２章　微分法　解答

２節　導関数

練習１

⑴　$\dfrac{f(5)-f(2)}{5-2}=\dfrac{5^2-2^2}{3}=7$

⑵　$\dfrac{f(1)-f(-3)}{1-(-3)}=\dfrac{1^2-(-3)^2}{4}=-2$

練習２

⑴　$f'(2)=\lim_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$

$=\lim_{h\to 0}\dfrac{\left\{(2+h)^2+3(2+h)\right\}-\left(2^2+3\cdot 2\right)}{h}=\lim_{h\to 0}\dfrac{7h+h^2}{h}=\lim_{h\to 0}(7+h)=7$

⑵　$f'(2)=\lim_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$

$=\lim_{h\to 0}\dfrac{\left\{-2(2+h)^2+1\right\}-\left(-2\cdot 2^2+1\right)}{h}=\lim_{h\to 0}\dfrac{-8h-2h^2}{h}=\lim_{h\to 0}(-8-2h)=-8$

練習３

$$\lim_{h\to +0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to +0}\frac{\left|(2+h)^2-2(2+h)\right|-\left|2^2-2\cdot 2\right|}{h}$$

$$=\lim_{h\to +0}\frac{\left|h^2+2h\right|}{h}=\lim_{h\to +0}\frac{h(h+2)}{h}=2$$

$$\lim_{h\to -0}\frac{f(2+h)-f(2)}{h}=\lim_{h\to -0}\frac{\left|(2+h)^2-2(2+h)\right|-\left|2^2-2\cdot 2\right|}{h}$$

$$=\lim_{h\to -0}\frac{\left|h(h+2)\right|}{h}=\lim_{h\to -0}\frac{-h(h+2)}{h}=-2$$

よって，$f'(2)$は存在しないから，$x=2$で微分可能ではない。

練習４

⑴　$f'(x)=\lim_{h\to 0}\dfrac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\dfrac{x^3+3x^2h+3xh^2+h^3-x^3}{h}=\lim_{h\to 0}\left(3x^2+3xh+h^2\right)=3x^2$

⑵　$f'(x)=\lim_{h\to 0}\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}=\lim_{h\to 0}\dfrac{1}{h}\cdot\dfrac{-h}{x(x+h)}=-\dfrac{1}{x^2}$

練習 5

⑴　$y' = \left(x^2 - 4x + 7\right)' = \left(x^2\right)' - 4(x)' + (7)' = 2x - 4 \cdot 1 = 2x - 4$

⑵　$y' = \left(-3x^2 + 2x + 3\right)' = -3\left(x^2\right)' + 2(x)' + (3)' = -3 \cdot 2x + 2 \cdot 1 = -6x + 2$

⑶　展開して整理すると　$y = 4x^2 + 20x + 25$　であるから

$y' = \left(4x^2 + 20x + 25\right)' = 4\left(x^2\right)' + 20(x)' + (25)' = 4 \cdot 2x + 20 \cdot 1 = 8x + 20$

⑷　展開して整理すると　$y = 1 - 9x^2$　であるから

$y' = \left(1 - 9x^2\right)' = (1)' - 9\left(x^2\right)' = -9 \cdot 2x = -18x$

⑸　$y' = \left(x^3 + \frac{1}{2}x^2 + \frac{2}{3}\right)' = \left(x^3\right)' + \left(\frac{1}{2}x^2\right)' + \left(\frac{2}{3}\right)' = 3x^2 + x$

⑹　$y' = \left(-\frac{1}{6}x^3 + \frac{3}{4}x^2 + 5x\right)' = \left(-\frac{1}{6}x^3\right)' + \left(\frac{3}{4}x^2\right)' + (5x)'$

$= -\frac{1}{2}x^2 + \frac{3}{2}x + 5$

⑺　展開して整理すると　$y = -2x^3 + 7x^2 - 3x$　であるから

$y' = \left(-2x^3 + 7x^2 - 3x\right)' = -2\left(x^3\right)' + 7\left(x^2\right)' - 3(x)' = -2 + 3x^2 \cdot 7 \cdot 2x - 3 = -6x^2 + 14x - 3$

