３章　積分法　解答

１－１　不定積分

A 問題

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⑴ $\int x\,dx = \frac{1}{2}x^2 + C$

⑵ $\int x^4\,dx = \frac{1}{5}x^5 + C$

⑶ $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx - \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + C = \frac{2}{3}x\sqrt{x} + C$

⑷ $\int t^3\sqrt{t}\,dt = \int t^{\frac{4}{3}}\,dt = \frac{1}{\frac{4}{3}+1}t^{\frac{4}{3}+1} + C = \frac{3}{7}t^2\sqrt[3]{t} + C$

⑸ $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{1}{-3+1}x^{-3+1} + C = \frac{1}{2x^2} + C$

⑹ $\int \frac{1}{\sqrt[3]{x}}\,dx = \int x^{-\frac{1}{3}}\,dx = \frac{1}{-\frac{1}{3}+1}x^{-\frac{1}{3}+1} + C = \frac{3}{2}\sqrt[3]{x^2} + C$

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⑴ $\int dx = x + C$

⑵ $\int (-6x+5)\,dx = -6\cdot\frac{1}{2}x^2 + 5x + C$

$= -3x^2 + 5x + C$

⑶ $\int (3x^2 - 4x + 1)\,dx = 3\cdot\frac{1}{3}x^3 - 4\cdot\frac{1}{2}x^2 + x + C$

$= x^3 - 2x^2 + x + C$

⑷ $\int x(x+3)\,dx = \int (x^2 + 3x)\,dx$

$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + C$

⑸ $\int (x-2)(2x-3)\,dx = \int (2x^2 - 7x + 6)\,dx$

$= \frac{2}{3}x^3 - \frac{7}{2}x^2 + 6x + C$

⑹ $\int (y-2)^2\,dy = \int (y^2 - 4y + 4)\,dy$

$= \frac{1}{3}y^3 - 2y^2 + 4y + C$

⑺ $\int (t+a)(t-a)\,dt = \int (t^2 - a^2)\,dt$

$= \frac{1}{3}t^3 - a^2t + C$

⑻ $\int (x+1)^2\,dx - \int (x-1)^2\,dx$

$= \int \left\{(x+1)^2 - (x-1)^2\right\}dx$

$= \int 4x\,dx = 2x^2 + C$

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(1) $\displaystyle\int \frac{x^3+1}{x^2}\,dx = \int\left(x+\frac{1}{x^2}\right)dx = \frac{1}{2}x^2 - \frac{1}{x} + C$

(2) $\displaystyle\int \frac{(x+2)^2}{x}\,dx = \int\left(x+4+\frac{4}{x}\right)dx = \frac{1}{2}x^2 + 4x + 4\log(x) + C$

(3) $\displaystyle\int \frac{(x-1)^2}{\sqrt{x}}\,dx = \int\left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx = \frac{2}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

$\displaystyle = \frac{2}{5}x^2\sqrt{x} - \frac{4}{3}x\sqrt{x} + 2\sqrt{x} + C$

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(1) $\displaystyle\int\left(\sin x + \frac{3}{\cos^2 x}\right)dx = -\cos x + 3\tan x + C$

(2) $\displaystyle\int\left(2+\frac{1}{\tan x}\right)\sin x\,dx = \int(2\sin x + \cos x)\,dx = -2\cos x + \sin x + C$

(3) $\displaystyle\int e^{x-1}\,dx = e^{x-1} + C$

(4) $\displaystyle\int\left(3^x\log 3 - 1\right) = \frac{3x}{\log 3}\cdot\log 3 - x + C = 3^x - x + C$

(5) $\displaystyle\int 5^{1-x}\,dx = \frac{5^{1-x}}{\log 5} + C$

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(1) $\displaystyle\int \sin\frac{1}{3}x\,dx = -3\cos\frac{1}{3}x + C$

(2) $\displaystyle\int \cos\pi\,dx = \frac{1}{\pi}\sin\pi x + C$

(3) $\displaystyle\int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C$

(4) $\displaystyle\int e^{\frac{x}{2}}\,dx = 2e^{\frac{x}{2}} + C$

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(1) $\displaystyle\int \frac{7}{3-5x}\,dx$

$3-5x=t$ とおくと $\displaystyle x=\frac{3-t}{5}$

$\displaystyle\frac{dx}{dt} = -\frac{1}{5}$ より $\displaystyle dx = \frac{1}{5}dt$

与式 $\displaystyle = \int\frac{1}{t}\left(-\frac{1}{5}\right)dt = -\frac{1}{5}\log|t| + C$

$\displaystyle = -\frac{1}{5}\log|3-5x| + C$

(2) $\int x(2x-1)^{3}\,dx$

$2x-1=t$ とおくと $x=\dfrac{t+1}{2}$, $dx=\dfrac{1}{2}dt$

$$与式=\int\frac{t+1}{2}\cdot t^{3}\cdot\frac{1}{2}dt=\frac{1}{4}\int\left(t^{4}+t^{3}\right)dt$$

$$=\frac{1}{4}\left(\frac{1}{5}t^{5}+\frac{1}{4}t^{4}\right)+C$$

$$=\frac{1}{20}t^{5}+\frac{1}{16}t^{4}+C$$

$$=\frac{1}{80}t^{4}\left(4t+5\right)+C=\frac{1}{80}\left(2x-1\right)^{4}\left(8x+1\right)+C$$

(3) $\int\dfrac{x}{\sqrt{x-4}}\,dx$

$x-4=t$ とおくと $x=t+4$, $dx=dt$

$$与式=\int\frac{t+4}{\sqrt{t}}dt=\int\left(t^{\frac{1}{2}}+4t^{-\frac{1}{2}}\right)dt=\frac{2}{3}t^{\frac{3}{2}}+8t^{\frac{1}{2}}+C$$

$$=\frac{2}{3}\left(x-4\right)\sqrt{x-4}+8\sqrt{x-4}+C=\frac{2}{3}\left(x+8\right)\sqrt{x-4}$$

(4) $\int\dfrac{x}{\left(x+2\right)^{2}}\,dx$

$x+2=t$ とおくと $x=t-2$, $dx=dt$

$$与式=\int\frac{t-2}{t^{2}}dt=\int\left(\frac{1}{t}-\frac{2}{t^{2}}\right)dt=\log\left|t\right|+2t^{-1}+C$$

$$=\log\left|x+2\right|+\frac{2}{x+2}+C$$

(5) $\int\sqrt[3]{2x-5}\,dx$

$2x-5=t$ とおくと $2\dfrac{dx}{dt}=1$ より $dx=\dfrac{1}{2}dt$

$$与式=\int\sqrt[3]{t}\cdot\frac{1}{2}dt=\frac{1}{2}\int t^{\frac{1}{3}}\,dt=\frac{3}{8}t^{\frac{4}{3}}+C$$

$$=\frac{3}{8}\left(2x-5\right)\sqrt[3]{2x-5}+C$$

(6) $\int\dfrac{1}{\left(2+3x\right)^{3}}\,dx$

$2+3x=t$ とおくと $3\dfrac{dx}{dt}-1$ より $dx=\dfrac{1}{3}dt$

$$与式=\int\frac{1}{t^{3}}\cdot\frac{1}{3}dt=\frac{1}{3}\int t^{3}\,dt=-\frac{1}{6}t^{-2}+C$$

$$=-\frac{1}{6\left(2+3x\right)^{2}}+C$$

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(1) $\int \left(x^2-4x+1 \right)^2 \left(x-2 \right) dx$

