３章　積分法　解答

１節　不定積分

練習１

(1) $\frac{1}{4}x^4+C$

(2) $\frac{5}{8}x^{\frac{8}{5}}+C$

(3) $-\frac{1}{3x^3}+C$

(4) $\frac{4}{7}x\sqrt[4]{x^3}+C$

(5) $\frac{2}{5}t^2\sqrt{t}+C$

(6) $2\sqrt{x}+C$

練習２

(1) $\int(2x+3)dx=2\int xdx+3\int dx$
$=2\cdot\frac{1}{2}x^2+3x+C$
$=x^2+3x+C$

(2) $\int(x^2+x-1)dx=\int x^2dx+\int xdx-\int dx$
$=\frac{1}{3}x^3+\frac{1}{2}x^2-x+C$

(3) $\int(-3)dx=-3\int dx=-3x+C$

(4) $\int(-6x^2+8x-5)dx=-6\int x^2dx+8\int xdx-5\int dx$
$=-6\cdot\frac{1}{3}x^3+8\cdot\frac{1}{2}x^2-5x+C$
$=-2x^3+4x^2-5x+C$

(5) $\int(x+3)(3x-1)dx=\int(3x^2+8x-3)dx$
$=x^3+4x^2-3x+C$

練習３

(1) 与式 $=\int\left(\frac{3}{x}-\frac{1}{x^2}\right)dx=3\log|x|+\frac{1}{x}+C$

(2) 与式 $=\int\left(\sqrt{x}-\frac{2}{\sqrt{x}}\right)dx=\frac{2}{3}x\sqrt{x}-4\sqrt{x}+C$

(3) 与式 $=\int\frac{x+2\sqrt{x}+1}{x}dx=\int\left(1+\frac{2}{\sqrt{x}}+\frac{1}{x}\right)dx$
$=x+4\sqrt{x}+\log x+C$

練習４

⑴　与式 $= -\cos x - 5\sin x + C$

⑵　与式 $= \int \left(\frac{1}{\cos^2 x} - 2 \right) dx = \tan x - 2x + C$

⑶　与式 $= \int \left(\frac{1}{\cos^2 x} + \frac{2}{\sin^2 x} \right) dx$

$= \tan x - \frac{2}{\tan x} + C$

⑷　与式 $= \int \frac{\cos^2 x}{\sin^2 x} dx = \int \frac{1 - \sin^2 x}{\sin^2 x} dx$

$= \int \left(\frac{1}{\sin^2 x} - 1 \right) dx = -\frac{1}{\tan x} - x + C$

練習５

⑴　与式 $= \frac{5^x}{\log 5} + C$

⑵　与式 $= 2e^x - \frac{2^x}{\log 2} + C$

⑶　与式 $= 10^x - \frac{x^5}{5} + C$

練習６

⑴　$\int e^{3x} dx = \frac{1}{3} e^{3x} + C$

⑵　$\int \sin 2x \, dx = -\frac{1}{2} \cos 2x + C$

⑶　$\int e^{-\frac{1}{2}} dx = -2e^{-\frac{1}{2}x} + C$

練習７

⑴　$3x - 5 = t$　とおくと

$x = \frac{t+5}{3}$，　$dx = \frac{1}{3} dt$

与式 $= \int t^4 \cdot \frac{1}{3} dt = \frac{1}{15} t^5 + C$

$= \frac{1}{15} (3x - 5)^5 + C$

⑵　$5x + 1 = t$　とおくと

$x = \frac{t-1}{5}$，　$dx = \frac{1}{5} dt$

与式 $= \int \sqrt{t} \cdot \frac{1}{5} dt = \frac{2}{15} t^{\frac{3}{2}} + C$

$= \frac{2}{15} (5x + 1) \sqrt{5x+1} + C$

練習８

⑴　与式 $= \frac{1}{2} \log | 2x + 3 | + C$

⑵　与式 $= 2\sin \left(\frac{1}{2} x - 5 \right) + C$

⑶　与式 $= -\frac{1}{3} e^{-3x+2} + C$

練習９

$2x + 1 = t$　すなわち　$x = \frac{t-1}{2}$　とおくと　$\frac{dx}{dt} = \frac{1}{2}$　より

与式 $= \int \frac{t-1}{2} \cdot \sqrt{t} \cdot \frac{1}{2} dt = \int \frac{t\sqrt{t} - \sqrt{t}}{4} dt$

$= \frac{1}{4} \int \left(t^{\frac{3}{2}} - t^{\frac{1}{2}} \right) dt = \frac{1}{4} \left(\frac{2}{5} t^{\frac{5}{2}} - \frac{2}{3} t^{\frac{3}{2}} \right) + C$

$= \frac{1}{30} (3t - 5) t^{\frac{3}{2}} + C = \frac{1}{15} (3x - 1)(2x + 1) \sqrt{2x+1} + C$

練習 10

⑴ $\sqrt{x-1}=t$ すなわち $x=t^2+1$ とおくと $\dfrac{dx}{dt}=2t$ より

$$与式 =\int\left(t^2+3\right)\cdot t\cdot 2t\,dt=\int\left(2t^4+6t^2\right)dt$$

$$=\frac{2}{5}t^5+2t^3+C=\frac{2}{5}t^3\left(t^2+5\right)+C=\frac{2}{5}(x+4)(x-1)\sqrt{x-1}+C$$

⑵ $\sqrt{x-1}=t$ すなわち $x=t^2+1$ とおくと $\dfrac{dx}{dt}=2t$ より

$$与式 =\int\frac{2\left(t^2+1\right)}{t}\cdot 2t\,dt=4\int\left(t^2+1\right)dt$$

$$=4\left(\frac{1}{3}t^3+t\right)+C=\frac{4}{3}t\left(t^2+3\right)+C=\frac{4}{3}(x+2)\sqrt{x-1}+C$$