⑻　展開して整理すると　$y = x^3 - 6x^2 + 12x - 8$　であるから

$y' = \left(x^3 - 6x^2 + 12x - 8\right)' = \left(x^3\right)' - 6\left(x^2\right)' + 12(x)' - (8)' = 3x^2 - 6 \cdot 2x + 12 \cdot 1 = 3x^2 - 12x + 12$

練習 6

⑴　$y' = (x + 2x)'(4x - 3) + (x + 2)(4x - 3)' = 1 \cdot (4x - 3) + (x + 2) \cdot 4 = 8x + 5$

⑵　$y' = \left(x^2 + 1\right)'(2x - 5) + \left(x^2 + 1\right)(2x - 5)' = 2x(2x - 5) + \left(x^2 + 1\right) \cdot 2 = 6x^2 - 10x + 2$

⑶　$y' = (x + 1)'\left(x^2 - x + 1\right) + (x + 1)\left(x^2 - x + 1\right)' = 1 \cdot \left(x^2 - x + 1\right) + (x + 1)(2x - 1) = 3x^2$

⑷　$y' = (x + 1)'(x + 2)(x + 3) + (x + 1)(x + 2)'(x + 3) + (x + 1)(x + 2)(x + 3)'$

$= \left(x^2 + 5x + 6\right) + \left(x^2 + 4x + 3\right) + \left(x^2 + 3x + 2\right) = 3x^2 + 12x + 11$

練習 7

⑴　$y' = -\dfrac{(3x + 2)'}{(3x + 2)^2} = -\dfrac{3}{(3x + 2)^2}$

⑵　$y' = \dfrac{(x - 1)'\left(x^2 - 3\right) - (x - 1)\left(x^2 - 3\right)'}{\left(x^2 - 3\right)^2} = \dfrac{x^2 - 3 - (x - 1) \cdot 2x}{\left(x^2 - 3\right)^2} = \dfrac{-x^2 + 2x - 3}{\left(x^2 - 3\right)^2}$

⑶　$y' = -\dfrac{\left(x^2 + 2\right)'\left(x^2 + 1\right) - \left(x^2 + 2\right)\left(x^2 + 1\right)'}{\left(x^2 + 1\right)^2} = \dfrac{2x\left(x^2 + 1\right) - \left(x^2 + 2\right) \cdot 2x}{\left(x^2 + 1\right)^2} = \dfrac{-2x}{\left(x^2 + 1\right)^2}$

練習 8

⑴　$y' = \left(\dfrac{1}{x}\right)' = \left(x^{-1}\right)' = -x^{-2} = -\dfrac{1}{x^2}$

⑵　$y' = \left(\dfrac{3}{x^2}\right)' = \left(3x^{-2}\right)' = -6x^{-3} = -\dfrac{6}{x^3}$

⑶　$y' = \left(-\dfrac{1}{2x^4}\right)' = \left(-\dfrac{1}{2}x^{-4}\right)' = 2x^{-5} = \dfrac{2}{x^5}$

練習 9

⑴　$y' = 2(3x-5)(3x-5)' = 6(3x-5)$

⑵　$y' = 3(1-2x)^2(1-2x)' = -6(1-2x)^2$

⑶　$y' = 4(x^2+x+1)^3(x^2+x+1)' = 4(2x+1)(x^2+x+1)^3$

⑷　$y' = -3(x+1)^{-4}\cdot(x+1)' = -\dfrac{3}{(x+1)^4}$

⑸　$y' = 3\left(x+\dfrac{1}{x}\right)^2\left(x+\dfrac{1}{x}\right)' = 3\left(x+\dfrac{1}{x}\right)^2\left(1-\dfrac{1}{x^2}\right) = \dfrac{3(x^2+1)^2(x+1)(x-1)}{x^4}$

⑹　$y' = 4\left(\dfrac{x-2}{x}\right)^3\left(\dfrac{x-2}{x}\right)' = 4\left(\dfrac{x-2}{x}\right)^3\cdot\dfrac{2}{x^2} = \dfrac{8(x-2)^3}{x^5}$

練習 10

⑴　$y = f(ax+b)$，$u = ax+b$　とおくと　$y = f(u)$

よって

$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = f'(u)\cdot a = af'(ax+b)$