$x^2-4x+1=t$ とおくと $\left(2x-4 \right) dx=dt$ より $\left(x-2 \right) dx=\dfrac{1}{2}dt$

与式 $=\int t^2 \cdot \dfrac{1}{2}dt=\dfrac{1}{6}t^3+C=\dfrac{1}{6}\left(x^2-4x+1 \right)^3+C$

(2) $\int \sin^4 \theta \cos\theta \, d\theta$

$\sin\theta=t$ とおくと $\cos\theta \, d\theta=dt$

与式 $=\int t^4 \, dt=\dfrac{1}{5}t^5+C=\dfrac{1}{5}\sin^5\theta+C$

(3) $\int x\sqrt{x^2+1}\, dx$

$x^2+1=t$ とおくと $2x\,dx=dt$ より $x\,dx=\dfrac{1}{2}dt$

与式 $=\int \sqrt{t}\cdot \dfrac{1}{2}dt=\dfrac{1}{3}t^{\frac{3}{2}}+C=\dfrac{1}{3}\left(x^2+1 \right)\sqrt{x^2+1}+C$

(4) $\int xe^{1-x^2}\, dx$

$1-x^2=t$ とおくと $-2x\,dx=dt$ より $x\,dx=-\dfrac{1}{2}dt$

与式 $=\int e^t \cdot \left(-\dfrac{1}{2}dt \right)=-\dfrac{1}{2}e^t+C=-\dfrac{1}{2}e^{1-x^2}+C$

(5) $\int \dfrac{\left(\log x \right)^2}{x}dx$

$\log x=t$ とおくと $\dfrac{1}{x}dx=dt$

与式 $=\int t^2\, dt=\dfrac{1}{3}t^2+C=\dfrac{1}{3}\left(\log x \right)^3+C$

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(1) $\int \dfrac{2x-1}{x^2-x}dx=\int \dfrac{\left(x^2-x \right)'}{x^2-x}dx=\log\left| x^2-x \right|+C$

(2) $\int \dfrac{e^x-e^{-x}}{e^x+e^{-x}}dx=\int \dfrac{\left(e^x+e^{-x} \right)'}{e^x+e^{-x}}dx=\log\left(e^x+e^{-x} \right)+C$

(3) $\int \dfrac{\cos x}{1+\sin x}dx=\int \dfrac{\left(1+\sin x \right)'}{1+\sin x}dx=\log\left(1+\sin x \right)+C$

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(1) $$\int (x+2)\sin x\,dx$$

$$= \int (x+2)(-\cos x)'\,dx = -(x+2)\cos x + \int \cos x\,dx$$

$$= -(x+2)\cos x + \sin x + C$$

(2) $$\int xe^{2x}\,dx = \int x\left(\frac{1}{2}e^{2x}\right)'dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

(3) $$\int x^2 \log x\,dx = \int\left(\frac{1}{3}x^3\right)'\log x\,dx$$

$$= \frac{1}{3}x^3\log x - \int\frac{1}{3}x^3\cdot\frac{1}{x}\,dx = \frac{1}{3}x^3\log x - \frac{1}{9}x^3 + C$$

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(1) $$\int x^2e^x\,dx = \int x^2\left(e^x\right)'dx = x^2e^x - \int 2xe^x\,dx$$

$$= x^2e^x - 2\int x\left(e^x\right)'dx$$

$$= x^2e^x - 2xe^x + 2\int e^x\,dx$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= \left(x^2 - 2x + 2\right)e^x + C$$

(2) $$\int x^2\cos x\,dx = \int x^2\left(\sin x\right)'dx = x^2\sin x - \int 2x\sin x\,dx$$

$$= x^2\sin x - 2\int x\left(-\cos x\right)'dx$$

$$= x^2\sin x + 2x\cos x - 2\int\cos x\,dx$$

$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

(3) $$\int\left(\log x\right)^2dx = \int x'\left(\log x\right)^2dx = x\left(\log x\right)^2 - \int x2\left(\log x\right)\cdot\frac{1}{x}\,dx$$

$$= x\left(\log x\right)^2 - 2\int\log x\,dx$$

$$= x\left(\log x\right)^2 - 2\int x'\log x\,dx$$

$$= x\left(\log x\right)^2 - 2x\log x + 2\int x\cdot\frac{1}{x}\,dx$$

$$= x\left(\log x\right)^2 - 2x\log x + 2x + C$$

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(1) $\displaystyle\int \frac{x^2}{x+1}\,dx = \int\left(x-1+\frac{1}{x+1}\right)dx = \frac{1}{2}x^2 - x + \log\left|x+1\right| + C$

(2) $\displaystyle\int \frac{1}{x(x-2)}\,dx = \frac{1}{2}\int\left(\frac{1}{x-2}-\frac{1}{x}\right)dx = \frac{1}{2}\left(\log\left|x-2\right| - \log\left|x\right|\right) + C$

$$= \frac{1}{2}\log\left|\frac{x-2}{x}\right| + C$$

(3) $\displaystyle\frac{3x+5}{(x-1)(x+3)} = \frac{9}{x-1} + \frac{b}{x+3}$　とおく。

両辺に $(x-1)(x+3)$ を掛けて

$3x+5 = a(x+3) + b(x-1) = (a+b)x + 3a - b$

$a+b=3$，$3a-b=5$ より $a=2$，$b=1$

与式 $\displaystyle = \int\left(\frac{2}{x-1}+\frac{1}{x+3}\right)dx = 2\log\left|x-1\right| + \log\left|x+3\right| + C$

$$= \log(x-1)^2\left|x+3\right| + C$$

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(1) $\displaystyle\int 2\sin^2 x\,dx = \int 2\cdot\frac{1-\cos 2x}{2}\,dx = \int(1-\cos 2x)\,dx$

$$= x - \frac{1}{2}\sin 2x + C$$

(2) $\displaystyle\int \sin 3x\cos x\,dx = \frac{1}{2}\int(\sin 4x + \sin 2x)\,dx = -\frac{1}{8}\cos 4x - \frac{1}{4}\cos x + C$

(3) $\displaystyle\int \cos x\cos 5x\,dx = \frac{1}{2}\int(\cos 6x + \cos 4x)\,dx = \frac{1}{2}\sin 6x + \frac{1}{8}\sin 4x + C$

(4) $\displaystyle\int(1+\cos x)^2\,dx = \int(1+2\cos x + \cos^2 x)\,dx$

$$= \int\left(1+2\cos x = \frac{1+\cos 2x}{2}\right)dx$$

$$= \frac{3}{2}x + 2\sin x + \frac{1}{4}\sin 2x + C$$

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(1) $\int x^3\sqrt{x+1}\,dx$

$x+1=t$ とおくと $dx=dt$

$$与式=\int (t-1)^3\sqrt{t}\,dt=\int\left(t^{\frac{1}{3}}-t^{\frac{1}{3}}\right)dt=\frac{3}{7}t^{\frac{7}{3}}-\frac{3}{4}t^{\frac{4}{3}}+C$$

$$=\frac{3}{7}t^2\sqrt[3]{t}-\frac{3}{4}t^3\sqrt{t}+C=\frac{3}{7}(x+1)^2\sqrt[3]{x+1}-\frac{3}{4}(x-1)\sqrt[3]{x+1}+C$$

