５章　指数関数・対数関数　解答

１節　指数関数

練習１

⑴　$5^{-1}=\dfrac{1}{5^1}=\dfrac{1}{5}$

⑵　$2^{-5}=\dfrac{1}{2^5}=\dfrac{1}{32}$

⑶　$10^0=1$

⑷　$(-5)^{-3}=\dfrac{1}{(-5)^3}=\dfrac{1}{-125}=-\dfrac{1}{125}$

練習２

⑴　$a^4a^{-2}\cdot a^{4+(-2)}=a^2$

⑵　$(a^{-3})^2=a^{(-3)\times 2}=a^{-6}$

⑶　$(a^{-2}b)^2=(a^{-2})^2\times b^2=a^{-2\times 2}\times b^2=a^{-4}b^2$

⑷　$a^3\div a^{-5}=\dfrac{a^3}{a^{-5}}=a^{3-(-5)}=a^8$

⑸　$a^4\times a^{-3}\div(a^2)^{-1}=a^{4+(-3)}\div a^{2\times(-1)}=a^1\div a^{-2}=a^{1-(-2)}=a^3$

⑹　$(a^3b^{-2})^2\div a^4b^{-3}=a^6b^{-4}\div a^4b^{-3}=\dfrac{a^6b^{-4}}{a^4b^{-3}}=\dfrac{a^6}{a^4}\times\dfrac{b^{-4}}{b^{-3}}=a^{6-4}\times b^{(-4)-(-3)}=a^2b^{-1}$

練習３

⑴　$2^6=64$ であるから 2 は 64 の6乗根　つまり $\sqrt[6]{64}=2$

⑵　$(-5)^4=625$ であるから -5 は 625 の負の方の4乗根　よって $-\sqrt[4]{625}=-5$

⑶　$(3)^5=-243$ であるから -3 は -243 の5乗根　よって $\sqrt[5]{-243}=-3$

練習４

性質２

$$\left(\frac{\sqrt[n]{a}}{\sqrt[n]{b}}\right)^n=\frac{(\sqrt[n]{a})^n}{(\sqrt[n]{b})^n}=\frac{a}{b}$$

また，$\sqrt[n]{a}>0$，$\sqrt[n]{b}>0$ から $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}>0$

よって，$\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ は $\dfrac{a}{b}$ の正の n 乗根である。

$$\therefore\ \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}$$

性質３

$$\left\{(\sqrt[n]{a})^m\right\}^n=(\sqrt[n]{a})^{mn}=\left\{(\sqrt[n]{a})^n\right\}^m=a^m$$

また，$(\sqrt[n]{a})^m>0$

よって，$(\sqrt[n]{a})^m$ は a^m の n 乗根である。

$$\therefore\ (\sqrt[n]{a})^m=(\sqrt[n]{a^m})$$

性質４

$$\left(\sqrt[m]{\sqrt[n]{a}}\right)^{mn} = \left\{\left(\sqrt[m]{\sqrt[n]{a}}\right)^{m}\right\}^{n} = \left(\sqrt[n]{a}\right)^{n} = a$$

これより，$\sqrt[m]{\sqrt[n]{a}}$ は a の mn 乗根といえる。

$$\therefore \quad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

練習５

(1) $\sqrt[5]{4}\sqrt[5]{8} = \sqrt[5]{4\times 8} = \sqrt[5]{32} = \sqrt[5]{2^5} = 2$

(2) $\dfrac{\sqrt[3]{54}}{\sqrt[3]{2}} = \sqrt[3]{\dfrac{54}{2}} = \sqrt[3]{27} = \sqrt[3]{3^3} = 3$

(3) $\sqrt[5]{\sqrt{1024}} = \sqrt[5]{\sqrt[2]{1024}} = \sqrt[10]{1024} = \sqrt[10]{2^{10}} = 2$

(4) $\sqrt[3]{27^4} = \sqrt[3]{\left(3^3\right)^4} = \sqrt[3]{\left(3^4\right)^3} = 3^4 = 81$

(5) $\sqrt[6]{125} = \sqrt[6]{5^3} = \sqrt{5}$

練習６

(1) $9^{\frac{1}{2}} = \sqrt[2]{9} = \sqrt{9} = 3$

(2) $27^{\frac{2}{3}} = \sqrt[3]{27^2} = \sqrt[3]{\left(3^3\right)^2} = \sqrt[3]{\left(3^2\right)^3} = 3^2 = 9$

(3) $125^{-\frac{1}{3}} = \dfrac{1}{125^{\frac{1}{3}}} = \dfrac{1}{\sqrt[3]{125}} = \dfrac{1}{\sqrt[3]{5^3}} = \dfrac{1}{5}$