⑵　$y = \{f(x)\}^n$，$u = f(x)$　とおくと　$y = u^n$

よって

$\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = nu^{n-1}\cdot f'(x) = n\{f(x)\}^{n-1}f'(x)$

練習 11

⑴　$y' = \left(-x^{\frac{2}{3}}\right)' = \dfrac{2}{3}x^{-\frac{1}{3}} = \dfrac{2}{3\sqrt[3]{x}}$

⑵　$y' = \left(\sqrt{x^2+x+1}\right)' = \left\{\left(x^2+x+1\right)^{\frac{1}{2}}\right\}' = \dfrac{1}{2}\left(x^2+x+1\right)^{-\frac{1}{2}}\left(x^2+x+1\right)' = \dfrac{2x+1}{2\sqrt{x^2+x+1}}$

⑶　$y' = \left(\sqrt[4]{2x+1}\right)' = \left\{(2x+1)^{\frac{1}{4}}\right\}' = \dfrac{1}{4}(2x+1)^{-\frac{3}{4}}(2x+1)' = \dfrac{2}{4\sqrt[4]{(2x+1)^3}} = \dfrac{1}{2\sqrt[4]{(2x+1)^3}}$

練習 12

$y = \sqrt[4]{x}$　より　$x = y^4$　であるから　$\dfrac{dx}{dy} = 4y^3$

よって　$\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}} = \dfrac{1}{4y^3} = \dfrac{1}{4\sqrt[4]{x^3}}$

練習 13

⑴　$y' = 1 \cdot \cos x + x(-\sin x) - \cos x = -x\sin x$

⑵　$y' = (\cos 3x) \cdot (3x)' = 3\cos 3x$

⑶　$y' = -\sin(1-x) \cdot (1-x)' = \sin(1-x)$

⑷　$y' = \dfrac{1}{\cos^2 2x} \cdot (2x)' = \dfrac{2}{\cos^2 2x}$

⑸　$y' = 3(\cos 2x)^2 \cdot (\cos 2x)' = 3(\cos 2x)^2 \cdot (-2\sin 2x) = -6\cos^2 2x \cdot \sin 2x$

⑹　$y' = 2\tan 3x \cdot (\tan 3x)' = 2\tan 3x \cdot \dfrac{1}{\cos^2 3x}(3x)' = \dfrac{6\tan 3x}{\cos^2 3x} = \dfrac{6\sin 3x}{\cos^3 3x}$

⑺　$y' = \left\{(1-\cos x)^{\frac{1}{2}}\right\}' = \dfrac{1}{2}(1-\cos x)^{-\frac{1}{2}}(1-\cos x)' = \dfrac{\sin x}{2\sqrt{1-\cos x}}$

⑻　$y' = -\dfrac{(1+\sin 2x)'}{(1+\sin 2x)^2} = -\dfrac{2\cos 2x}{(1+\sin 2x)^2}$

⑼　$y' = \dfrac{(\sin x)'x - \sin x \cdot (x)'}{(x)^2} = \dfrac{x\cos x - \sin x}{x^2}$

練習 14　$y = \mathrm{Cos}^{-1}x$

⑴　$x = \cos y$　であるから　$1 = \dfrac{d}{dx}(\cos y) = -\sin y\dfrac{dy}{dx}$

したがって，　$\dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} = -\dfrac{1}{\sqrt{1-x^2}}$

⑵　$x = \cos y$　の両辺を y で微分して　$\dfrac{dx}{dy} = -\sin y$

したがって　$\dfrac{dy}{dx} = -\dfrac{1}{\sin y} = -\dfrac{1}{\sqrt{1-\cos^2 y}} = -\dfrac{1}{\sqrt{1-x^2}}$

練習 15

⑴　$\dfrac{1}{x} = u$　とおくと　$y = \mathrm{Cos}^{-1}u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{\sqrt{1-u^2}}\left(\frac{1}{x}\right)' = -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\left(-\frac{1}{x^2}\right) = \frac{1}{x\sqrt{x^2-1}}$$