(2) $\int \frac{1}{e^x-2}\,dx$

$e^x=t$ とおくと $\frac{dt}{dx}=e^x$ より $dx=\frac{dt}{e^t}=\frac{dt}{t}$

$$与式=\int\frac{1}{t-2}\cdot\frac{1}{t}\,dt=\frac{1}{2}\int\left(\frac{1}{t-2}-\frac{1}{t}\right)dt=\frac{1}{2}\left(\log|t-2|-\log|t|\right)+C$$

$$=\frac{1}{2}\log\left|\frac{t-2}{t}\right|+C=\frac{1}{2}\log\left|\frac{e^x-2}{e^x}\right|+C$$

(3) $\int (2x-1)e^{x^2-x+3}\,dx$

$x^2-x+3=t$ とおくと $\frac{dt}{dx}=2x-1$ より $(2x-1)dx=dt$

$$与式=\int e^t\,dt=e^t+C=e^{x^2-x+2}+C$$

(4) $\int \sin^5\theta\,d\theta=\int \sin^4\theta\sin\theta\,d\theta=\int\left(1-\cos^2\theta\right)^2\sin\theta\,d\theta$

$\cos\theta=t$ とおくと $\frac{dt}{d\theta}=-\sin\theta$ より $dt=-\sin\theta\,d\theta$

$$与式=\int\left(1-t^2\right)^2(-dt)=-\int\left(t^4-2t^2+1\right)dt$$

$$=-\frac{1}{5}t^5+\frac{2}{3}t^3-t+C=-\frac{1}{5}\cos^5\theta+\frac{2}{3}\cos^3\theta-\cos\theta+C$$

(5) $\int\frac{\cos^3 x}{\sqrt{\sin x}}\,dx-\int\frac{\left(1-\sin^2\theta\right)\cos\theta}{\sqrt{\sin\theta}}\,d\theta$

$\sin\theta=t$ とおくと $\frac{dt}{d\theta}=\cos\theta$ より $\cos\theta\,d\theta=dt$

$$与式=\int\frac{\left(1-t^2\right)}{\sqrt{t}}\,dt=\int\left(t^{\frac{1}{2}}-t^{\frac{3}{2}}\right)dt=2t^{\frac{1}{2}}-\frac{2}{5}t^{\frac{5}{2}}+C$$

$$=2\sqrt{\sin x}-\frac{2}{5}\sin^2 x\sqrt{\sin x}+C$$

(6)　$\displaystyle\int\frac{\log x}{x(\log x+1)}dx$

$\log x=t$ とおくと $\dfrac{dt}{dx}=\dfrac{1}{x}$ より $\dfrac{1}{x}dx=dt$

与式 $\displaystyle=\int\frac{t}{t+1}dt=\int\left(1-\frac{1}{t+1}\right)dt=t-\log|t+1|+C$

$=\log x-\log|\log x+1|+C$

B 問題

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(1)　$\sin x=t$ とおくと $\dfrac{dt}{dx}=\cos x$ より

$\cos x\,dx=dt$

$$\int\frac{\cos x}{4-\sin^2 x}dx=\int\frac{dt}{4-t^2}=\frac{1}{4}\int\left(\frac{1}{t+2}-\frac{1}{t-2}\right)dt$$

$$=\frac{1}{4}\{\log|t+2|-\log|t-2|\}+C$$

$$=\frac{1}{4}\log\left|\frac{t+2}{t-2}\right|+C$$

$$=\frac{1}{4}\log\left|\frac{\sin x+2}{\sin x-2}\right|+C$$

$$=\frac{1}{4}\log\left(\frac{2+\sin x}{2-\sin x}\right)+C$$

(2)　$\displaystyle\int\frac{1}{\cos x}dx=\int\frac{\cos x}{\cos^2 x}dx=\int\frac{\cos x}{1-\sin^2 x}dx$

$\sin x=t$ とおけば $\dfrac{dt}{dx}=\cos x$ より

$\cos x\,dx=dt$

$$\int\frac{1}{\cos x}dx=\int\frac{1}{1-t^2}dt=\frac{1}{2}\int\left(\frac{1}{t+1}-\frac{1}{t-1}\right)dt$$

$$=\frac{1}{2}(\log|t+1|-\log|t-1|)+C$$

$$=\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|+C$$

$$=\frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right)+C$$

(3)　$e^x+1=t$ とおくと $\dfrac{dt}{dx}=e^x$ より

$dx=\dfrac{1}{e^x}dt=\dfrac{1}{t-1}dt$

$$\int\frac{e^{3x}}{(e^x+1)^2}dx=\int\frac{(t-1)^3}{t^2(t-1)}dt$$

$$=\int\frac{(t^2-2t+1)(t-1)}{t^2(t-1)}dt=\int\left(1-\frac{2}{t}+\frac{1}{t^2}\right)dt$$

$$=t-2\log|t|-\frac{1}{t}+C$$

$$=e^x-2\log(e^x+1)-\frac{1}{e^x+1}+C$$

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(1) $$\int \frac{x}{\sqrt{2x+1}-\sqrt{x+1}}\,dx = \int \frac{x\left(\sqrt{2x+1}+\sqrt{x+1}\right)}{(2x+1)-(x+1)}\,dx$$

$$= \int \left(\sqrt{2x+1}+\sqrt{x+1}\right)dx = \frac{2}{3}\cdot\frac{1}{2}(2x+1)^{\frac{3}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(2x+1)\sqrt{2x+1} + \frac{2}{3}(x+1)\sqrt{x+1} + C$$

(2) $$\int \frac{2x^2+x-1}{x^3+1}\,dx = \int \frac{\cancel{(x+1)}(2x-1)}{\cancel{(x+1)}-(x^2-x+1)}\,dx$$

$$= \int \frac{(x^2-x+1)}{x^2-x+1}\,dx = \log(x^2-x+1)+C$$

(3) $$\int \left(\tan x + \frac{1}{\tan x}\right)^2 dx = \int \left(\tan^2 x + 2 + \frac{1}{\tan^2 x}\right)dx$$

$$= \int \left(1+\tan^2 x + 1 + \frac{1}{\tan^2 x}\right)dx = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)dx$$

$$= \tan x - \frac{1}{\tan x} + C$$

(4) $$\int \cos^4 x\,dx = \int (\cos^2 x)^2 dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \cos^2 2x\right)dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

別解 $$\int \cos^4 x\,dx = \int \cos^3 x \cos x\,dx = \int \cos^3 x (\sin x)'\,dx$$

$$= \cos^3 x \sin x - \int (\cos^3 x)' \sin x\,dx$$

$$= \cos^3 x \sin x - \int (-3\cos^2 x \sin x)\sin x\,dx$$

$$= \cos^3 x \sin x + 3\int (\sin x \cos x)^2\,dx$$

$$= \cos^3 x \sin x + \frac{3}{4}\int \sin^2 2x\,dx$$

$$= \cos^3 x \sin x + \frac{3}{4}\int \frac{1-\cos 4x}{2}\,dx$$

$$= \cos^3 x \sin x + \frac{3}{8}x - \frac{3}{32}\sin 4x + C$$ でもよい。

（参考）　別解の答えは次のように変形することによって
同じであることがわかる。

$$\frac{3}{8}x+\left(\sin x\cos x\right)\cos^2 x-\frac{3}{32}\sin 4x+C$$

$$=\frac{3}{8}x+\frac{1}{2}\sin 2x\cdot\frac{1+\cos 2x}{2}-\frac{3}{32}\sin 4x+C$$

$$=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{4}\sin 2x\cos 2x-\frac{3}{32}\sin 4x+C$$

$$=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{8}\sin 4x-\frac{3}{32}\sin 4x+C$$

$$=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

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$$\frac{x^2+4x-1}{\left(x^2+1\right)\left(x+2\right)}=\frac{ax+b}{x^2+1}+\frac{c}{x+2}$$

