　$yy' = \pm\sqrt{1 - y^2}$　$(x = 0,\ \ y = 1)$

$$\int \frac{y}{\pm\sqrt{1 - y^2}}\,dy = \int dx$$

$$\mp\sqrt{1 - y^2} = x + C$$

$$1 - y^2 = (x + C)^2$$

$x = 0,\ \ y = 1$ を代入して　$0 = C^2$　$\therefore\ C = 0$　$\therefore\ x^2 + y^2 = 1$

⑵　$(1 + x)y' + (1 + y) = 0$　$(x = 0,\ \ y = 0)$

$$\int \frac{1}{1 + y}\,dy = -\int \frac{1}{1 + x}\,dx$$

$$\log|1 + y| = -\log|1 + x| + C$$

$$\log|(1 + x)(1 + y)| = C$$

$$(1 + x)(1 + y) = \pm e^C = C$$

$x = 0,\ \ y = 0$ を代入して　$C = 1$　$\therefore\ (1 + x)(1 + y) = 1$

⑶　$y' - y\cot x = 0 \quad \left(x = \dfrac{\pi}{2},\ \ y = 1\right)$

$$\int \frac{1}{y}dy = \int \cot x\,dx$$

$$\log|y| = \log|\sin x| + C$$

$$\log\left|\frac{y}{\sin x}\right| = C$$

$$\frac{y}{\sin x} = \pm e^{c} = C$$

$$y = C\sin x$$

$x = \dfrac{\pi}{2}$，　$y = 1$　を代入して　$1 = C$　　∴　$y = \sin x$

練習 3　C は任意定数

⑴　$xyy' = x^2 + y^2$

$$y' = \frac{x^2 + y^2}{xy} = \frac{1 + \left(\dfrac{y}{x}\right)^2}{\dfrac{y}{x}}$$

$\dfrac{y}{x} = u$　とおくと　$y' = u + xu'$　　∴　$u + xu' = \dfrac{1 + u^2}{u}$

$$xu' = \frac{1}{u}$$

$$\int u\,du = \int \frac{1}{x}dx$$

$$\frac{1}{2}u^2 = \log|x| + C$$

$\dfrac{1}{2}\left(\dfrac{y}{x}\right)^2 = \log|x| + C$　∴　$y^2 = x^2(2\log|x| + C)$

⑵ $xy^2y' = x^3 + y^3$

$$y' = \frac{x^3 + y^3}{xy^2} = \frac{1+\left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$$

$\frac{y}{x} = u$ とおくと　$u + xu' = \frac{1+u^3}{u^2}$

$$xu' = \frac{1}{u^2}$$

$$\int u^2\,du = \int \frac{1}{x}\,dx$$

$$\frac{1}{3}u^3 = \log|x| + C$$

$$\left(\frac{y}{x}\right)^3 = 3\log|x| + C$$

$$\therefore\ y^3 = x^3(3\log|x| + C)$$

⑶ $xy' = -x\tan\frac{y}{x} + y$

$$y' = -\tan\frac{y}{x} + \frac{y}{x}$$

$\frac{y}{x} = u$ とおくと　$y' = u + xu'$　より

$$u + xu' = -\tan u + u$$

$$xu' = -\tan u \quad \cot u\frac{du}{dx} = -\frac{1}{x}$$

$$\int \frac{\cos u}{\sin u}\,du = -\int \frac{1}{x}\,dx$$

$$\log|\sin u| = -\log|x| + C$$

$$\log|x\sin u| = C$$

$$x\sin u = \pm e^c = C$$

$$\therefore\ x\sin\frac{y}{x} = C$$

⑷　$(3x^2+y^2)y'-2xy=0$

$$y'=\frac{2xy}{3x^2+y^2}=\frac{2\dfrac{y}{x}}{3+\left(\dfrac{y}{x}\right)^2}$$

$\dfrac{y}{x}=u$ とおくと　$y'=u+xu'$　より

$$u+xu'=\frac{2u}{3+u^2}$$

$$xu'=\frac{-u-u^3}{3+u^2} \qquad \frac{3+u^2}{u^3+u}\cdot\frac{du}{dx}=-\frac{1}{x}$$