（別解） $x-1=t$ とおくと $\dfrac{dx}{dt}=1$ より

⑴ $$与式 =\int(t+3)\sqrt{t}\,dt=\frac{2}{5}t^{\frac{5}{2}}+2t^{\frac{3}{2}}+C$$

$$=\frac{2}{5}t^{\frac{3}{2}}(t+5)+C=\frac{2}{5}(x+4)(x-1)\sqrt{x-1}+C$$

⑵ $$与式 =\int\frac{2(t+1)}{\sqrt{t}}\,dt=2\left(\frac{2}{3}t^{\frac{3}{2}}+2t^{\frac{1}{2}}\right)+C$$

$$=\frac{4}{3}t^{\frac{1}{2}}(t+3)+C=\frac{4}{3}(x+2)\sqrt{x-1}+C$$

練習 11

⑴ $x^2+5x+1=t$ とおくと $(2x+5)\,dx=dt$

$$与式 =\int t^3\,dt=\frac{1}{4}t^4+C=\frac{1}{4}\left(x^2+5x+1\right)^4+C$$

⑵ $\cos x=t$ とおくと $-\sin x\,dx=dt$

よって

$$与式 =\int t^5(-dt)=-\frac{1}{6}t^6+C=-\frac{1}{6}\cos^6 x+C$$

⑶ $x^2=t$ とおくと $2x\,dx=dt$

よって

$$与式 =\int e^t\cdot\frac{1}{2}\,dt=\frac{1}{2}e^t+C=\frac{1}{2}e^{x^2}+C$$

⑷ $2e^x-1=t$ とおくと $2e^x dx=dt$

よって

$$与式 =\int t^2\cdot\frac{1}{2}\,dt=\frac{1}{6}t^3+C=\frac{1}{6}\left(2e^x-1\right)^3+C$$

練習 12

$f(x)=t$ とおくと $f'(x)dx=dt$

よって，$\alpha \neq -1$ のとき

$$\int \{f(x)\}^{\alpha} f'(x)dx = \int t^{\alpha}dt = \frac{1}{\alpha+1}t^{\alpha+1}+C = \frac{1}{\alpha+1}\{f(x)\}^{\alpha+1}+C$$

練習 13

⑴ 与式 $= \displaystyle\int \frac{(x^2-1)'}{x^2-1}\times\frac{1}{2}dx = \frac{1}{2}\log|x^2-1|+C$

⑵ 与式 $= \displaystyle\int \frac{\cos x}{\sin x}dx = \int \frac{(\sin x)'}{\sin x}dx = \log|\sin x|+C$

⑶ 与式 $= \displaystyle\int \frac{(e^x-1)'}{e^x-1}dx = \log|e^x-1|+C$

⑷ 与式 $= \displaystyle\int \frac{(\log x)'}{\log x}dx = \log|\log x|+C$

練習 14

⑴ 与式 $= \displaystyle\int x(-\cos x)'\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$

⑵ 与式 $= \displaystyle\int x\left(\frac{1}{2}\sin 2x\right)'dx = \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$

⑶ 与式 $= \displaystyle\int x(e^x)'\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$

⑷ 与式 $= \displaystyle\int xe^{-x}\,dx = \int x(-e^{-x})'\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{x} - e^{-x} + C$

練習 15

⑴ 与式 $= \displaystyle\int (x)'\log 3x\,dx = x\log 3x - \int x\cdot\frac{3}{3x}\,dx = x\log 3x - \int dx = x\log 3x - x + C$

⑵ 与式 $= \displaystyle\int \left(\frac{1}{2}x^2\right)'\log x\,dx = \frac{1}{2}x^2\log x - \int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx$

$\displaystyle = \frac{1}{2}x^2\log x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C$

⑶ 与式 $= \displaystyle\int (x+1)'\log(x+1)dx$

$\displaystyle = (x+1)\log(x+1) - \int (x+1)\cdot\frac{1}{x+1}\,dx$

$\displaystyle = (x+1)\log(x+1) - \int dx = (x+1)\log(x+1) - x + C$

練習 16

$$\int e^x \cos x\,dx = \int (e^x)' \cos x\,dx$$

$$= e^x \cos x - \int e^x (\cos x)'\,dx = e^x \cos x + \int e^x \sin x\,dx$$

$$= e^x \cos x + \int (e^x)' \sin x\,dx = e^x \cos x + \left\{ e^x \sin x - \int e^x (\sin x)'\,dx \right\}$$

$$\therefore\ \int e^x \cos x\,dx = e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$$

したがって

$$2\int e^x \cos x\,dx = e^x(\sin x + \cos x) + C_1$$

よって　$\int e^x \cos x\,dx = \frac{1}{2} e^x(\sin x + \cos x) + C$　（ただし，$C = \frac{1}{2}C_1$）

練習 17

(1)　右の割り算より

$$\text{与式} = \int \left(x + 2 + \frac{5}{x-2} \right) dx = \frac{1}{2}x^2 + 2x + 5\log|x-2| + C$$

$$\begin{array}{r@{\,}l} & x+2 \\ x-2 \,\big)\! & \overline{x^2 \qquad +1} \\ & x^2 - 2x \\ \hline & 2x+1 \\ & 2x-4 \\ \hline & 5 \end{array}$$

(2)　$\text{与式} = \int \frac{1}{x} - \frac{1}{x+1}\,dx = \log|x| - \log|x+1| + C = \log\left|\frac{x}{x+1}\right| + C$