両辺に$\left(x^2+1\right)\left(x-2\right)$を掛けて

$$x^2+4x-1=\left(ax+b\right)\left(x+2\right)+c\left(x^2+1\right)$$

$$=\left(a+c\right)x^2+\left(2a+b\right)x+2b+c$$

$a+c=1$，$2a+b=4$，$2b+c=-1$ より
$a=2$，$b=0$，$c=-1$
このとき

$$与式=\int\frac{2x}{x^2+1}dx-\int\frac{1}{x+2}dx$$

$$=\int\frac{\left(x^2+1\right)}{x^2+1}dx-\int\frac{1}{x+2}dx$$

$$=\log\left(x^2+1\right)-\log\left(x+2\right)+C$$

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(1) $\int e^{-x}\sin x\,dx=\int\left(-e^{-x}\right)'\sin x\,dx=-e^{-x}\sin x+\int e^{-x}\cos x\,dx$

$=-e^{x}\sin x+\int\left(-e^{-x}\right)'\cos x\,dx$

$=-e^{-x}\sin x+-e^{-x}\cos x-\int e^{-x}\sin x\,dx$

$2\int e^{-x}\sin x\,dx=-e^{-x}\sin x-e^{-x}\cos x$

よって，$\int e^{-x}\sin x\,dx=-\frac{1}{2}e^{-x}\left(\sin x+\cos x\right)+C$

(2) $\int\log\left(x+\sqrt{x^2+1}\right)dx=\int(x)'\log\left(x+\sqrt{x^2+1}\right)dx$

$=x\log\left(x+\sqrt{x^2+1}\right)-\int x\left\{\log\left(x+\sqrt{x^2+1}\right)\right\}'dx$

ここで $\left\{\log\left(x+\sqrt{x^2+1}\right)\right\}'=\dfrac{1+\dfrac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$

$=\dfrac{1}{\sqrt{x^2+1}}$

よって $\int\log\left(x+\sqrt{x^2+1}\right)dx=x\log\left(x+\sqrt{x^2+1}\right)-\int\dfrac{x}{\sqrt{x^2+1}}dx$

$=x\log\left(x+\sqrt{x^2+1}\right)-\sqrt{x^2+1}+C$

160 点$(x,\ y)$における接線の傾きは $f'(x)$だから

$f'(x)=6x^2-2x+3$

両辺積分して

$f(x)=\int\left(6x^2-2x+3\right)dx$

$=2x^3-x^2+3x+C$

ここで，点$(-1,\ 3)$を通るから $f(-1)=3$ より

$-2-1-3+C=3\quad\therefore\ C=9$

したがって $f(x)=2x^3-x^2+3x+9$

$\therefore\ y=2x^3-x^2+3x+9$

発展問題

161 $x+\sqrt{x^2+1}=t$ より $\dfrac{dt}{dx}=1+\dfrac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}\cdot 2x=\dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$

$$\frac{dx}{\sqrt{x^2+1}}=\frac{dt}{\sqrt{x^2+1}+x}=\frac{dt}{t}$$

与式 $=\displaystyle\int\frac{dt}{t}=\log\left|t\right|+C=\log\left|x+\sqrt{x^2+1}\right|+C$

162

(1) $\displaystyle\int\cos^n x\,dx=\int\cos^{n-1}x\cos x\,dx=\int\cos^{n-1}x\left(\sin x\right)dx$

$$=\cos^{n-1}x\sin x-\int\left(n-1\right)\cos^{n-2}x\left(-\sin^2 x\right)dx$$

$$=\cos^{n-1}x\sin x+\left(n-1\right)\int\cos^{n-2}x\left(1-\cos^2 x\right)dx$$

$$=\cos^{n-1}x\sin x+\left(n-1\right)\int\cos^{n-2}x\,dx-\left(n-1\right)\int\cos^n x\,dx$$

$\therefore\ n\displaystyle\int\cos^n x\,dx=\cos^{n-1}x\sin x+\left(n-1\right)\int\cos^{n-2}x\,dx$

よって, $\displaystyle\int\cos^n x\,dx=\frac{\cos^{n-1}x\sin x}{n}+\frac{n-1}{n}\int\cos^{n-2}x\,dx$

(2) $\displaystyle\int\tan^n x\,dx=\int\tan^{n-2}x\tan^2 x\,dx=\int\tan^{n-2}x\left(\frac{1}{\cos^2 x}-1\right)dx$

$$=-\int\tan^{n-2}x\,dx+\int\tan^{n-2}x\frac{1}{\cos^2 x}dx$$

$$=-\int\tan^{n-2}x\,dx+\int\tan^{n-2}x\left(\tan x\right)'dx$$

$$=-\int\tan^{n-2}x\,dx+\tan^{n-1}x-\int\left(n-2\right)\tan^{n-3}x\cdot\frac{1}{\cos^2 x}\cdot\tan x\,dx$$

$$=\tan^{n-1}x-\int\tan^{n-2}x\,dx-\left(n-2\right)\int\tan^{n-2}x\left(1+\tan^2 x\right)dx$$

$$=\tan^{n-1}x-\left(n-1\right)\int\tan^{n-2}x\,dx-\left(n-2\right)\int\tan^n x\,dx$$

$\therefore\ \left(n-1\right)\displaystyle\int\tan^n x\,dx=\tan^{n-1}x-\left(n-1\right)\int\tan^{n-2}x\,dx$

よって, $\displaystyle\int\tan^n x\,dx=\frac{\tan^{n-1}x}{n-1}-\int\tan^{n-2}x\,dx$

１－２　定積分

A 問題

163

⑴ $\int_0^1 (3x^2-1)dx = \left[x^3-x\right]_0^1$

$= (1-1)-0 = 0$

⑵ $\int_1^4 (x-2)(2x+1)dx = \int_1^4 (2x^2-3x-2)dx$

$= \left[\frac{2}{3}x^3-\frac{3}{2}x^2-2x\right]_1^4$

$= \frac{2}{3}(64-1)-\frac{3}{2}(16-1)-2(4-1)$

$= 42-\frac{45}{2}-6 = \frac{27}{2}$

⑶ $\int_{-3}^2 (x-1)^2 dx = \int_{-3}^2 (x^2-2x+1)dx$

$= \left[\frac{1}{3}x^3-x^2+x\right]_{-3}^2$

$= \left(\frac{8}{3}-4+2\right)-(-9-9-3) = \frac{65}{3}$

⑷ $\int_2^1 (6x^2+2x-1)dx = \left[2x^3+x^2-x\right]_2^1$

$= (2+1-1)-(16+4-2)$

$= -16$

（別解） $\int_2^1 (6x^2+2x-1)dx = -\int_1^2 (6x^2+2x-1)dx$

$= -\left[2x^3+x^2-x\right]_1^2$

$= -\{(16+4-2)-(2+1-1)\}$

$= -16$

⑸ $\int_1^3 (4x^3+2x)dx = \left[x^4+x^2\right]_1^3$

$= (81+9)-(1+1) = 88$

(6) $\int_{-1}^{2}(x^4+3x^2+2)\,dx=\left[\frac{1}{5}x^5+x^3+2x\right]_{-1}^{2}$

$=\left(\frac{32}{5}+8+4\right)-\left(-\frac{1}{5}-1-2\right)$

$=\frac{108}{5}$

164

(1) $\int_0^4 x\sqrt{x}\,dx=\int_0^4 x^{\frac{3}{2}}dx=\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^4=\frac{64}{5}$