(2) $\dfrac{1}{x}=u$ とおくと　$y=\mathrm{Tan}^{-1}u$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{1+x^2}\left(\frac{1}{x}\right)'=\frac{1}{1+u^2}\left(-\frac{1}{x^2}\right)=\frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)=-\frac{1}{x^2+1}$$

(3) $\sqrt{x}=u$ とおくと　$y=\mathrm{Sin}^{-1}u$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{\sqrt{1-u^2}}\left(\sqrt{x}\right)'=\frac{1}{\sqrt{1-x}}\,\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x(1-x)}}$$

練習 16

(1) $y'=\dfrac{1}{3x}\cdot(3x)'=\dfrac{1}{x}$

(2) $y'=\dfrac{1}{2-3x}\cdot(2-3x)'=\dfrac{-3}{2-3x}=\dfrac{3}{3x-2}$

(3) $y'=\dfrac{1}{x\log 10}$

別解　底を e に変換して

$y=\dfrac{\log x}{\log 10}$ より $y'=\dfrac{1}{\log 10}\cdot\dfrac{1}{x}=\dfrac{1}{x\log 10}$

(4) $y'=(x^2)'\log x+x^2(\log x)'=2x\log x+x^2\cdot\dfrac{1}{x}=2x\log x+x$

(5) $y'=2(\log x)(\log x)'=\dfrac{2}{x}\log x$

(6) $y'=-\dfrac{(\log x)'}{(\log x)^2}=-\dfrac{1}{x(\log x)^2}$

練習 17

(1) $y'=\dfrac{(3x+1)'}{3x+1}=\dfrac{3}{3x+1}$

(2) $y'=\dfrac{(x^2-3x+2)'}{x^2-3x+2}=\dfrac{2x-3}{x^2-3x+2}$

(3) $y'=\dfrac{(\sin x)'}{\sin x}=\dfrac{\cos x}{\sin x}$

練習 18

両辺の絶対値の自然対数をとると

$\log|y| = \log|x+1| + 2\log|x-2| - 3\log|x-1|$

両辺を x で微分すると

$$\frac{y'}{y} = \frac{1}{x+1} + \frac{2}{x-2} - \frac{3}{x-1} = \frac{(x-2)(x-1)+2(x-1)(x-1)-3(x+1)(x-2)}{(x+1)(x-2)(x-1)}$$

$$= \frac{6}{(x+1)(x-2)(x-1)}$$

ゆえに　$y' = \dfrac{6}{(x+1)(x-2)(x-1)} \cdot \dfrac{(x+1)(x-2)^2}{(x-1)^3} = \dfrac{6(x-2)}{(x-1)^4}$

練習 19

$y = x^\alpha$ とおいて，両辺の自然対数をとると　$\log y = \alpha \log x$

両辺を x で微分すると　$\dfrac{y'}{y} = \dfrac{\alpha}{x}$

よって　$y' = \dfrac{\alpha}{x} \cdot y = \dfrac{\alpha}{x} \cdot x^\alpha = \alpha x^{\alpha-1}$

練習 20

(1) $y' = e^{2x} \cdot (2x)' = 2e^{2x}$

(2) $y' = (x+1)'e^{-x} + (x+1)(e^{-x})' = 1 \cdot e^{-x} + (x+1)(-e^{-x}) = -xe^{-x}$

(3) $y' = 3^{x+2}\log 3 \cdot (x+2)' = 3^{x+2}\log 3$

(4) $y' = (x)'a^{3x} + x(a^{3x})' = 1 \cdot a^{3x} + x \cdot a^{3x}\log a \cdot (3x)' = (1+3x\log a)a^{3x}$

練習 21

(1) $y' = e^x \log x + e^x \cdot \dfrac{1}{x} = \left(\log x + \dfrac{1}{x}\right)e^x$

(2) $y' = -e^{-x}\sin x + e^{-x}\cos x = (\cos x - \sin x)e^{-x}$

(3) $y' = e^{\sin x}(\sin x)' = e^{\sin x}\cos x$

(4) $y' = e^{\frac{1}{x}} \cdot \left(\dfrac{1}{x}\right)' = -\dfrac{1}{x^2}e^{\frac{1}{x}}$