(3)　$\frac{-x+7}{(x-2)(x+3)} = \frac{a}{x-2} + \frac{b}{x+3}$　とおく。

両辺に$(x-2)(x+3)$　を掛けて

$$-x+7 = a(x+3) + b(x-2)$$
$$= (a+b)x + 3a - 2b$$

両辺の係数を比較して　$a+b=-1$，$3a-2b=7$

これより　$a=1$，$b=-2$

したがって

$$\text{与式} = \int \frac{-x+7}{(x-2)(x+3)}\,dx = \int \frac{(x+3)-2(x-2)}{(x-2)(x+3)}\,dx = \int \left(\frac{1}{x-2} - \frac{2}{x+3} \right) dx$$

$$= \log|x-2| - 2\log|x+3| + C = \log\frac{|x-2|}{(x+3)^2} + C$$

練習 18

(1) 与式 $= \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

(2) 与式 $= \frac{1}{2}\int (\sin 4x - \sin 2x) dx = -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + C$

(3) 与式 $= -\frac{1}{2}\int (\cos 3x - \cos x) dx = -\frac{1}{6}\sin 3x + \frac{1}{2}\sin x + C$

練習 19

(1) $\cos^3 x = \cos^2 x \cos x = (1-\sin^2 x)\cos x$ より $\sin x = t$ とおくと $\cos x\, dx = dt$

よって　与式 $= \int (1-t^2) dt = -\frac{1}{3}t^3 + t + C = -\frac{1}{3}\sin^3 x + \sin x + C$

(2) $e^{2x} - 1 = t$ とおくと $2e^{2x}dx = dt$

すなわち　$dx = \frac{1}{2e^{2x}} dt = \frac{1}{2(t+1)} dt$

よって

与式 $= \int \frac{1}{t} \cdot \frac{1}{2(t+1)} dt = \frac{1}{2}\int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt$

$= \frac{1}{2}\left(\log|t| - \log|t+1|\right) + C = \frac{1}{2}\log\left|e^{2x} - 1\right| - x + C$

練習 20

(1) $\int_0^2 x^3 dx = \left[\frac{1}{4}x^4\right]_0^2 = 4$

(2) $\int_{-2}^2 (x+2) dx = \left[\frac{1}{2}x^2 + 2x\right]_{-2}^2 = (2+4) - (2-4) = 8$

(3) $\int_1^2 (3x^2 - 8x + 5) dx = \left[x^3 - 4x^2 + 5x\right]_1^2 = (8 - 16 + 10) - (1 - 4 + 5) = 2 - 2 = 0$

(4) $\int_{-1}^1 x(x-2) dx = \int_{-1}^1 (x^2 - 2x) dx = \left[\frac{1}{3}x^3 - x^2\right]_{-1}^1 = \left(\frac{1}{3} - 1\right) - \left(-\frac{1}{3} - 1\right) = \frac{2}{3}$

(5) $\int_{-1}^3 (x+1)(x-3) dx = \int_{-1}^3 (x^2 - 2x - 3) dx$

$= \left[\frac{1}{3}x^3 - x^2 - 3x\right]_{-1}^3 = (9 - 9 - 9) - \left(-\frac{1}{3} - 1 + 3\right) = -\frac{32}{3}$

(6) $\int_0^3 (t-1)^2 dt = \int_0^3 (t^2 - 2t + 1) dt = \left[\frac{1}{3}t^3 - t^2 + t\right]_0^3 = (9 - 9 + 3) - 0 = 3$

別解

$\int_0^3 (t-1)^2 dx = \left[\frac{1}{3}(t-1)^3\right]_0^3 = \frac{8}{3} - \left(-\frac{1}{3}\right) = 3$

(7) $\int_{-3}^1 (t+1)^3 dt = \left[\frac{1}{4}(t+1)^4\right]_{-3}^1 = \frac{1}{4}(2^4 - 2^4) = 0$

(8) $\int_{-2}^{2} x(x+1)(x-1)\,dt = \int_{-2}^{2}(x^3 - x)\,dt = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-2}^{2} = 2-2=0$

練習 21

(1) 与式 $= \left[2\sqrt{x}\right]_1^4 = 4-2=2$

(2) 与式 $= \left[\sin t\right]_0^{\pi} = 0-0=0$

(3) 与式 $= \left[\log u\right]_1^{e} = 1-0=1$

(4) （与式）$= \left[\frac{3}{5}x^{\frac{5}{3}}\right]_0^{1} = \frac{3}{5}$

(5) 与式 $= \left[\tan x\right]_0^{\frac{\pi}{4}} = 1$

(6) （与式）$= \int_{-1}^{0} e^{-x}\,dx = \left[-e^{x}\right]_{-1}^{0} = -1+e$

(7) （与式）$= \int_0^1 \frac{1}{e}\cdot e^x\,dx = \frac{1}{e}\left[e^x\right]_0^1 = 1-\frac{1}{e}$

(8) （与式）$= \left[\frac{2^x}{\log 2}\right]_1^2 = \frac{4-2}{\log 2} = \frac{2}{\log 2}$

(9) （与式）$= \int_0^{2} \frac{1-\sin 2x}{2}\,dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

練習 22

(1) 与式 $= \int_0^{\pi}(\cos x - 3\sin x)\,dx = \left[\sin x + 3\cos x\right]_0^{\pi} = -3-3=-6$

(2) 与式 $= \int_1^2 \frac{4x-2}{x^2-x+1}\,dx = 2\int_1^2 \frac{(x^2-x+1)'}{x^2-x+1}\,dx = 2\left[\log(x^2-x+1)\right]_1^2 = 2\log 3$