(2) $\int_1^e \frac{1}{x}\,dx=\left[\log x\right]_1^e=1$

(3) $\int_{-1}^{0}\frac{x+3}{x+2}\,dx=\int_{-1}^{0}\left(\frac{1}{x+2}+1\right)dx$

$=\left[\log|x+2|+x\right]_{-1}^{0}$

$=\log 2+1$

(4) $\int_4^5\frac{1}{(x-3)(x-2)}\,dx=\int_4^5\left(\frac{1}{x-3}-\frac{1}{x-2}\right)dx$

$=\left[\log\left|\frac{x-3}{x-2}\right|\right]_4^5$

$=\log\frac{2}{3}-\log\frac{1}{2}=\log\frac{4}{3}$

(5) $\int_0^8\frac{x-5}{\sqrt[3]{x}}\,dx=\int_0^8\left(x^{\frac{2}{3}}-5x^{-\frac{1}{3}}\right)dx$

$=\left[\frac{3}{5}x^{\frac{5}{3}}-\frac{15}{2}x^{\frac{2}{3}}\right]_0^8$

$=-\frac{54}{5}$

(6) $\int_0^1\frac{dx}{\sqrt{x+1}-\sqrt{x}}=\int_0^1\left(\sqrt{x+1}+\sqrt{x}\right)dx$

$=\left[\frac{2}{3}(x+1)^{\frac{3}{2}}+\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$

$=\frac{4\sqrt{2}}{3}$

165

(1) $\int_0^{\frac{\pi}{4}}\sin^2 x\,dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\cos 2x)\,dx$

$=\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}=\frac{\pi}{8}-\frac{1}{4}$

(2) $\int_0^{\frac{\pi}{3}}\tan^2 x\,dx=\int_0^{\frac{\pi}{3}}\left(\frac{1}{\cos^2 x}-1\right)dx$

$=\left[\tan x-x\right]_0^{\frac{\pi}{3}}=\sqrt{3}-\frac{\pi}{3}$

(3) $\int_{-2}^{2}\left(e^x+e^{-x}\right)^2dx=\int_{-2}^{2}\left(e^{2x}+2+e^{-2x}\right)dx$

$=\left[\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}\right]_{-2}^{2}$

$=e^4-\frac{1}{e^4}+8$

(4) $\int_0^1\left(5^x+e^x\right)dx=\left[\frac{5^x}{\log 5}+e^x\right]_0^1$

$=\frac{4}{\log 5}+e-1$

(5) $\int_0^{\frac{\pi}{4}}\left(\frac{1}{\cos^2 x}-\sin x\right)dx=\left[\tan x+\cos x\right]_0^{\frac{\pi}{4}}=\frac{\sqrt{2}}{2}$

(6) $\int_0^{\frac{\pi}{3}}\frac{\sin 2x}{\cos x}\,dx=\int_0^{\frac{\pi}{3}}\frac{2\sin x\cos x}{\cos x}\,dx$

$=\left[-2\cos x\right]_0^{\frac{\pi}{3}}=1$

166

(1) $\int_0^1 \left(1-e^x\right)^2 dx+\int_0^1 \left(1+e^x\right)^2 dx$

$=\int_0^1 \left\{\left(1-2e^x+e^{2x}\right)+\left(1+2e^x+e^{2x}\right)\right\} dx$

$=2\int_0^1 \left(1+e^{2x}\right) dx=2\left[x+\frac{1}{2}e^{2x}\right]_0^1$

$=2\left\{\left(1+\frac{1}{2}e^2\right)-\frac{1}{2}\right\}=1+e^2$

(2) $\int_1^e \log\left(x^2+4x\right) dx-\int_1^e \left(x+4\right) dx$

$=\int_1^e \left\{\log\left(x^2+4x\right)-\log\left(x+4\right)\right\} dx$

$=\int_1^e \log\frac{x\left(x+4\right)}{x+4} dx=\int_1^e \log x\, dx$

$=\int_1^e x\log x\, dx=\left[x\log x\right]_1^e-\int_1^e x\cdot\frac{1}{x} dx$

$=e-\left[x\right]_1^e=e-\left(e-1\right)=1$

167

(1) 与式 $=\int_1^2 \left(x^3-2x\right) dx-\int_1^2 \left(x^3+x^2+5\right) dx$

$=\int_1^2 \left(-3x^2-5\right) dx=\left[-x^3-5x\right]_1^2=\left(-8+10\right)-\left(-1-5\right)=-12$

(2) 与式 $=\int_0^{\frac{\pi}{12}} \sin 2x\, dx-\int_0^{\frac{\pi}{12}} 2\sin x\cos x\, dx=\int_0^{\frac{\pi}{12}} \left(\sin 2x-\sin 2x\right) dx=0$

(3) 与式 $=\int_0^1 \left(e^x+e^{-x}\right) dx+\int_1^2 \left(e^x+e^{-x}\right) dx$

$=\int_0^2 \left(e^x+e^{-x}\right) dx=\left[e^x-e^{-x}\right]_0^2=e^2-e^{-2}=e^2-\frac{1}{e^2}$

(4) 与式 $=\int_0^{\frac{\pi}{3}} \sin^2 x+\int_0^{\frac{\pi}{3}} \cos^2 x\, dx=\int_0^{\frac{\pi}{3}} \left(\sin^2 x+\cos^2\right) dx=\int_0^{\frac{\pi}{3}} dx=\left[x\right]_0^{\frac{\pi}{3}}=\frac{\pi}{3}$

⑴　$1-x=t$　とおくと　$\dfrac{dt}{dx}=-1$　より

$dx=-dt$

x	$0 \to 1$
t	$1 \to 0$

$$\int_0^1 (x+1)\sqrt{1-x}\,dx=\int_1^0 (2-t)\sqrt{t}\,(-dt)=\int_0^1\left(2t^{\frac{1}{2}}-t^{\frac{3}{2}}\right)dt$$

$$=\left[\frac{4}{3}t^{\frac{3}{2}}-\frac{2}{5}t^{\frac{5}{2}}\right]_0^1=\frac{4}{3}-\frac{2}{5}=\frac{14}{15}$$

⑵　$x^2-1=t$　とおくと　$\dfrac{dt}{dx}=2x$　より

$xdx=\dfrac{1}{2}dt$

x	$1 \to 3$
t	$0 \to 8$

$$\int_1^3 x\sqrt{x^2-1}\,dx=\int_0^8\sqrt{t}\cdot\frac{1}{2}\,dt=\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^8=\frac{16}{3}\sqrt{2}$$

⑶　$x^2+1=t$　とおくと　$\dfrac{dt}{dx}=2x$　より

$xdx=\dfrac{1}{2}dt$

x	$-1 \to 0$
t	$2 \to 1$

$$\int_{-1}^0 x\left(x^2+1\right)^3 dx=\int_2^1 t^3\cdot\frac{1}{2}\,dt=\frac{1}{2}\left[\frac{1}{4}t^4\right]_2^1=\frac{1}{8}(1-16)=-\frac{15}{8}$$

169

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \sin 3x \cos x\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\sin 4x + \sin 2x)\, dx$

$\displaystyle = \frac{1}{2}\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} \right]_0^{\frac{\pi}{2}}$

$\displaystyle = \frac{1}{2}$

(2) $t = \cos x$ とおくと $\dfrac{dt}{dx} = -\sin x$ より

$\sin x\, dx = -dt$

x	$0 \to \frac{\pi}{3}$
t	$1 \to \frac{1}{2}$

$\displaystyle\int_0^{\frac{\pi}{3}} \sin x \cos^2 x\, dx = -\int_1^{\frac{1}{2}} t^2 dt$