(4) $16^{-\frac{3}{2}} = \dfrac{1}{16^{\frac{3}{2}}} = \dfrac{1}{\sqrt[2]{16^3}} = \dfrac{1}{\sqrt[2]{\left(4^2\right)^3}} = \dfrac{1}{\sqrt[2]{\left(4^3\right)^2}} = \dfrac{1}{4^3} = \dfrac{1}{64}$

練習７

(1) $\sqrt[5]{a^2} = a^{\frac{2}{5}}$

(2) $\sqrt{a^{-3}} = \sqrt[2]{a^{-3}} = a^{-\frac{3}{2}}$

(3) $\dfrac{1}{\sqrt[7]{a^5}} = \dfrac{1}{a^{\frac{5}{7}}} = a^{-\frac{5}{7}}$

(4) $\dfrac{1}{\left(\sqrt[3]{a}\right)^5} = \dfrac{1}{\sqrt[3]{a^5}} = \dfrac{1}{a^{\frac{5}{3}}} = a^{-\frac{5}{3}}$

練習８

$$\left(a^r\right)^s = \left(a^{\frac{1}{3}}\right)^{\frac{2}{3}} = \left(\sqrt[3]{a}\right)^{\frac{2}{3}} = \left(\sqrt[3]{\left(\sqrt[3]{a}\right)}\right)^2 = \left(\sqrt[9]{a}\right)^2 = a^{\frac{2}{9}} = a^{\frac{1}{3}\times\frac{2}{3}} = a^{rs}$$

指数法則３

$$\left(ab\right)^r = \left(ab\right)^{\frac{1}{3}} = \sqrt[3]{ab} = \sqrt[3]{a}\times\sqrt[3]{b} = a^{\frac{1}{3}}\times b^{\frac{1}{3}} = a^r b^r$$

練習 9

⑴ $2^{\frac{3}{4}} \times 2^{\frac{1}{12}} \div 2^{\frac{1}{3}} = 2^{\frac{3}{4}+\frac{1}{12}-\frac{1}{3}} = 2^{\frac{9+1-4}{12}} = 2^{\frac{1}{2}}$

⑵ $\left(27^{\frac{4}{9}}\right)^{-\frac{3}{2}} = 27^{\frac{4}{9}\times\left(-\frac{3}{2}\right)} = 27^{-\frac{2}{3}} = \left(3^3\right)^{-\frac{2}{3}} = 3^{3\times\left(-\frac{2}{3}\right)} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

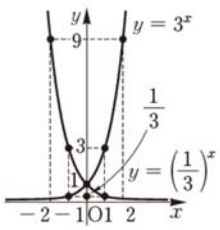
⑶ $\sqrt[4]{25} \times \sqrt[6]{125} = 25^{\frac{1}{4}} \times 125^{\frac{1}{6}} = \left(5^2\right)^{\frac{1}{4}} \times \left(5^3\right)^{\frac{1}{6}} = 5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5$

⑷ $\sqrt{a} \times \sqrt[6]{a^5} \times \sqrt[3]{a} = a^{\frac{1}{2}} \div a^{\frac{5}{6}} \times a^{\frac{1}{3}} = a^{\frac{1}{2}-\frac{5}{6}+\frac{1}{3}} = a\,\frac{3-5+2}{6} = a^0 = 1$

⑸ $ab^2 \times \sqrt[4]{a^{-3}b} \div \sqrt{ab^5} = ab^2 \times \left(a^{-3}b\right)^{\frac{1}{4}} \div \left(ab^5\right)^{\frac{1}{2}} = ab^2 \times a^{-\frac{3}{4}}b^{\frac{1}{4}} \div \left(a^{\frac{1}{2}}b^{\frac{5}{2}}\right)$

$= a^{1-\frac{3}{4}-\frac{1}{2}} \times b^{2+\frac{1}{4}-\frac{5}{2}} = a^{-\frac{1}{4}}b^{-\frac{1}{4}} = (ab)^{-\frac{1}{4}} = \frac{1}{(ab)^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{ab}}$

練習 10



練習 11

⑴ $\sqrt[4]{3^3} = 3^{\frac{3}{4}}$

$\sqrt[3]{3^4} = 3^{\frac{3}{4}}$

$\sqrt{3} = \sqrt[2]{3} = 3^{\frac{1}{2}}$

指数を比較すると $\frac{1}{2} < \frac{3}{4} < \frac{4}{3}$

底 3 は 1 より大きいので $3^{\frac{1}{2}} < 3^{\frac{3}{4}} < 3^{\frac{4}{3}}$

$\therefore\ \sqrt{3} < \sqrt[4]{3^3} < \sqrt[3]{3^4}$

⑵ $\sqrt{\frac{1}{2}} = \sqrt[2]{\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}$