$$\int\frac{3+u^2}{u^3+u}du=-\int\frac{1}{x}dx$$

$$\int\left(\frac{3}{u}-\frac{2u}{u^2+1}\right)du=-\log|x|+C$$

$$3\log|u|-\log|u^2+1|=-\log|x|+C$$

$$\log\left|\frac{u^3x}{u^2+1}\right|=C$$

$$\therefore\ \frac{u^3x}{u^2+1}=\pm e^C=C$$

$$u^3x=C(u^2+1)$$

$$\left(\frac{y}{x}\right)^3x=C\left(\left(\frac{y}{x}\right)^2+1\right)$$

$$\frac{y^3}{x^2}=C\left(\frac{y^2+x^2}{x^2}\right)$$

$$y^3=C(y^2+x^2)$$

ここで $C=\dfrac{1}{C}$ とおけば　$x^2+y^2=Cy^3$

練習 4

(1)　$(y-x)y' = y + x$　$(x=1,\ y=1)$

$$y' = \frac{y+x}{y-x} = \frac{\frac{y}{x}+1}{\frac{y}{x}-1}$$

$\dfrac{y}{x} = u$　とおくと　$u + xu' = \dfrac{u+1}{u-1}$

$$xu' = \frac{-u^2+2u+1}{u-1}$$

$$\int \frac{u-1}{-u^2+2u+1}\,du = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\log|-u^2+2u+1| = \log|x| + C$$

$$\log|-u^2+2u+1| = -2\log|x| + C$$

$$\log|x^2(-u^2+2u+1)| = C$$

$$x^2(-u^2+2u+1) = \pm e^c = C$$

$$x^2\left(-\frac{y^2}{x^2}+\frac{2y}{x}+1\right) = C$$

$$-y^2+2xy+x^2 = C$$

$x=1,\ y=1$　を代入して

$-1+2+1 = C$　より　$C=2$

$\therefore\ x^2+2xy-y^2 = 2$

(2)　$xy' = x + y$　$(x=1,\ y=2)$

$$y' = 1 + \frac{y}{x}$$

$\dfrac{y}{x} = u$　とおくと　$u + xu' = 1 + u$

$$xu' = 1$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\int du = \int \frac{1}{x}\,dx$$

$$u = \log|x| + C$$

$$\frac{y}{x} = \log|x| + C$$

$$y = x(\log|x| + C)$$

$x=1,\ y=2$　を代入して　$2 = C$

$\therefore\ y = x(\log|x|+2)$

(3)　$(xy - x^2)y' = y^2$

$$y' = \frac{y^2}{xy-x^2} = \frac{\frac{y^2}{x^2}}{\frac{y}{x}-1}$$

$\dfrac{y}{x} = u$　とおくと　$u + xu' = \dfrac{u^2}{u-1}$

$$xu' = \frac{u}{u-1}$$

$$\int \frac{u-1}{u}\,du = \int \frac{1}{x}\,dx$$

$$u - \log|u| = \log|x| + C$$

$$\log|xu| = u - C$$

$$xu = \pm e^{u-C} = Ce^u$$

$x\dfrac{y}{x} = Ce^{\frac{y}{x}}$　　$\therefore\ y = Ce^{\frac{y}{x}}$

$x=1,\ y=1$　を代入して　$C = e^{-1}$

$\therefore\ y = e^{\frac{y}{x}-1}$

練習 5　C は任意定数

(1)　$y' + \frac{1}{x} y = 1$

$y' + \frac{1}{x} y = 0$ を変形して　$y' = -\frac{1}{x} y$

$\int \frac{1}{y} dy = -\int \frac{1}{x} dx$

$\log|y| = -\log|x| + C$

$\log|xy| = C$

$xy = \pm e^C = C$

$y = \frac{C}{x}$

C を x の関数 u とみなして　$y = \frac{u(x)}{x}$　とおく

$y' = \frac{u'}{x} - \frac{u}{x^2}$　　$\therefore\ \frac{u'}{x} - \frac{u}{x^2} + \frac{u}{x^2} = 1$

$\frac{u'}{x} = 1$

$u' = x$

$u = \frac{1}{2}x^2 + C$

$\therefore\ y = \frac{1}{x}\left(\frac{1}{2}x^2 + C\right) = \frac{1}{2}x + \frac{C}{x}$