練習 22

(1) $y' = 4x^3 - 6x$ より　$y'' = 12x^2 - 6$

(2) $y' = \dfrac{1}{x}$ より　$y'' = -\dfrac{1}{x^2}$

(3) $y' = e^{2x} + 2xe^{2x} = (2x+1)e^{2x}$

$y'' = 2e^{2x} + (4x+2)e^{2x} = (4x+4)e^{2x}$

練習 23

(1)　$y' = 3x^2 \quad y'' = 6x$　より　$y''' = 6$

(2)　$y' = e^x \quad y'' = e^x$　より　$y''' = e^x$

(3)　$y' = -\sin x \quad y'' = -\cos x$　より　$y''' = \sin x$

練習 24

$$y' = \frac{1}{x},\ y'' = -\frac{1}{x^2},\ y^{(3)} = \frac{1\cdot 2}{x^3} = \frac{2!}{x^3}$$

$$y^{(4)} = -\frac{3\cdot 2!}{x^4} = -\frac{3!}{x^4}$$

$$y^{(5)} = \frac{4\cdot 9!}{x^5} = \frac{4!}{x^5}$$

これより $y^{(n)} = (-1)^{n-1}\dfrac{(n+1)!}{x^n}$

練習 25

$$y' = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x)$$

$$y'' = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x) = -2e^x \sin x$$

ゆえに $y'' - 2y' + 2y = -2e^x \sin x - 2e^x(\cos x - \sin x) + 2e^x \cos x = 0$

節末問題

1.　$f(x) = x^2 + x$

(1)　$\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{(9 + 3) - (1 + 1)}{3 - 1} = 5$

(2)　$\dfrac{f(2 + h) - f(2)}{2 + h - 2} = \dfrac{\left\{(2 + h)^2 + (2 + h)\right\} - (2^2 + 2)}{h} = \dfrac{5h + h^2}{h} = 5 + h$

(3)　$f'(2) = \lim\limits_{h\to 0}\dfrac{f(2 + h) - f(2)}{h} = \lim\limits_{h\to 0}(5 + h) = 5$

2.

(1)　$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x + h) - f(x)}{h} = \lim\limits_{h\to 0}\dfrac{\left\{(x + h)^3 - 2(x + h)\right\} - (x^3 - 2x)}{h}$

$$= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3 - 2h}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 - 2 = 3x^2 - 2$$

(2)　$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x + h) - f(x)}{h} = \lim\limits_{h\to 0}\dfrac{\sqrt{x + h} - \sqrt{x}}{h}$

$$= \lim_{h\to 0}\frac{\left(\sqrt{x + h} - \sqrt{x}\right)\left(\sqrt{x + h} + \sqrt{x}\right)}{h\left(\sqrt{x + h} + \sqrt{x}\right)} = \lim_{h\to 0}\frac{h}{h\left(\sqrt{x + h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

3.

$f(a+h)-f(a-h)=\{f(a+h)-f(a)\}-\{f(a-h)-f(a)\}$ より

$$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{h}=\lim_{h\to 0}\left\{\frac{f(a+h)-f(a)}{h}-\frac{f(a-h)-f(a)}{h}\right\}$$

$$=\lim_{h\to 0}\left\{\frac{f(a+h)-f(a)}{h}+\frac{f(a-h)-f(a)}{h}\right\}$$

$$=f'(a)+f'(a)=2f'(a)$$

$f'(x)=e^x$ より $f'(a)=e^a$ であるから $\displaystyle\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{h}=2e^a$

4. $f(x)=ax^2+bx+c$ とおくと $f'(x)=2ax+b$ であるから

$$\begin{cases} f(1)=a+b+c=0 & \cdots① \\ f'(1)=2a+b=1 & \cdots② \\ f'(2)=4a+b=3 & \cdots③ \end{cases}$$

①，②，③を解いて $a=1,\ b=-1,\ c=0$

よって， $f(x)=x^2-x$

5.

(1) $\dfrac{dV}{dr}=\left(\dfrac{4}{3}\pi r^3\right)'=4\pi r^2$

(2) $\dfrac{ds}{dt}=\left(t^3+at^2+b\right)'=3t^2+2at$

6.