練習 23

(1) 与式 $= \int_0^{\frac{\pi}{2}} \sin x\,dx = \left[-\cos x\right]_0^{\frac{\pi}{2}} = 1$

(2) 与式 $= \int_1^2 (x^3 - x + 1 - x^3 + x)\,dx = \int_1^2 1\,dx = \left[x\right]_1^2 = 1$

(3) 与式 $= \int_1^{2e}\left(x + \frac{1}{x}\right)dx = \left[\frac{1}{2}x^2 + \log x\right]_1^{2e}$

$= 2e^2 + \log 2e - \frac{1}{2} = 2e^2 + \log e + \log 2 - \frac{1}{2}$

$= 2e^2 + \log 2 + \frac{1}{2}$

練習 24

⑴ $2x-1=t$ すなわち $x=\frac{1}{2}(t+1)$ とおくと $dx=\frac{1}{2}dt$

よって

x	$0\to 1$
t	$-1\to 1$

$$与式 =\int_{-1}^{1}\frac{1}{2}(t+1)\cdot t^4\cdot\frac{1}{2}dt$$

$$=\frac{1}{4}\int_{-1}^{1}(t^5+t^4)dt=\frac{1}{4}\left[\frac{1}{6}t^6+\frac{1}{5}t^5\right]_{-1}^{1}=\frac{1}{10}$$

⑵ $t+1=s$ すなわち $t=s-1$ とおくと $dt=ds$

よって

t	$0\to 1$
s	$1\to 2$

$$与式 =\int_{1}^{2}\frac{s-1}{s^3}ds=\int_{1}^{2}\left(\frac{1}{s^2}-\frac{1}{s^3}\right)ds$$

$$=\left[-\frac{1}{s}+\frac{1}{2s^2}\right]_{1}^{2}=\frac{1}{8}$$

練習 25

⑴ $\sin x=t$ とおくと $\cos x\,dx=dt$

x	$0\to\frac{\pi}{2}$
t	$0\to 1$

$$与式 =\int_{0}^{\frac{\pi}{2}}\sin^4 x\cos x\,dx=\int_{0}^{1}t^4\,dt=\left[\frac{1}{5}t^5\right]_{0}^{1}=\frac{1}{5}$$

⑵ $\cos x=1$ とおくと $-\sin x\,dx=dt$

x	$0\to\frac{\pi}{2}$
t	$1\to 0$

$$与式 =-\int_{0}^{\frac{\pi}{2}}e^{\cos x}(-\sin x)\,dx=-\int_{1}^{0}e^t dt=\int_{0}^{1}e^t dt$$

$$=\left[e^t\right]_{0}^{1}=e-1$$

練習 26

(1) $x = \sin\theta$ とおくと $dx = \cos\theta\, d\theta$

よって

x	$0 \to \frac{1}{2}$
θ	$0 \to \frac{\pi}{6}$

$$\text{与式} = \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta = \int_0^{\frac{\pi}{6}} \cos^2\theta\, d\theta = \int_0^{\frac{\pi}{6}} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} = \frac{1}{2}\left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right) = \frac{\pi}{12}+\frac{\sqrt{3}}{8}$$

(2) $t = 2\sin\theta$ とおくと $dt = 2\cos\theta\, d\theta$

よって

t	$0 \to 1$
θ	$0 \to \frac{\pi}{6}$

$$\text{与式} = \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\, d\theta = \int_0^{\frac{\pi}{6}} \frac{1}{2\cos\theta}\cdot 2\cos\theta\, d\theta = \int_0^{\frac{\pi}{6}} d\theta$$

$$= \Big[\theta\Big]_0^{\frac{\pi}{6}} = \frac{\pi}{6}$$

(3) $x = 2\sin\theta$ とおくと $dx = 2\cos\theta\, d\theta$

よって

x	$-1 \to \sqrt{2}$
θ	$-\frac{\pi}{6} \to \frac{\pi}{4}$

$$\text{与式} = \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} 4\cos^2\theta\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} 2(1+\cos 2\theta)\, d\theta$$

$$= \Big[2\theta + \sin 2\theta\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{5}{6}\pi + \frac{2+\sqrt{3}}{2}$$

練習 27

(1) $x = 3\tan\theta$ とおくと $dx = \dfrac{3}{\cos^2\theta}\, d\theta$

よって

x	$0 \to 3$
θ	$0 \to \frac{\pi}{4}$

$$\text{与式} = \int_0^{\frac{\pi}{4}} \frac{1}{9+9\tan^2\theta}\cdot\frac{3}{\cos^2\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos^2\theta}{9}\cdot\frac{3}{\cos^2\theta}\, d\theta = \frac{1}{3}\Big[\theta\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{12}$$

(2) $x = \sqrt{3}\tan\theta$ とおくと $dx = \dfrac{\sqrt{3}}{\cos^2\theta}\, d\theta$

よって

x	$-1 \to \sqrt{3}$
θ	$-\frac{\pi}{6} \to \frac{\pi}{4}$

$$\text{与式} = \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{3\tan^2\theta+3}\cdot\frac{\sqrt{3}}{\cos^2\theta}\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos^2\theta}{3}\cdot\frac{\sqrt{3}}{\cos^2\theta}\, d\theta = \frac{\sqrt{3}}{3}\Big[\theta\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{5\sqrt{3}}{36}\pi$$

練習 28

⑴ $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = 2\tan\frac{x}{2}\cdot\frac{1}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$

$\cos x = \cos 2\cdot\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 = \frac{2}{1+\tan^2\frac{x}{2}} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$