$\displaystyle = \int_{\frac{1}{2}}^1 t^2 dt = \left[\frac{1}{3} t^3 \right]_{\frac{1}{2}}^1 = \frac{7}{24}$

(3) $e^x = t$ とおくと $e^x \dfrac{dx}{dt} = 1$ より

$e^x dx = dt$

x	$0 \to 1$
t	$1 \to e$

$\displaystyle\int_0^1 \frac{e^{2x}}{e^x + 1} dx = \int_1^e \frac{t}{t+1} dt$

$\displaystyle = \int_1^e \left(1 - \frac{1}{t+1}\right) dt$

$\displaystyle = \Big[t - \log|t+1| \Big]_1^e$

$\displaystyle = e - 1 + \log \frac{2}{e+1}$

(4) $e^x = t$ とおくと $e^x \dfrac{dx}{dt} = 1$ より

$e^x dx = dt$

x	$1 \to 2$
t	$e \to e^2$

$\displaystyle\int_1^2 \frac{1}{e^x - 1} dx = \int_1^2 \frac{e^x}{e^x(e^x - 1)} dx$

$\displaystyle = \int_e^{e^2} \frac{1}{t(t-1)} dt = \int_e^{e^2} \left(\frac{1}{t-1} - \frac{1}{t} \right) dt$

$\displaystyle = \left[\log \left| \frac{t-1}{t} \right| \right]_e^{e^2} = \log \frac{e^2 - 1}{e^2} - \log \frac{e-1}{e}$

$\displaystyle = \log \frac{e+1}{e}$

(5) $\log x = t$ とおくと $\dfrac{1}{x}\dfrac{dx}{dt} = 1$ より

$\dfrac{1}{x} dx = dt$

x	$e^2 \to e^3$
t	$2 \to 3$

$\displaystyle\int_{e^2}^{e^3} \frac{1}{x \log x} dx = \int_2^3 \frac{1}{t} dt = \Big[\log|t| \Big]_2^3$

$\displaystyle = \log 3 - \log 2 = \log \frac{3}{2}$

(6) $x^2 = t$ とおくと $2x \dfrac{dx}{dt} = 1$ より

$x\, dx = \dfrac{1}{2} dt$

x	$0 \to 1$
t	$0 \to 1$

$\displaystyle\int_0^1 x e^{x^2} dx = \frac{1}{2}\int_0^1 e^t dt$

$\displaystyle = \frac{1}{2}\Big[e^t \Big]_0^1 = \frac{1}{2}(e-1)$

(1) $x = 3\sin\theta$ とおくと $\dfrac{dx}{d\theta} = 3\cos\theta$ より

$dx = 3\cos\theta\, d\theta$

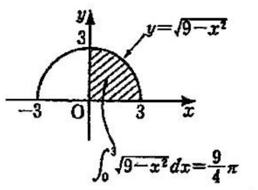
x	$0 \to 3$
t	$0 \to \frac{\pi}{2}$

$$\int_0^3 \sqrt{9-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\, d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} \left(1+\cos 2\theta\right) d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{9}{4}\pi$$



(2) $x = 3\sin\theta$ とおくと $\dfrac{dx}{d\theta} = 3\cos\theta$ より

$dx = 3\cos\theta\, d\theta$

$x = 0$ のとき $\sin\theta = 0$

よって $\theta = 0$

x	$0 \to \frac{3}{2}$
θ	$0 \to \frac{\pi}{6}$

$x = \dfrac{3}{2}$ のとき $\sin\theta = \dfrac{1}{2}$

よって $\theta = \dfrac{\pi}{6}$

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} = \int_0^{\frac{\pi}{6}} \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{3\cos\theta}{3\sqrt{1-\sin^2\theta}}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}$$

(3) $x = \sqrt{2}\tan\theta$ とおくと $\dfrac{dx}{d\theta} = \dfrac{\sqrt{2}}{\cos^2\theta}$ より

$dx = \dfrac{\sqrt{2}}{\cos^2\theta}\, d\theta$

$x = 0$ のとき $\tan\theta = 0$

よって $\theta = 0$

x	$0 \to \sqrt{6}$
θ	$0 \to \frac{\pi}{3}$

$x = \sqrt{6}$ のとき $\tan\theta = \sqrt{3}$

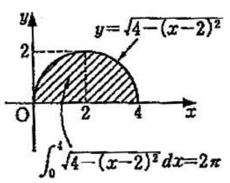
よって $\theta = \dfrac{\pi}{3}$

$$\int_0^{\sqrt{6}} \frac{1}{x^2+2}\, dx = \int_0^{\frac{\pi}{3}} \frac{1}{2\tan^2\theta + 2}\cdot\frac{\sqrt{2}}{\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{2\left(\tan^2\theta + 1\right)}\cdot\frac{\sqrt{2}}{\cos^2\theta}\, d\theta$$

$$= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{3}} \cos^2\theta\cdot\frac{1}{\cos^2\theta}\, d\theta$$

$$= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{3}} d\theta = \frac{\sqrt{2}}{6}\pi$$



(4) $\displaystyle\int_1^2 \frac{dx}{x^2-2x+2} = \int_1^2 \frac{1}{\left(x-1\right)^2+1}\, dx$ より

$x - 1 = \tan\theta$ とおくと

$\dfrac{dx}{d\theta} = \dfrac{1}{\cos^2\theta}$ より $dx = \dfrac{1}{\cos^2\theta}\, d\theta$

x	$1 \to 2$
θ	$0 \to \frac{\pi}{4}$

だから

$$\int_1^2 \frac{1}{\left(x-1\right)^2+1}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{\tan^2\theta + 1}\cdot\frac{1}{\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta\cdot\frac{1}{\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{4}$$

(5) $\int_0^4 \sqrt{4x-x^2}\,dx = \int_0^4 \sqrt{4-(x-2)^2}\,dx$ より $x-2 = 2\sin\theta$ とおくと

$\dfrac{dx}{d\theta} = 2\cos\theta$ より $dx = 2\cos\theta\,d\theta$

x	$0 \to 4$
θ	$-\frac{\pi}{2} \to \frac{\pi}{2}$

だから

$$\int_0^4 \sqrt{4-(x-2)^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4-\sin^2\theta} \cdot 2\cos\theta\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{1-\sin^2\theta} \cdot 2\cos\theta\,d\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta = 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\pi$$

(6) $x = \tan\theta$ とおくと

$\dfrac{dx}{d\theta} = \dfrac{1}{\cos^2\theta}$ より $dx = \dfrac{1}{\cos^2\theta}\,d\theta$

x	$0 \to 1$
θ	$0 \to \frac{\pi}{4}$

だから

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{1}{1+\tan^2\theta}} \cdot \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos\theta \cdot \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\cos^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos\theta}{1-\sin^2\theta}\,d\theta$$

ここで $t = \sin\theta$ とおくと

$\dfrac{dt}{d\theta} = \cos\theta$ より $\cos\theta\,d\theta = dt$

θ	$0 \to \frac{\pi}{4}$
t	$0 \to \frac{1}{\sqrt{2}}$

だから

$$\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{1-\sin^2\theta}\,d\theta = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{1-t^2}\,dt$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \frac{-1}{(t+1)(t-1)}\,dt = \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{t+1} - \frac{1}{t-1}\right)dt$$

$$= \frac{1}{2}\left[\log\left|\frac{t+1}{t-1}\right|\right]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{2}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \frac{1}{2}\log\left(\sqrt{2}+1\right)^2 = \log\left(\sqrt{2}+1\right)$$