(1) $y'=3x^2(x+4)^3+x^3\cdot 3(x+4)^2$

$=3x^2(x+4)^2(2x+4)=6x^2(x+2)(x+4)^2$

(2) $y'=2(2x^3-6x+3)\cdot(2x^3-6x+3)'$

$=2(2x^3-6x+3)(6x^2-6)=12(x+1)(x-1)(2x^3-6x+3)$

(3) $y'=-\dfrac{(x+3)'}{(x+3)^2}=-\dfrac{1}{(x+3)^2}$

(4) $y'=\dfrac{(1+x^2)'(1-x^2)-(1+x^2)(1-x^2)'}{(1-x^2)^2}=\dfrac{2x(1-x^2)-(1+x^2)(-2x)}{(1-x^2)^2}=\dfrac{4x}{(x^2-1)^2}$

(5) $y'=\left\{(4x+3)^{\frac{1}{2}}\right\}'=\dfrac{1}{2}(4x+3)^{-\frac{1}{2}}\cdot 4=\dfrac{2}{\sqrt{4x+3}}$

(6) $y'=\left\{(3x-2)^{\frac{2}{3}}\right\}'=\dfrac{2}{3}(3x-2)^{-\frac{1}{3}}\cdot 3=\dfrac{2}{\sqrt[3]{3x-2}}$

7.

(1) $y = \sqrt{\dfrac{x^2+1}{2x+1}}$ の両辺の自然対数をとると $\log y = \dfrac{1}{2}\left\{ \log\left| x^2+1 \right| - \log\left| 2x+1 \right| \right\}$

両辺を x で微分すると $\dfrac{y'}{y} = \dfrac{1}{2}\left(\dfrac{2x}{x^2+1} - \dfrac{2}{2x+1} \right) = \dfrac{x^2+x-1}{\left(x^2+1\right)\left(2x+1\right)}$

したがって $y' = \dfrac{x^2+x-1}{\left(x^2+1\right)\left(2x+1\right)}\sqrt{\dfrac{x^2+1}{2x+1}} = \dfrac{x^2+x-1}{\sqrt{\left(2x+1\right)^3\left(x^2+1\right)}}$

(2) $y = x^x$ の両辺の自然対数をとると $\log y = x\log x$

両辺を微分すると $\dfrac{y'}{y} = \log x + x \cdot \dfrac{1}{x} = \log x + 1$

したがって $y' = \left(\log x + 1\right) y = x^x\left(\log x + 1\right)$

8.

(1) $y' = \dfrac{-\sin x\left(1+\sin x\right) - \cos x \cdot \cos x}{\left(1+\sin x\right)^2} = \dfrac{-\sin x - \left(\sin^2 x + \cos^2 x\right)}{\left(1+\sin x\right)^2}$

$= -\dfrac{1+\sin x}{\left(1+\sin x\right)^2} = -\dfrac{1}{1+\sin x}$

(2) $y' = \dfrac{-\dfrac{1}{\cos^2 x}\left(1+\tan x\right) - \left(1-\tan x\right)\cdot\dfrac{1}{\cos^2 x}}{\left(1+\tan x\right)^2} = \dfrac{-\left(1+\tan x\right) - \left(1-\tan x\right)}{\cos^2 x\left(1+\tan x\right)^2}$

$= \dfrac{-2}{\left\{\cos x\left(1+\tan x\right)\right\}^2} = -\dfrac{2}{\left(\cos x + \sin x\right)^2}$

(3) $y = e^{\frac{1}{x}} + x \cdot e^{\frac{1}{x}} \cdot \left(-\dfrac{1}{x^2}\right) = e^{\frac{1}{x}}\left(1 - \dfrac{1}{x}\right)$

(4) $y' = \dfrac{\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right) - \left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)}{\left(e^x+e^{-x}\right)^2} = \dfrac{e^{2x}+2+e^{-2x} - \left(e^{2x}-2+e^{-2x}\right)}{\left(e^x+e^{-x}\right)^2} = \dfrac{4}{\left(e^x+e^{-x}\right)^2}$

(5) $y' = \left(\log 2\right)\cdot 2^{\log x}\cdot\left(\log x\right)' = \dfrac{\left(\log 2\right)\cdot 2^{\log x}}{x}$