$\sqrt[3]{\frac{1}{4}} = \sqrt[3]{\left(\frac{1}{2}\right)^2} = \left(\frac{1}{2}\right)^{\frac{2}{3}}$

$\sqrt[4]{\frac{1}{8}} = \sqrt[4]{\left(\frac{1}{2}\right)^3} = \left(\frac{1}{2}\right)^{\frac{3}{4}}$

指数を比較すると $\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$

底 $\frac{1}{2}$ は 1 より小さいので $\left(\frac{1}{2}\right)^{\frac{1}{2}} > \left(\frac{1}{2}\right)^{\frac{2}{3}} > \left(\frac{1}{2}\right)^{\frac{3}{4}}$

$\therefore\ \sqrt{\frac{1}{2}} > \sqrt[3]{\frac{1}{4}} > \sqrt[4]{\frac{1}{8}}$

練習 12

(1) $27^x = 81$

$\left(3^3\right)^x = 3^4$

$3^{3x} = 3^4$　したがって $3x = 4$

$\therefore\ x = \frac{4}{3}$

(2) $25^x = \frac{1}{\sqrt{5}}$

$\left(5^2\right)^x = \frac{1}{5^{\frac{1}{2}}}$

$5^{2x} = 5^{-\frac{1}{2}}$　したがって $2x = -\frac{1}{2}$

$\therefore\ x = -\frac{1}{4}$

(3) $\left(\frac{1}{2}\right)^{3x} = 16^{-x+2}$

$\left(2^{-1}\right)^{3x} = \left(2^4\right)^{-x+2}$

$2^{-3x} = 2^{4(-x+2)}$　したがって $-3x = -4x + 8$

$\therefore\ x = 8$

(4) $3^{-x} < 3\sqrt{3}$

$3^{-x} < 3^1 \times 3^{\frac{1}{2}}$

$3^{-x} < 3^{\frac{3}{2}}$

底3は1より大きいので $-x < \frac{3}{2}$

$\therefore\ x > -\frac{3}{2}$

(5) $4^{1-x} \geqq 128$

$\left(2^2\right)^{1-x} \geqq 2^7$

$2^{2(1-x)} \geqq 2^7$

底2は1より大きいので $2 - 2x \geqq 7$

$-2x \geqq 5$

$\therefore\ x \leqq -\frac{5}{2}$

(6) $5 \times \left(\frac{1}{5}\right)^{2x} \geqq \frac{1}{\sqrt[3]{5}}$

$5^1 \times \left(5^{-1}\right)^{2x} \geqq \frac{1}{5^{\frac{1}{3}}}$

$5^{1-2x} \geqq 5^{-\frac{1}{3}}$

底5は1より大きいので $1 - 2x \geqq -\frac{1}{3}$

$-2x \geqq -\frac{4}{3}$　$\therefore\ x \leqq \frac{2}{3}$

練習 13

(1) $2^{2x} - 9 \times 2^x + 8 = 0$

$\left(2x\right)^2 - 9 \times 2^x + 8 = 0$

$2^x = t\left(t > 0\right)$ とおく

$t^2 - 9t + 8 = 0$

$\left(t - 8\right)\left(t - 1\right) = 0$

$t > 0$ より $t = 8, 1$

$t = 8$ のとき $2^x = 8$

$2^x = 2^3$　$\therefore x = 3$

$t = 1$ のとき $2^x = 1$

$2^x = 2^0$　$\therefore x = 0$

よって $x = 0, 3$

(2) $9^x - 3^{x+1} - 54 = 0$

$\left(3^2\right)^x - 3^x \times 3^1 - 54 = 0$

$\left(3^x\right)^2 - 3 \times 3^x - 54 = 0$

$3^x = t$ とおく　$\left(t > 0\right)$

$t^2 - 3t - 54 = 0$

$\left(t - 9\right)\left(t + 6\right) = 0$

$t > 0$ より　$t = 9$

$3^x = 9$

$3^x = 3^2$　　$\therefore x = 2$

(3) $4^x-3\cdot 2^x+2 \leqq 0$

$\left(2^x\right)^2-3\cdot 2^x+2 \leqq 0$

$2^x=t\left(t>0\right)$ とおく

$t^2-3t+2 \leqq 0$

$\left(t-1\right)\left(t-2\right) \leqq 0$

$1 \leqq t \leqq 2$ より

$1 \leqq 2^x \leqq 2$