(2)　$y' + y = e^x$

$y' + y = 0$

$y' = -y$

$\int \frac{1}{y} dy = -\int dx$

$\log|y| = -x + C$

$y = \pm e^{-x+C} = Ce^{-x}$

C を x の関数 u として　$y = ue^{-x}$

$y' = u'e^{-x} - ue^{-x}$

$u'e^{-x} - ue^{-x} + ue^{-x} = e^x$

$u'e^{-x} = e^x$

$u' = e^{2x}$

$u = \frac{1}{2}e^{2x} + C$

$\therefore\ y = \left(\frac{1}{2}e^{2x} + C\right)e^{-x} = \frac{1}{2}e^x + Ce^{-x}$

(3)　$y' \cos x + y \sin x = 1$

$y' \cos x + y \sin x = 0$

$\int \frac{1}{y} dy = -\int \frac{\sin x}{\cos x} dx$

$\log|y| = \log|\cos x| + C$

$\log\left|\frac{y}{\cos x}\right| = C$

$y = C \cdot \cos x$

C を x の関数 u として　$y = u \cos x$

$y' = u' \cos x - u \sin x$

$(u' \cos x - u \sin x)\cos x + u \cos x \sin x = 1$

$u' = \frac{1}{\cos^2 x}$

$\therefore u = \tan x + C$　$\therefore y = (\tan x + C)\cos x$

$= \sin x + C \cos x$

(4)　$xy' + y = \log x$

$xy' + y = 0$

$\int \frac{1}{y} dy = -\int \frac{1}{x} dx$

$\log|y| = -\log|x| + C$

$\log|xy| = C$

$xy = \pm e^c = C$

$y = \frac{C}{x}$

C を x の関数 u として　$y = \frac{u}{x}$

$y' = -\frac{u}{x^2} + \frac{u'}{x}$

$x\left(-\frac{u}{x^2} + \frac{u'}{x}\right) + \frac{u}{x} = \log x$

$u' = \log x$

$u = x \log x - x + C$

$\therefore y = \frac{1}{x}(x \log x - x + C)$

$= \log x - 1 + \frac{C}{x}$

練習 6　$y' = x + y$　（$x = 0$,　$y = 0$）

$y' - y = 0$

$$\int \frac{1}{y} dy = \int dx$$

$\log|y| = x + C$

$y = \pm e^{x+C} = C \cdot e^{x}$

Cをxの関数uとおく　$y = ue^{x}$

$y' = u'e^{x} + ue^{x}$

$u'e^{x} + ue^{x} = x + ue^{x}$

$u'e^{x} = x$

$u' = x \cdot e^{-x}$　　$\therefore$　$u = -xe^{-x} - e^{-x} + C$

$\therefore$　$y = (-xe^{-x} - e^{-x} + C)e^{x}$

$= -x - 1 + Ce^{x}$

$x = 0$,　$y = 0$ を代入して　$C = 1$

$\therefore$　$y = e^{x} - x - 1$

練習 7　C は任意定数

⑴　$\dfrac{dv}{dt} + kv = g$

同次方程式　$\dfrac{dv}{dt} + kv = 0$

$$\frac{dv}{dt} = -kv$$

$$\int \frac{1}{v}\,dv = -\int k\,dt$$

$$\log\left|\, v \,\right| = -kt + C_1 \quad （C_1 \text{は任意定数}）$$

$$v = \pm e^{-kt+C_1} = \pm e^{C_1}e^{-kt} = Ce^{-kt}$$