171

(1) $\int_0^{\pi} x\sin x\,dx = \int_0^{\pi} x(-\cos x)'dx$

$= \left[x(-\cos x)\right]_0^{\pi} + \int_0^{\pi}\cos x\,dx$

$= \pi + \left[\sin x\right]_0^{\pi} = \pi$

(2) $\int_{-1}^{0} xe^{-x}dx = \int_{-1}^{0} x(-e^{-x})'dx$

$= \left[x(-e^{-x})\right]_{-1}^{0} + \int_{-1}^{0} e^{-x}dx$

$= -e - \left[e^{-x}\right]_{-1}^{0} = -1$

(3) $\int_0^2 \log(x+1)dx = \int_0^2 (x+1)'\cdot\log(x+1)dx$

$= \left[(x+1)\log(x+1)\right]_0^2 - \int_0^2 \frac{x+1}{x+1}dx$

$= 3\log 3 - 2$

(4) $\int_1^e x\log x\,dx = \left[\frac{x^2}{2}\log x\right]_1^e - \int_1^e \frac{1}{2}x\,dx$

$= \frac{1}{2}e^2 - \frac{1}{4}\left[x^2\right]_1^e = \frac{1}{4}(e^2+1)$

(5) $\int_0^{\frac{\pi}{2}} (x+1)\cos x\,dx = \int_0^{\frac{\pi}{2}} (x+1)(\sin x)'dx$

$= \left[(x+1)\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\sin x\,dx$

$= \frac{\pi}{2} + 1 + \left[\cos x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

(6) $\int_{\frac{1}{e}}^{1} x^2\log x\,dx = \int_{\frac{1}{e}}^{1}\left(\frac{1}{3}x^3\right)'\log x\,dx$

$= \left[\frac{1}{3}x^3\log x\right]_{\frac{1}{e}}^{1} - \frac{1}{3}\int_{\frac{1}{e}}^{1} x^2dx$

$= \frac{1}{3e^3} - \frac{1}{9}\left[x^3\right]_{\frac{1}{e}}^{1} = \frac{1}{9}\left(\frac{4}{e^3}-1\right)$

172

(1) $F'(x) = x\sin x$

(2) $F'(x) = \int_0^x \sin t\,dt + x\sin x$

$= \left[-\cos t\right]_0^x + x\sin x$

$= 1 - \cos x + x\sin x$

(3) $F'(x) = (2x\sin 2x)\times(2x)' = 4x\sin 2x$

(4) $F'(x) = (3x\sin 3x)\times(3x)' - (x\sin x)\times(x)'$

$= 9x\sin 3x - x\sin x$

B 問題

173

⑴ $\int_1^{e^2} |\log x - 1| \, dx$

$= \int_1^e (1 - \log x) \, dx + \int_e^{e^2} (\log x - 1) \, dx$

ここで, $\int \log x \, dx = \int x' \log x \, dx = x \log x - \int x \cdot \frac{1}{x} \, dx$

$= x \log x - x + c$

∴ 与式 $= \Big[x - x \log x + x \Big]_1^e + \Big[x \log x - x - x \Big]_e^{e^2}$

$= (2e - e) - 2 + (2e^2 - 2e^2) - (e - 2e)$

$= 2e - 2$

⑵ $|\sin x - \cos x| = \sqrt{2} \sin\left(x - \frac{\pi}{4} \right)$ だから

$0 \leqq x \leqq \frac{\pi}{4}$ のとき $\sin\left(x - \frac{\pi}{4} \right) \leqq 0$

$\frac{\pi}{4} \leqq x \leqq \pi$ のとき $\sin\left(x - \frac{\pi}{4} \right) \geqq 0$ である。

∴ 与式 $= \sqrt{2} \int_0^{\frac{\pi}{4}} \sin\left(x - \frac{\pi}{4} \right) dx + \sqrt{2} \int_{\frac{\pi}{4}}^{\pi} \sin\left(x - \frac{\pi}{4} \right) dx$

$= -\sqrt{2} \left[-\cos\left(x - \frac{\pi}{4} \right) \right]_0^{\frac{\pi}{4}} + \sqrt{2} \left[-\cos\left(x - \frac{\pi}{4} \right) \right]_{\frac{\pi}{4}}^{\pi}$

$= \left(\sqrt{2} - \sqrt{2} \cdot \frac{\sqrt{2}}{2} \right) + \left(\sqrt{2} \cdot \frac{\sqrt{2}}{2} + \sqrt{2} \right) = 2\sqrt{3}$

(1) $\sin^2 x \cos^3 x = \sin^2 x \left(1 - \sin^2 x\right) \cos x$

より $\sin x = t$ とおくと

$\frac{dt}{dx} = \cos x$ より $\cos x\, dx = dt$

x	$0 \to \frac{\pi}{2}$
t	$0 \to 1$

だから

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x\, dx = \int_0^{\frac{\pi}{2}} \sin^2 x \left(1 - \sin^2 x\right) \cos x\, dx$$

$$= \int_0^1 t^2 \left(1 - t^2\right) dt = \int_0^1 \left(t^2 - t^4\right) dt$$

$$= \left[\frac{1}{3} t^3 - \frac{1}{5} t^5\right]_0^1 = \frac{2}{15}$$

(2) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{dx}{\cos x} = \int_0^{\frac{\pi}{3}} \frac{\cos x}{\cos^2 x} dx = \int_0^{\frac{\pi}{3}} \frac{\cos x}{1 - \sin^2 x} dx$

$\sin x = t$ とおくと $\frac{dt}{dx} = \cos x$ より

$\cos x\, dx = dt$

x	$0 \to \frac{\pi}{3}$
t	$0 \to \frac{\sqrt{3}}{2}$

だから

$$\int_0^{\frac{\pi}{3}} \frac{dx}{\cos x} = \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1-t^2} dt = \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{(1+t)(1-t)} dt$$

$$= \frac{1}{2} \int_0^{\frac{3}{2}} \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt = \frac{1}{2} \left[\log(1+t) - \log(1-t)\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2} \left(\log \frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}}\right) = \frac{1}{2} \log \frac{2+\sqrt{3}}{2+\sqrt{3}} = \frac{1}{2} \log\left(2+\sqrt{3}\right)^2$$

$$= \log\left(2+\sqrt{3}\right)$$

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(1) $\int_0^1 \left(1-x^2\right)e^x dx = \int_0^1 \left(1-x^2\right)\left(e^x\right)' dx$

$$= \Big[\left(1-x^2\right)e^x \Big]_0^1 + 2\int_0^1 xe^x dx$$

$$= -1 + 2\int_0^1 xe^x dx$$

また $\int_0^1 xe^x dx = \int_0^1 x\left(e^x\right)' dx$

$$= \Big[xe^x \Big]_0^1 - \int_0^1 e^x dx = e - \Big[e^x \Big]_0^1 = 1$$

よって $\int_0^1 \left(1-x^2\right)e^x dx = -1 + 2\cdot 1 = 1$

(2) $\int_0^{\frac{\pi}{2}} x\sin^2 x\, dx = \int_0^{\frac{\pi}{2}} x\cdot \dfrac{1-\cos 2x}{2} dx$

$$= \int_0^{\frac{\pi}{2}} x\cdot \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)' dx$$

$$= \left[x\cdot \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(x - \frac{1}{2}\sin 2x\right)dx$$

$$= \frac{\pi^2}{8} - \frac{1}{2}\left[\frac{1}{2}x^2 + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{16}\pi^2 + \frac{1}{4}$$