$\frac{dt}{dx} = \frac{1}{2}\cdot\frac{1}{\cos^2\frac{x}{2}} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) = \frac{1+t^2}{2}$

よって，$\frac{dx}{dt} = \frac{2}{1+t^2}$　（終）

（別解）

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1} = \frac{2t}{1+t^2}$

また，次のように求めることもできる。

$\tan x = \tan 2\cdot\frac{x}{2} = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2}{1+t^2}$　から　$\sin x = \cos x\tan x = \frac{1-t^2}{1+t^2}\times\frac{2t}{1-t^2} = \frac{2t}{1+t^2}$

⑵ ⑴の結果から，$\tan\frac{x}{2} = t$　とおくと

$\frac{1}{1+\sin x} = \frac{1}{1+\frac{2t}{1+t^2}} = \frac{1+t^2}{1+t^2+2t} = \frac{1+t^2}{(t+1)^2}$

x	$0 \to \frac{2}{3}\pi$
t	$0 \to \sqrt{3}$

よって，$\int_0^{\frac{2}{3}\pi}\frac{dx}{1+\sin x} = \int_0^{\sqrt{3}}\frac{1+t^2}{(t+1)^2}\cdot\frac{2}{1+t^2}\,dt = \int_0^{\sqrt{3}}\frac{2}{(t+1)^2}\,dt$

$= \left[-\frac{2}{t+1}\right]_0^{\sqrt{3}} = -\left(\frac{2}{\sqrt{3}+1} - 2\right) = -\left(\sqrt{3}-1\right)+2 = 3-\sqrt{3}$

練習 29

⑴　与式 $= \int_0^{\frac{\pi}{2}} x(\sin x)' \, dx = \Big[x\sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx = \frac{\pi}{2} + \Big[\cos x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$

⑵　与式 $= \int_0^1 x(e^x)' \, dx = \Big[xe^x\Big]_0^1 - \int_0^1 e^x \, dx = e - \Big[e^x\Big]_0^1 = e - (e-1) = 1$

⑶　与式 $= \int_1^e \left(\frac{1}{3}x^3\right)' \log x \, dx$

$$= \left[\frac{1}{3}x^3 \log x\right]_1^e - \int_1^e \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx$$

$$= \frac{1}{3}e^3 - \frac{1}{3}\int_1^e x^2 \, dx = \frac{1}{3}e^3 - \frac{1}{3}\left[\frac{1}{3}x^3\right]_1^e = \frac{1}{3}e^3 - \frac{1}{9}\left(e^3 - 1\right) = \frac{1}{9}\left(2e^3 + 1\right)$$

⑷　与式 $= \int_1^e \left(-\frac{1}{x}\right)' \log x \, dx$

$$= \left[-\frac{1}{x}\log x\right]_1^e + \int_1^e \frac{1}{x} \cdot \frac{1}{x} \, dx = -\frac{1}{e} + \left[-\frac{1}{x}\right]_1^e = -\frac{1}{e} + \left(-\frac{1}{e} + 1\right) = 1 - \frac{2}{e}$$

練習 30

⑴　与式 $= \int_\alpha^\beta (x-\alpha)\left\{\frac{1}{2}(x-\beta)^2\right\}' dx$

$$= \left[\frac{1}{2}(x-\alpha)(x-\beta)^2\right]_\alpha^\beta - \frac{1}{2}\int_\alpha^\beta (x-\beta)^2 \, dx$$

$$= -\frac{1}{2}\left[\frac{1}{3}(x-\beta)^3\right]_\alpha^\beta = -\frac{1}{6} \times (\alpha-\beta)^3$$ （終）

⑵　与式 $= \int_\alpha^\beta (x-a)\left\{\frac{1}{3}(x-\beta)^3\right\}' dx$

$$= \left[\frac{1}{3}(x-\alpha)(x-\beta)^3\right]_\alpha^\beta - \int_\alpha^\beta \frac{1}{3}(x-\beta)^3 \, dx$$

$$= 0 - \frac{1}{3}\left[\frac{(x-\beta)^4}{4}\right]_\alpha^\beta = -\frac{1}{3} \times \left\{-\frac{(\alpha-\beta)^4}{4}\right\} = \frac{1}{12}(\beta-\alpha)^4$$ （終）

練習 31

$\frac{\pi}{2} - x = t$　とおくと　$x = \frac{\pi}{2} - t$,　$dx = -dt$

x	$0 \longrightarrow \frac{\pi}{2}$
t	$\frac{\pi}{2} \longrightarrow 0$

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_{\frac{\pi}{2}}^0 \sin^n\left(\frac{\pi}{2} - t\right)(-dt)$$

$$= \int_0^{\frac{\pi}{2}} \cos^n t \, dt = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$ （終）

練習 32

(1) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^6 x\,dx$

$\displaystyle= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5}{32}\pi$

(2) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^7 x\,dx$

$\displaystyle= \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 1 = \frac{16}{35}$

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^8 x\,dx$

$\displaystyle= \int_0^{\frac{\pi}{2}} \sin^8 x\,dx$

$\displaystyle= \frac{8}{7}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35}{256}\pi$

練習 33

(1) $\displaystyle F(x) = \int_{-1}^{x} 2t^2\,dt - x\int_{-1}^{x} t\,dt$ より

$\displaystyle F'(x) = 2x^2 - \left(\int_{-1}^{x} t\,dt + x\cdot x\right)$

$\displaystyle = 2x^2 - \left[\frac{1}{2}t^2\right]_{-1}^{x} - x^2 = \frac{1}{2}x^2 + \frac{1}{2}$

(2) $\displaystyle F'(x) = \left(e^x\int_{-3}^{x} e^t\,dt\right)' = e^x\int_{-3}^{x} e^t\,dt + e^x\cdot e^x$

$\displaystyle = e^x\Big[e^t\Big]_{-3}^{x} + e^{2x} = e^x\left(e^x - e^{-3}\right) + e^{2x}$

$= 2e^{2x} - e^{x-3}$

節末問題

1.