C を t の関数 u とおいて

$$v = ue^{-kt}$$

$\dfrac{dv}{dt} = \dfrac{du}{dt}e^{-kt} - uke^{-kt}$　を元の微分方程式に代入して

$$\frac{du}{dt}e^{-kt} - uke^{-kt} + kue^{-kt} = g$$

$$\frac{du}{dt} = ge^{kt}$$

$$u = \frac{g}{k}e^{kt} + C$$

したがって　$v = \left(\dfrac{g}{k}e^{kt} + C\right)e^{-kt}$

$$= \frac{g}{k} + Ce^{-kt} \quad （C\text{は任意定数}）$$

⑵　$\dfrac{dv}{dt} = g - kv$

$$\int \frac{1}{g-kv}\,dv = \int dt$$

$$-\frac{1}{k}\log\left|\, g-kv \,\right| = t + C$$

$$g - kv = \pm e^{-kt+C} = Ce^{-kt}$$

$$-kv = -g + Ce^{-kt}$$

$$v = \frac{g}{k} + Ce^{-kt} \quad （C\text{は任意定数}）$$

節末問題（p.172）

1　C は任意定数

⑴　$yy' + 4x = 0$

$$\int y\,dy = -\int 4x\,dx$$

$$\frac{1}{2}y^2 = -2x^2 + C$$

$$\therefore\ 4x^2 + y^2 = C$$

⑵　$xy' = y\log x$

$$\int \frac{1}{y}dy = \int \frac{\log x}{x}dx$$

$$\log|\ y\ | = \frac{1}{2}(\log x)^2 + C$$

$$\therefore\ y = Ce^{\frac{(\log x)^2}{2}}$$

⑶　$(x - y)y' = 2y$

$$y' = \frac{2y}{x-y} = \frac{2\frac{y}{x}}{1-\frac{y}{x}}$$

$\dfrac{y}{x} = u$　とおくと　$y = xu$　$y' = u + xu'$

$$u + xu' = \frac{2u}{1-u}$$

$$xu' = \frac{u+u^2}{1-u}$$

$$\int \frac{1-u}{u+u^2}du = \int \frac{1}{x}dx$$

$$\int \left(\frac{1}{u} - \frac{2}{u+1}\right)du = \log|\ x\ | + C$$

$$\log|\ u\ | - 2\log|\ u+1\ | = \log|\ x\ | + C$$

$$\therefore \frac{u}{(u+1)^2 x} = C$$

したがって　$\dfrac{y}{(y+x)^2} = C$

⑷　$y' = \dfrac{y}{x} + \dfrac{1}{\cos\frac{y}{x}}$

$\dfrac{y}{x} = u$　とおく

$$u + xu' = u + \frac{1}{\cos u}$$

$$xu' = \frac{1}{\cos u}$$

$$\int \cos u\,du = \int \frac{1}{x}dx + \sin u = \log|\ x\ | + C$$

$$\therefore\ \sin\frac{y}{x} - \log|\ x\ | = C$$

(5) $xy' + 3y + x = 0$

$xy' + 3y = 0$

$xy' = -3y$

$\int \frac{1}{y}\,dy = \int \frac{-3}{x}\,dx$

$\log|\ y\ | = -3\log|\ x\ | + C$

$\log|\ yx^3\ | = C$

$yx^3 = \pm e^c = C$

$y = \frac{C}{x^3}$

Cをxの関数uとすると　$y = \frac{u}{x^3}$

$y' = \frac{u'}{x^3} - \frac{3u}{x^4}$

$x \cdot \left(\frac{u'}{x^3} - \frac{3u}{x^4} \right) + 3 \cdot \frac{u}{x^3} + x = 0$