(3) $\int_0^{\log 2} xe^{2x}\, dx = \int_0^{\log 2} x\left(\dfrac{1}{2}e^{2x}\right)' dx$

$$= \left[\frac{1}{2}xe^{2x}\right]_0^{\log 2} - \int_0^{\log 2}\frac{1}{2}e^{2x}\, dx$$

$$= \frac{1}{2}(\log 2)e^{2\log 2} - \frac{1}{4}\Big[e^{2x}\Big]_0^{\log 2}$$

$$= \frac{1}{2}(\log 2)e^{\log 4} - \frac{1}{4}e^{2\log 2} + \frac{1}{4}$$

$$= \frac{1}{2}(\log 2)\cdot 4 - \frac{1}{4}\cdot 4 + \frac{1}{4} = 2\log 2 - \frac{3}{4}$$

(4) $\int_1^e x(\log x)^2 dx = \int_1^e \left(\dfrac{x^2}{2}\right)'(\log x)^2 dx$

$$= \left[\frac{x^2}{2}(\log x)^2\right]_1^e - \int_1^e \frac{x^2}{2}\cdot 2(\log x)\frac{1}{x}dx$$

$$= \frac{1}{2}e^2 - \int_1^e x\log x\, dx$$

$$= \frac{1}{2}e^2 - \int_1^e \left(\frac{x^2}{2}\right)'\log x\, dx$$

$$= \frac{1}{2}e^2 - \left[\frac{x^2}{2}\cdot\log x\right]_1^e + \int_1^e \frac{x^2}{2}\cdot\frac{1}{x}dx$$

$$= \frac{1}{2}e^2 - \frac{1}{2}e^2 + \left[\frac{1}{4}x^2\right]_1^e = \frac{1}{4}\left(e^2 - 1\right)$$

176　〔部分積分〕

$$I=\int_0^1\left\{\frac{1}{2}\log\left(x^2+1\right)\right\}'\log\left(x^2+1\right)dx$$

$$=\left[\frac{1}{2}\log\left(x^2+1\right)\cdot\log\left(x^2+1\right)\right]_0^1$$

$$-\int_0^1\frac{1}{2}\log\left(x^2+1\right)\cdot\frac{2x}{x^2+1}dx$$

$$=\frac{1}{2}(\log 2)^2-\int_0^1\frac{x}{x^2+1}\log\left(x^2+1\right)dx$$

したがって

$$I=\frac{1}{2}(\log 2)^2-I$$

$$2I=\frac{1}{2}(\log 2)^2$$ よって $$I=\frac{1}{4}(\log 2)^2$$

〔置換積分〕

$\log\left(x^2+1\right)=t$ とおくと $\dfrac{2x}{x^2+1}\cdot\dfrac{dx}{dt}=1$ より

$$\frac{x}{x^2+1}dx=\frac{1}{2}dt$$

x	$0\to1$
t	$0\to\log2$

$$I=\int_0^1\log\left(x^2+1\right)\cdot\frac{x}{x^2+1}dx$$

$$=\int_0^{\log2}t\left(\frac{1}{2}dt\right)=\frac{1}{2}\int_0^{\log2}t\,dt$$

$$=\frac{1}{2}\left[\frac{1}{2}t^2\right]_0^{\log2}=\frac{1}{4}(\log 2)^2$$

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(1)

$$F(x)=\int_0^x(x-t)\cos t\,dt$$

$$=x\int_0^x\cos t\,dt-\int_0^x t\cos t\,dt$$ より

$$F'(x)=\int_0^x\cos t\,dt+x\cos x-x\cos x$$

$$=\Big[\sin t\Big]_0^x=\sin x$$

(2) $F(x)=\int_0^x t\sin(x-t)dt$ において

$x-t=u$ とおくと $t=x-u$

$\dfrac{du}{dt}=-1$ より $dt=(-1)du$

t	$0\to x$
u	$x\to0$

$$F(x)=\int_x^0(x-u)\sin u\cdot(-1)du$$

$$=\int_0^x(x-u)\sin u\,du$$

$$=x\int_0^x\sin u\,du-\int_0^x u\sin u\,du$$

よって $$F'(x)=\int_0^x\sin u\,du+x\sin x-x\sin x$$

$$=\Big[-\cos u\Big]_0^x=-\cos x+1$$

(3)

$$F(x)=\int_0^x e^t\log\frac{x+1}{t+1}dt$$

$$=\int_0^x e^t\{\log(x+1)-\log(t+1)\}dt$$

$$=\log(x+1)\int_0^x e^t dt-\int_0^x e^t\log(t+1)dt$$

よって $$F'(x)=\frac{1}{x+1}\int_0^x e^t dt+e^x\log(x+1)$$

$$-e^x\log(x+1)$$

$$=\frac{1}{x+1}\Big[e^t\Big]_0^x=\frac{e^x-1}{x+1}$$

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$f'(x)=\int_0^x \sin^2 t\,dt+x\sin^2 x-x\sin^2 x$

$=\int_0^x \sin^2 t\,dt$

ゆえに $f''(x)=\sin^2 x$

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$\int_0^x (x-t)f(t)dt=a\cos x-2\cdots$①

$x\int_0^x f(t)dt-\int_0^x tf(t)dt=a\cos x-2$

両辺を x で微分すると

$\int_0^x f(t)dt+xf(x)-xf(x)=-a\sin x$

$\int_0^x f(t)dt=-a\sin x$

さらに両辺をx で微分すると

$f(x)=-a\cos x\cdots$②

また，①において $x=0$ とおくと

$0=a\cos 0-2$ より $a=2$

$a=2$ を②に代入して $f(x)=-2\cos x$

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⑴ $\int_{-a}^{a} f(x)dx=\int_{-a}^{0} f(x)dx+\int_0^a f(x)dx$

$=\int_{-a}^{0} f(x)dx$ について

$x=-t$ とおくと

$dx=-dt$

x	$-a\to 0$
t	$a\to 0$

$\int_{-a}^{0} f(x)dx=\int_a^0 f(-t)(-dt)=\int_0^a f(-t)dt$

$\therefore\ \int_{-a}^{a} f(x)dx=\int_0^a f(x)dx+\int_0^a f(-x)dx$

が成り立つ。

⑵ $\int_0^1 f(1-x)dx$ について

$1-x=t$ とおくと

$dx=-dt$

x	$0\to 1$
t	$1\to 0$

$\int_0^1 f(1-x)dx=\int_1^0 f(t)(-dt)=\int_0^1 f(t)dt$

$\therefore\ \int_0^1 \{f(x)+f(1-x)\}dx=2\int_0^1 f(x)dx$

が成り立つ。

発展問題

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(1) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx$ において

$x=\dfrac{\pi}{2}-t$ とおくと

x	$0 \to \frac{\pi}{2}$
t	$\frac{\pi}{2} \to 0$

$ax=-dt$ だから

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx=\int_{\frac{\pi}{2}}^{0} \frac{\sin\left(\frac{\pi}{2}-t\right)}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)}(-dt)$$

$$=\int_{\frac{\pi}{2}}^{0} \frac{\cos t}{\cos t+\sin t}(-dt)-\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}dt$$

よって，$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}dx$

(2) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx+\int \frac{\cos x}{\sin x+\cos x}dx$

$$=\int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sin x+\cos x}dx-\int_0^{\frac{\pi}{2}}dx=\Big[\,x\,\Big]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$

$\therefore\ \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx=\frac{\pi}{4}$