⑴ 与式 $= \int \frac{x - \sqrt{x} - 2}{x} dx = \int \left(1 - \frac{1}{\sqrt{x}} - \frac{2}{x} \right) dx = x - 2\sqrt{x} - 2\log x + C$

⑵ $2x + 3 = t$ とおくと $2dx = dt$

与式 $= \int 10^t \cdot \frac{1}{2} dt = \frac{1}{2} \cdot \frac{10^t}{\log 10} + C = \frac{10^{2x+3}}{2\log 10} + C$

⑶ $\sqrt{x-1} = t$ とおくと $x = t^2 + 1$ から $dx = 2t\,dt$

よって

与式 $= \int \left(t^2 + 1 \right) \cdot t \cdot 2t\,dt = \int \left(2t^4 + 2t^2 \right) dt = \frac{2}{5} t^5 + \frac{2}{3} t^3 + C$

$= \frac{2}{15} t^3 \left(3t^2 + 5 \right) + C = \frac{2}{15} \left(3x + 2 \right) \left(x - 1 \right) \sqrt{x-1} + C$

⑷ $\sqrt{1-x} = t$ とおくと $1 - x = t^2$

$x = 1 - t^2$ から $dx = -2t\,dt$

与式 $= \int \frac{\left(1 - t^2 \right)^2}{t} \cdot \left(-2t \right) dt = -2\int \left(1 - t^2 \right)^2 dt$

$= -2\int \left(t^4 - 2t^2 + 1 \right) dt = -2\left(\frac{1}{5} t^5 - \frac{2}{3} t^3 + t \right) + C$

$= -\frac{2}{15} t \left(3t^4 - 10t^2 + 15 \right) + C = -\frac{2}{15} \sqrt{1-x} \left\{ 3\left(1 - x \right)^2 - 10\left(1 - x \right) + 15 \right\} + C$

$= -\frac{2}{15} \left(3x^2 + 4x + 8 \right) \sqrt{1-x} + C$

⑸ $\log x = t$ とおくと $\frac{1}{x} dx = dt$

よって

与式 $= \int t\,dt = \frac{1}{2} t^2 + C = \frac{\left(\log x \right)^2}{2} + C$

⑹ $\sin x = t$ とおくと $\cos x\,dx = dt$

よって

与式 $= \int \frac{1}{1+t} dt = \log\left| 1 + t \right| + C = \log\left(1 + \sin x \right) + C$

2.

⑴　与式 $= \int x\left(-\dfrac{1}{3}\cos 3x\right)' dx = -\dfrac{1}{3}x\cos 3x + \int \dfrac{1}{3}\cos 3x\,dx = -\dfrac{1}{3}x\cos 3x + \dfrac{1}{9}\sin 3x + C$

⑵　与式 $= \int x\left(-\dfrac{1}{3}e^{-3x}\right)' dx = -\dfrac{1}{3}xe^{-3x} + \dfrac{1}{3}\int e^{-3x}\,dx$

$= -\dfrac{1}{3}xe^{-3x} - \dfrac{1}{9}e^{-3x} + C = -\dfrac{1}{9}(3x+1)e^{-3x} + C$

⑶　与式 $= \int\left(\dfrac{1}{3}x^3\right)' \log x\,dx = \dfrac{1}{3}x^3\log x - \int \dfrac{1}{3}x^3 \cdot \dfrac{1}{x}\,dx$

$= \dfrac{1}{3}x^3\log x - \int \dfrac{1}{3}x^2\,dx = \dfrac{1}{3}x^3\log x - \dfrac{x^3}{9} + C$

⑷　与式 $= \int\left(\dfrac{x^2}{2}\right)' \log(x+1)\,dx = \dfrac{x^2}{2}\log(x+1) - \int \dfrac{x^2}{2}\cdot\dfrac{1}{x+1}\,dx$

$= \dfrac{x^2}{2}\log(x+1) - \dfrac{1}{2}\int\left(x - 1 + \dfrac{1}{x+1}\right)dx$

$= \dfrac{x^2}{2}\log(x+1) - \dfrac{1}{4}x^2 + \dfrac{1}{2}x - \dfrac{1}{2}\log(x+1) + C$

$= \dfrac{x^2-1}{2}\log(x+1) - \dfrac{1}{4}x^2 + \dfrac{1}{2}x + C$

（別解）

与式 $= \int\{(x+1)\log(x+1) - \log(x+1)\}dx$

$= \dfrac{(x+1)^2}{2}\log(x+1) - \int \dfrac{(x+1)^2}{2}\cdot\dfrac{1}{x+1}\,dx - \{(x+1)\log(x+1) - x\} + C$

$= \dfrac{x^2-1}{2}\log(x+1) - \dfrac{1}{4}x^2 + \dfrac{1}{2}x + C$

3.