$\frac{u'}{x^2} + x = 0$

$u' = -x^3$

$\therefore\ u = -\frac{1}{4}x^4 + C$

$\therefore\ y = \frac{1}{x^3}\left(-\frac{1}{4}x^4 + C \right)$

$= -\frac{1}{4}x + \frac{C}{x^3}$

(6) $y' - y = 2\sin 2x$

$y' - y = 0$

$y' = y$

$\int \frac{1}{y}\,dy = \int dx$

$\log|\ y\ | = x + C$

$y = Ce^x$

Cをxの関数uとして　$y = ue^x$　$y' = u'e^x + ue^x$

$u'e^x + ue^x - ue^x = 2\sin 2x$

$u' = 2e^{-x}\sin 2x$

$u = -\frac{1}{5}e^{-x}(2\sin 2x + 4\cos 2x) + C$

したがって　$y = -\frac{1}{5}(2\sin 2x + 4\cos 2x) + Ce^x$

2　C は任意定数

(1)　$y' = e^{2x-y}$　$(x = 0,\ y = 0)$

$$\int e^y\, dy = \int e^{2x}\, dx$$

$$e^y = \frac{1}{2}e^{2x} + C$$

$x = 0,\ y = 0$ を代入して

$1 = \frac{1}{2} + C$ より $C = \frac{1}{2}$

$$\therefore e^y = \frac{1}{2}e^{2x} + \frac{1}{2}$$

(2)　$xy' = y(\log y - \log x + 1)$　$(x = 1,\ y = e^2)$

$$y' = \frac{y}{x}\left(\log\frac{y}{x} + 1\right)$$

$\frac{y}{x} = u$ とおくと $y = xu,\ y' = u + xu'$

$$u + xu' = u(\log u + 1)$$

$$xu' = u\log u$$

$$\int \frac{1}{u\log u}\, du = \int \frac{1}{x}\, dx$$

$$\log\left|\ \log u\ \right| = \log\left|\ x\ \right| + C$$

$$\log\left|\ \frac{\log u}{x}\ \right| = C$$

$$\frac{\log u}{x} = \pm e^C = C$$

$$\log u = Cx$$

$\log\frac{y}{x} = Cx$　$\therefore$　$y = xe^{Cx}$

$x = 1,\ y = e^2$ を代入して $e^2 = e^C$

$\therefore C = 2$　　$\therefore$　$y = xe^{2x}$

(3) $y' + y = \sin x \quad \left(x = \dfrac{\pi}{2},\ \ y = 0\right)$

$y' + y = 0$

$y' = -y$

$\displaystyle\int \frac{1}{y}\,dy = -\int dx$

$\log|\ y\ | = -x + C$

$y = \pm e^{-x+C} = \pm e^{C} e^{-x} = Ce^{-x}$

C を x の関数 u として　$y = ue^{-x},\ \ y' = u'e^{-x} - ue^{-x}$　より

$u'e^{-x} - ue^{-x} + ue^{-x} = \sin x$

$u'e^{-x} = \sin x$

$u' = e^{x} \sin x$

$\therefore\ u = \displaystyle\int e^{x} \sin x\, dx + C$

$= \dfrac{1}{2} e^{x}(\sin x - \cos x) + C$

したがって　$y = \dfrac{1}{2}(\sin x - \cos x) + Ce^{-x}$

$x = \dfrac{\pi}{2},\ \ y = 0$　を代入して

$0 = \dfrac{1}{2} + Ce^{-\frac{\pi}{2}}$　より　$c = -\dfrac{1}{2} e^{\frac{\pi}{2}}$

$\therefore\ y = \dfrac{1}{2}(\sin x - \cos x - e^{\frac{\pi}{2} - x})$

3　法線の方程式は，$Y - y = -\dfrac{1}{y'}(X - x)$

つねに原点を通るので　$X = 0,\ \ Y = 0$　より

$-y = -\dfrac{1}{y'}(-x)$

$\therefore\ \ yy' = -x$

$\displaystyle\int y\, dy = -\int x\, dx$

$\dfrac{1}{2} y^2 = -\dfrac{1}{2} x^2 + C$

$\therefore\ \ x^2 + y^2 = C$

$x = 1,\ \ y = 1$　を代入して　$C = 2$

$\therefore\ \ x^2 + y^2 = 2$

4　接線の方程式は　$Y-y=y'(X-x)$

y 軸との交点の y 座標は　$X=0$　のときの Y

すなわち　$Y=y-xy'$

題意より　$y=-2xy'$

$$\int \frac{1}{y}dy=-\int \frac{1}{2x}dx$$

$$\log|y|=-\frac{1}{2}\log|x|+C$$

$$\log|y\sqrt{x}|=C$$

$$y\sqrt{x}=\pm e^{c}=C$$

$$y=\frac{C}{\sqrt{x}}$$

$x=1,\ y=2$ を代入して $2=\dfrac{C}{\sqrt{1}}$ より $C=2$

$$\therefore\quad y=\frac{2}{\sqrt{x}}$$

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⑴　$u=x+y$ とおくと $u'=1+y'$ より $y'=u'-1$