(6) $y = e^{-x}\cos x$

$y' = -e^{-x}\cos x + e^{-x}\left(-\sin x\right) = -e^{-x}\left(\sin x + \cos x\right)$

(7) $y = \log\left(x + \sqrt{x^2+a}\right)$

$y' = \dfrac{\left(x+\sqrt{x^2+a}\right)'}{x+\sqrt{x^2+a}} = \dfrac{1+\dfrac{1}{2}\left(x^2+a\right)^{-\frac{1}{2}}\cdot 2x}{x+\sqrt{x^2+a}} = \dfrac{\sqrt{x^2+a}+x}{\left(x+\sqrt{x^2+a}\right)\sqrt{x^2+a}} = \dfrac{1}{\sqrt{x^2+a}}$

(8) $y = \log\left|\dfrac{x-a}{x+a}\right| \log\left(x-a\right) - \log\left(x+a\right)$

$y' = \dfrac{\left(x-a\right)'}{x-a} - \dfrac{\left(x+a\right)'}{x+a} = \dfrac{1}{x-a} - \dfrac{1}{x+a} = \dfrac{2a}{x^2-a^2}$

9.

(1) 両辺を x で微分すると

$$2x-y-x\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$(x-2y)\frac{dy}{dx}=2x-y$$

よって，$x-2y\neq 0$ すなわち

$x\neq 2y$ のとき $\dfrac{dy}{dx}=\dfrac{2x-y}{x-2y}$

(2) 両辺を x で微分すると

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$$

ゆえに，$x\neq 0$，$y\neq 0$ のとき

$$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}=-\sqrt{\frac{y}{x}}$$

10.

$y=A\cos kx+B\sin kx$

$y'=-Ak\sin kx+Bk\cos kx$

$y''=-Ak^2\cos kx-Bk^2\sin kx$ だから

$y''+k^2y=-Ak^2\cos kx-Bk^2\sin kx+k^2(A\cos kx+B\sin kx)=0$

11.

(1) $y'=e^x-e^{-x}$

$y''=e^x+e^{-x}$

$y'''=e^x-e^{-x}$ より

$\vdots$

$y^{(n)}=e^x+(-1)^n e^{-x}$

(2) $y'=-1\cdot x^{-2}$

$y''=(-1)(-2)x^{-3}=2!x^{-3}$

$y'''=(-1)(-2)(-3)x^{-4}=-3!x^{-4}$

$y^{(4)}=(-1)(-2)(-3)(-4)x^{-5}=4!x^{-5}$

これより

$$y^{(n)}=(-1)^n\cdot n!x^{-(n+1)}=\frac{(-1)^n\cdot n!}{x^{n+1}}$$

12.

（Ⅰ） $n=1$ のとき $y'=\cos x$

一方 $y^{(1)}=\sin\left(x+\dfrac{\pi}{2}\right)=\cos x$

よって，$n=1$ のとき与えられた等式が成り立つ。

（Ⅱ） $n=k$ のとき $y^{(k)}=\sin\left(x+\dfrac{k\pi}{2}\right)$ であると仮定すると，

$n=k+1$ のとき $y^{(k+1)}=\left\{y^{(k)}\right\}'=\cos\left(x+\dfrac{k\pi}{2}\right)$

$=\sin\left\{\left(x+\dfrac{k\pi}{2}\right)+\dfrac{\pi}{2}\right\}$ $\left(\sin\left(\theta\dfrac{\pi}{2}\right)=\cos\theta\ より\right)$

$=\sin\left(x+\dfrac{k+1}{2}\pi\right)$

よって，$n=k+1$ のときも与えられた等式が成り立つ。

したがって，（Ⅰ)，(Ⅱ）より，すべての自然数 n に対して与えられた等式が成り立つ。

（終）

13.

⑴　初項 1，　公比 $x(\neq 1)$，　項数 $n+1$ の等比数列の和であるから

$$1+x+x^2+x^3+\cdots\cdots+x^n=\frac{x^{n+1}-1}{x-1}$$

⑵　⑴の両辺を x で微分すると

$$1+2x+3x^2+\cdots\cdots+nx^{n-1}=\frac{(n+1)x^n(x-1)-(x^{n+1}-1)}{(x-1)^2}$$

$$=\frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$$