⑴　右の割り算より　与式 $=\int\left(x-3-\frac{1}{x+2}\right)dx$

$$=\frac{1}{2}x^2-3x-\log|x+2|+C$$

$$\begin{array}{r}
x-3 \\
x+2\,\overline{)\,x^2-\ \ x-7} \\
\underline{x^2+\ 2x\quad} \\
-3x-7 \\
\underline{-3x-6} \\
5
\end{array}$$

⑵　$\frac{3x+3}{x^2-9}=\frac{3x+3}{(x+3)(x-3)}=\frac{a}{x+3}+\frac{b}{x-3}$　とおくと

両辺に $(x+3)(x-3)$ を掛けて

$$3x+3=a(x-3)+b(x+3)$$
$$=(a+b)x-3a+3b$$

両辺の係数を比較して

$a+b=3,-3a+3b=3$

これより $a=1,\ \ b=2$

よって,

$$与式=\int\left(\frac{2}{x-3}+\frac{1}{x+3}\right)dx$$
$$=2\log|x-3|+\log|x+3|+C=\log(x-3)^2|x+3|+C$$

4.

⑴　与式 $=\frac{1}{2}\int(\cos 6x+\cos 4x)\,dx=\frac{1}{12}\sin 6x+\frac{1}{8}\sin 4x+C$

⑵　与式 $=\int x\cdot\frac{1+\cos 2x}{2}\,dx=\int\left(\frac{1}{2}x+\frac{1}{2}x\cos 2x\right)dx=\frac{1}{4}x^2+\frac{1}{2}\int x\left(\frac{1}{2}\sin 2x\right)'\,dx$

$$=\frac{1}{4}x^2+\frac{1}{2}\left(\frac{1}{2}x\sin 2x-\frac{1}{2}\int\sin 2x\,dx\right)=\frac{1}{4}x^2+\frac{1}{4}x\sin 2x-\frac{1}{4}\int\sin 2x\,dx$$

$$=\frac{1}{4}x^2+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x+C$$

⑶　$\cos^4 x\sin^3 x=\cos^4 x\left(1-\cos^2 x\right)\sin x$　から　$\cos x=t$　とおくと

$-\sin x\,dx=dt$　よって

与式 $=\int t^4\left(1-t^2\right)\cdot(-1)\,dt=\int\left(t^6-t^4\right)dt=\frac{1}{7}t^7-\frac{1}{5}t^5+C=\frac{1}{7}\cos^7 x-\frac{1}{5}\cos^5 x+C$

⑷　$e^x+1=t$　とおくと　$e^x dx=dt$　よって　$dx=\frac{1}{t}\,dt$

与式 $=\int\frac{t^3}{t+1}\cdot\frac{1}{t}\,dt=\int\frac{t^2}{t+1}\,dt=\int\left(t-1+\frac{1}{t+1}\right)dt$

$$=\frac{1}{2}t^2-t+\log(t+1)+C=\frac{1}{2}e^{2x}-e^x+\log\left(e^x+1\right)+C$$

（別解）

$e^x = t$ とおくと $e^x dx = dt$ よって $dx = \frac{1}{t}\,dt$

$$与式 = \int \frac{t^3}{t+1}\cdot\frac{1}{t}\,dt = \int \frac{t^2}{t+1}\,dt = \int\left(t-1+\frac{1}{t+1}\right)dt$$

$$= \frac{1}{2}t^2 - t + \log(t+1) + C$$

$$= \frac{1}{2}e^{2x} - e^x + \log(e^x+1) + C$$

5.

(1) $\frac{1}{\sin x} = \frac{\sin x}{\sin^2 x} = \frac{\sin x}{1-\cos^2 x}$ だから $\int \frac{1}{\sin x}\,dx = \int \frac{\sin x}{1-\cos^2 x}\,dx$

ここで, $\cos x = t$ とおくと $-\sin x\,dx = dt$

よって

$$与式 = \int \frac{1}{1-t^2}\cdot(-1)\,dt = \int \frac{1}{(t-1)(t+1)}\,dt = \frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$= \frac{1}{2}\left(\log|t-1| - \log|t+1|\right) + C = \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{\cos x - 1}{\cos x + 1}\right| + C = \frac{1}{2}\log\frac{1-\cos x}{1+\cos x} + C$$

(2) $\frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1-\sin^2 x}$ だから $\int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx$

ここで, $\sin x = t$ とおくと $\cos x\,dx = dt$

よって

$$与式 = \int \frac{1}{1-t^2}\,dt = \int \frac{-1}{(t-1)(t+1)}\,dt = \frac{1}{2}\int\left(\frac{1}{t+1}-\frac{1}{t-1}\right)dt$$

$$= \frac{1}{2}\left(\log|t+1| - \log|t-1|\right) + C = \frac{1}{2}\log\left|\frac{t+1}{t-1}\right| + C$$

$$= \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} + C$$

6.

(1) $\log x = t$ とおくと $\frac{1}{x}dx = dt$ よって

x	$1 \to e$
t	$0 \to 1$

与式 $= \int_0^1 t\,dt = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}$

(2) $x = \sqrt{3}\sin\theta$ とおくと $dx = \sqrt{3}\cos\theta\,d\theta$

x	$-\frac{\sqrt{3}}{2} \to \frac{3}{2}$
θ	$-\frac{\pi}{6} \to \frac{\pi}{3}$

$\sqrt{3 - x^2} = \sqrt{3(1 - \sin^2\theta)} = \sqrt{3\cos^2\theta} = \sqrt{3}\,|\cos\theta|$

$-\frac{\pi}{6} \leqq \theta \leqq \frac{\pi}{3}$ のとき $\cos\theta > 0$ だから

よって　与式 $= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sqrt{3}\cos\theta} \cdot \sqrt{3}\cos\theta\,d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta = \left[\ \theta\ \right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{2}$