$\therefore\ u(u'-1)=1$

$uu'-u=1$

⑵　$uu'-u=1$

$$uu'=1+u$$

$$u'=\frac{1+u}{u}$$

$$\int \frac{u}{1+u}du=\int dx$$

$1+u=t$ とおくと $du=dt$

$$\int \frac{t-1}{t}dt=\int dx$$

$$t-\log|t|=x+C$$

$$1+u-\log(1+u)=x+C$$

$$1+x+y-\log|1+x+y|=x+C$$

$\therefore\ y=\log|1+x+y|+C$　(Cは任意定数)

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⑴ $\dfrac{dN(t)}{dt} = k\,(a - N(t))$

⑵ $\dfrac{dN(t)}{dt} + kN(t) = ka$

$\dfrac{dN(t)}{dt} + kN(t) = 0$ を変形して $\dfrac{dN(t)}{dt} = -kN(t)$

$\int \dfrac{1}{N(t)}\,dN = -\int k\,dt$

$\log\left|\,N(t)\,\right| = -kt + C$

$N(t) = \pm e^{-kt+C} = C \cdot e^{-kt}$

C を t の関数 u として $N(t) = u \cdot e^{-kt}$

$\dfrac{dN(t)}{dt} = u'e^{-kt} - kue^{-kt}$

$\left(\dfrac{du}{dt} e^{-kt} - ku\,e^{-kt}\right) + k \cdot u \cdot e^{-kt} = ka$

$\dfrac{du}{dt} e^{-kt} = ka$

$\dfrac{du}{dt} = k \cdot a \cdot e^{kt}$

$\therefore\ u = ae^{kt} + C$

したがって $N(t) = (a \cdot e^{kt} + C) \cdot e^{-kt}$

$= a + Ce^{-kt}$（Cは任意定数）

⑶ $N(0) = N_0$ のとき

$N_0 = a + c$ より $c = N_0 - a$

$\therefore\ N(t) = a + (N_0 - a)e^{-kt}$

演習 1　$(x^2+y+3)\,dx+(x-y^2+1)\,dy=0$

$P(x,\ y)=x^2+y+3$，$Q(x,\ y)=x-y^2+1$ とおくと

$\dfrac{\partial}{\partial y}P(x,\ y)=1$

$\dfrac{\partial}{\partial x}Q(x,\ y)=1$　より

$\dfrac{\partial P(x,\ y)}{\partial y}=\dfrac{\partial Q(x,\ y)}{\partial x}$ となるので完全微分方程式である。

$$F(x,\ y)=\int(x^2+y+3)\,dx+\int\left(x-y^2+1-\frac{\partial}{\partial y}\int(x^2+y+3)\,dx\right)dy$$

$$=\frac{1}{3}x^3+xy+3x+\int\left(x-y^2+1-\frac{\partial}{\partial y}\left(\frac{1}{3}x^3+xy+3x\right)\right)dy$$

$$=\frac{1}{3}x^3+xy+3x+\int(x-y^2+1-x)\,dy$$

$=\dfrac{1}{3}x^3+xy+3x-\dfrac{1}{3}y^3+y=C$（$C$は任意定数）$3C=D$とおくと

$x^3-y^3+3xy+9x+3y=D$（Dは任意定数）

演習 2　$xy'-2y=x^2y^3$　は　$y'-\dfrac{2}{x}=xy^3$　㋐なので，ベルヌーイの微分方程式のタイプである。

$z=y^{-2}$　とおくと　$z'=-2y^{-3}y'$　より　$y'=\dfrac{1}{-2}y^3z'$。よって㋐は $\dfrac{1}{-2}y^2z'-\dfrac{2}{x}y=xy^3$ となり

$z'+\dfrac{4}{x}y^{-2}=2x$　つまり　$z'+\dfrac{4}{x}z=-2x$　…㋑であるから線形微分方程式である。

(i)　$z'+\dfrac{4}{x}z=0$　の一般解を求める。

これより　$z'=-\dfrac{4}{x}z$，$\dfrac{1}{z}\dfrac{dz}{dx}=-\dfrac{4}{x}$

両辺を x で積分して

$\displaystyle\int\frac{1}{z}dz=-\int\frac{4}{x}dx$

$\log|\ z\ |=-4\log|\ x\ |+C$

$\log|\ x^4z\ |=C$

$x^4z=\pm e^c=D$ とおくと

$z=\dfrac{D}{x^4}$

(ii)　D を x の関数 u とおくと　$z = \dfrac{u}{x^4} = ux^{-4}$ より

$$z' = \frac{u'}{x^4} - \frac{4u}{x^5}$$

㋑に代入して

$$\frac{u'}{x^4} - \frac{4u}{x^5} + \frac{4}{x}\left(\frac{u}{x^4}\right) = -2x$$

$$\frac{u'}{x^4} = -2x$$

$$u' = -2x^5$$

$$\therefore\ u = -\frac{1}{3}x^6 + C$$

$$\therefore\ z = \frac{1}{x^4}\left(-\frac{1}{3}x^6 + C\right) = -\frac{1}{3}x^2 + \frac{C}{x^4}$$

したがって　$\dfrac{1}{y^2} = -\dfrac{1}{3}x^2 + \dfrac{C}{x^4}$　（Cは任意定数）

演習 3　$y = xy' + (y')^3$

$y' = p$ とおくと $y = xp + p^3$　…①

両辺を x で微分して

$y' = p = p + xp' + 3p^2p'$ より $(x + 3p^2)p' = 0$

したがって

$$\begin{cases} x + 3p^2 = 0 \\ p' = 0 \end{cases}$$

が成り立つ。

$p' = 0$ のとき $p = C$ （Cは任意定数）

したがって（①に代入して） 一般解は $y = Cx + C^3$

$x + 3p^2 = 0$ のとき

$$\begin{cases} x + 3p^2 = 0 \\ y = xp + p^3 \end{cases}$$

より p を消去すると

$x + 3p^2 = 0$ より $p^2 = -\dfrac{x}{3}$　　$p = \pm\sqrt{-\dfrac{x}{3}}$

$$y = p(x + p^2) = \pm\sqrt{-\frac{x}{3}}\left(x - \frac{x}{3}\right) = \pm\sqrt{-\frac{x}{3}} \cdot \frac{2}{3}x$$

したがって特異解は

$$y^2 = -\frac{x}{3} \cdot \frac{4}{9}x^2$$

$$27y^2 = -4x^3$$

演習 4　$y = 2xy' + (y')^2$

$y' = p$ とおくと　$y = 2xp + p^2$　…①

両辺を x で微分すると　$p = 2p + 2xp' + 2pp'$

p を独立変数，x を従属変数とみなすと

$$\frac{dx}{dp} + \frac{2x}{p} + \frac{2p}{p} = 0 \qquad \frac{dx}{dp} + \frac{2x}{p} + 2 = 0 \quad \cdots ②$$

同次方程式　$\dfrac{dx}{dp} + \dfrac{2}{p}x = 0$　を考えて

$$\int \frac{1}{x}\,dx = -\int \frac{2}{p}\,dp$$

$$\log|x| = -2\log|p| + C$$

$$\log|xp^2| = C$$

$$xp^2 = \pm e^c = C \qquad x = \frac{C}{p^2}$$

C を p の関数 u とおくと　$x = \dfrac{u}{p^2}$，$x' = \dfrac{u'}{p^2} - \dfrac{2u}{p^3}$　を②に代入して

$$\frac{u'}{p^2} - \frac{2u}{p^3} + \frac{2}{p}\frac{u}{p^2} + 2 = 0$$

$$u' = -2p^2$$

$$\therefore\ u = -\frac{2}{3}p^3 + C$$

$$\therefore\ x = \frac{1}{p^2}\left(-\frac{2}{3}p^3 + C\right) = -\frac{2}{3}p + \frac{C}{p^2}$$

$$3xp^2 = -2p^3 + 3C$$

$$p^2(3x + 2p) = C \quad \cdots ③$$

①，③から p を消去する

①より　$p^2 + 2xp - y = 0$　　$\therefore$　$p = -x \pm \sqrt{x^2 + y^2}$

これを③　$p^2(3x + 2p) = C$　に代入すると

$$\left(-x \pm \sqrt{x^2 + y^2}\right)^2 \left(3x + 2\left(-x \pm \sqrt{x^2 + y^2}\right)\right) = C$$

$$\left(2x^2 \mp 2x\sqrt{x^2 + y} + y\right)\left(x \pm 2\sqrt{x^2 + y}\right) = C$$ （複号同順）

（C は任意定数）