(3) $x = \tan\theta$ とおくと $dx = \frac{1}{\cos^2\theta}d\theta$

x	$0 \to 1$
θ	$0 \to \frac{\pi}{4}$

ここで

$(x^2 + 1)^{\frac{3}{2}} = (\tan^2\theta + 1)^{\frac{3}{2}} = \left(\frac{1}{\cos^2\theta}\right)^{\frac{3}{2}} = \frac{1}{\cos^3\theta}$

よって

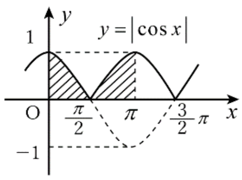
与式 $= \int_0^{\frac{\pi}{4}} \cos^3\theta \cdot \frac{1}{\cos^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \cos\theta\,d\theta = \left[\ \sin\theta\ \right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}$

(4) 与式 $= \int_0^{\frac{\pi}{4}} x(\tan x)'\,dx = \left[x\tan x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{\pi}{4} + \int_0^{\frac{\pi}{4}} \frac{(\cos x)'}{(\cos x)}dx$

$= \frac{\pi}{4} + \left[\log(\cos x)\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} = \frac{\pi}{4} - \log\sqrt{2}$

7.

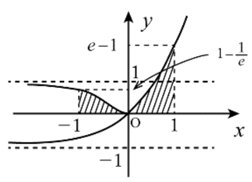
(1) 与式 $= \int_0^{\frac{\pi}{2}} \cos x\,dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\,dx$



$= \left[\sin x\right]_0^{\frac{\pi}{2}} - \left[\sin x\right]_{\frac{\pi}{2}}^{\pi}$

$= 1 - (-1) = 2$

(2) 与式 $= \int_{-1}^{0} (-e^x + 1)\,dx + \int_0^1 (e^x - 1)\,dx$



$= \left[-e^x + x\right]_{-1}^{0} - \left[e^x - x\right]_0^1$

$= -1 - \left(-\frac{1}{e} - 1\right) + (e - 1) - 1$

$= e + \frac{1}{e} - 2$

8.

⑴　$x=\dfrac{e^t-e^{-t}}{2}$　とおくと　$dx=\dfrac{e^t+e^{-t}}{2}dt$

また　$\sqrt{x^2+1}=\sqrt{\left(\dfrac{e^t-e^{-t}}{2}\right)^2+1}=\sqrt{\dfrac{e^{2t}+2+e^{-2t}}{4}}=\sqrt{\left(\dfrac{e^t+e^{-t}}{2}\right)^2}=\dfrac{e^t+e^{-t}}{2}$

したがって

$$A=\int\frac{2}{e^t+e^{-t}}\cdot\frac{e^t+e^{-t}}{2}\,dt=\int dt=t+C$$

⑵　$e^t-\dfrac{1}{e^t}=2x$　から　$\left(e^t\right)^2-2xe^t-1=0$

$$e^t=x\pm\sqrt{x^2+1}$$

$e^t>0$　より適するのは　$e^t=x+\sqrt{x^2+1}$

よって　$t=\log\left(x+\sqrt{x^2+1}\right)$

ゆえに　$\displaystyle\int\frac{1}{\sqrt{x^2+1}}dx=\log\left(x+\sqrt{x^2+1}\right)+C$　が成り立つ。（終）

9.

$\displaystyle\int_a^x(x-t)f'(t)\,dt=\int_a^x xf'(t)\,dt-\int_a^x xf'(t)\,dt$ だから

$$\begin{aligned}\frac{d}{dx}\int_a^x(x-t)f'(t)\,dt&=\left(x\int_a^x f'(t)\,dt\right)'-\left(\int_a^x tf'(t)\,dt\right)\\&=\int_a^x f'(t)\,dt+xf'(x)-xf'(x)\\&=\int_a^x f'(t)\,dt=\Big[f(t)\Big]_a^x=f(x)-f(a)\end{aligned}$$

（終り）

10.

与式の両辺を x で微分すると

$$f(x)=2(\log x)\cdot\frac{1}{x}-\frac{1}{x}=\frac{2\log x-1}{x}$$

与式に $x=a$　を代入すると　$\displaystyle\int_0^a f(x)\,dt=0$　だから

$$(\log a)^2-\log a-6=0$$

$(\log a-3)(\log a+2)=0$　∴　$\log a=3,\ -2$

したがって，$3=\log e^3$，$-2=\log\dfrac{1}{e^2}$　より　$a=e^3,\ \dfrac{1}{e^2}$

11.

$$I_n = \int_0^e (x)'(\log x)^n\, dx = \Big[x(\log x)^n \Big]_0^e - \int_0^e x \cdot n(\log x)^{n-1} \cdot \frac{1}{x}\, dx = e - n\int_0^{\frac{\pi}{2}} (\log x)^{n-1}\, dx$$

よって， $I_n = e - nI_{n-1}$ （終）

12. $\cos\alpha\cos\beta = \dfrac{1}{2}\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}$ より

$I = \int_0^{2\pi} \cos mx \cos nx\, dx$ とおくと

$$I = \frac{1}{2}\int_0^{2\pi} \{\cos(m+n)x + \cos(m-n)x\}\, dx$$

⑴ $m \neq n$ のとき，$m - n \neq 0$ であるから

$$I = \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_0^{2\pi} = 0$$

⑵ $m = n$ のとき，$\cos(m-n)x = \cos 0 = 1$ であるから

$$I = \frac{1}{2}\int_0^{2\pi} (\cos 2mx + 1)\, dx = \frac{1}{2}\left[\frac{1}{2m}\sin 2mx + x \right]_0^{2\pi} = \pi$$

以上から

$$\int_0^{2\pi} \cos mx \cos nx\, dx = \begin{cases} 0\ (m \neq n) \\ \pi\ (m = n) \end{cases}$$