$2^0 \leqq 2^x \leqq 2^1$

底2は1より大きいので

$0 \leqq x \leqq 1$

(4) $3^{2x+1}+5\cdot 3^x-2 > 0$

$3\cdot\left(3^x\right)^2+5\cdot 3^x-2 > 0$

$3^x=t\left(t>0\right)$ とおく

$3t^2+5t-2>0$

$\left(3t-1\right)\left(t+2\right)>0$

$t+2>0$ だから $3t-1>0$

$t>\dfrac{1}{3}$ より $3^x>3^{-1}$

底3は1より大きいので

$x>-1$

節末問題

1．

⑴ $a^{\frac{5}{3}}\div a^{\frac{1}{6}}\times a^{\frac{1}{2}}=a^{\frac{5}{3}-\frac{1}{6}+\frac{1}{2}}=a^{\frac{10-1+3}{6}}=a^2$

⑵ $\sqrt[3]{a^2}\times\sqrt[4]{a}\div\sqrt[6]{a^5}=a^{\frac{2}{3}}\times a^{\frac{1}{4}}\div a^{\frac{5}{6}}=a^{\frac{2}{3}+\frac{1}{4}-\frac{5}{6}}=a^{\frac{8+3-10}{12}}=a^{\frac{1}{12}}$

⑶ $\sqrt[3]{\sqrt[4]{a\sqrt[3]{a}}}=\sqrt[12]{a\times\sqrt[3]{a}}=\sqrt[12]{a^1\times a^{\frac{1}{3}}}=\sqrt[12]{a^{\frac{4}{3}}}=\left(a^{\frac{4}{3}}\right)^{\frac{1}{12}}=a^{\frac{1}{9}}$

⑷ $\dfrac{a\sqrt[4]{a}}{\sqrt{a\sqrt{a}}}=\dfrac{a^1\times a^{\frac{1}{4}}}{\left(a^1\times a^{\frac{1}{2}}\right)^{\frac{1}{2}}}=\dfrac{a^{\frac{5}{4}}}{a^{\frac{3}{4}}}=a^{\frac{5}{4}-\frac{3}{4}}=a^{\frac{1}{2}}$

2．

⑴ $2^{\frac{1}{2}}\times 8^{\frac{1}{2}}\times 4^{\frac{1}{12}}=2^{\frac{1}{3}}\times\left(2^3\right)^{\frac{1}{2}}\times\left(2^2\right)^{\frac{1}{12}}=2^{\frac{1}{3}}\times 2^{\frac{3}{2}}\times 2^{\frac{2}{12}}=2^{\frac{1}{3}+\frac{3}{2}+\frac{1}{6}}=2^{\frac{12}{6}}=4$

⑵ $6^{\frac{1}{2}}\times 12^{\frac{1}{4}}\div 9^{\frac{1}{4}}=\left(2\times 3\right)^{\frac{1}{2}}\times\left(2^2\times 3\right)^{\frac{1}{4}}\div\left(3^2\right)^{\frac{1}{4}}$

$=2^{\frac{1}{2}}\times 3^{\frac{1}{2}}\times 2^{\frac{2}{4}}\times 3^{\frac{1}{4}}\div 3^{\frac{2}{4}}=2^{\frac{1}{2}+\frac{1}{2}}\times 3^{\frac{1}{2}+\frac{1}{4}-\frac{1}{2}}=2^1\times 3^{\frac{1}{4}}=2\sqrt[4]{3}$

⑶ $\left\{\left(\dfrac{16}{25}\right)^{\frac{3}{4}}\right\}^{-\frac{2}{3}}=\left(\dfrac{16}{25}\right)^{\frac{3}{4}\times\left(-\frac{2}{3}\right)}=\dfrac{1}{\left(\dfrac{16}{25}\right)^{\frac{1}{2}}}=\dfrac{1}{\sqrt{\dfrac{16}{25}}}=\dfrac{1}{\left(\dfrac{4}{5}\right)}=\dfrac{5}{4}$

⑷ $\left(\sqrt[3]{2}+1\right)\left(\sqrt[3]{4}-\sqrt[3]{2}+1\right)=\left(\sqrt[3]{2}+1\right)\left\{\left(\sqrt[3]{2}\right)^2-\sqrt[3]{2}+1\right\}=\left(\sqrt[3]{2}\right)^3+1^3=2+1=3$

3．

(1) $8^{\frac{1}{2}}=\left(2^3\right)^{\frac{1}{2}}=2^{\frac{3}{2}}$

$4^{-\frac{2}{3}}=\left(2^2\right)^{-\frac{2}{3}}=2^{-\frac{4}{3}}$

$\left(2\sqrt{2}\right)^3=\left(2^1\times 2^{\frac{1}{2}}\right)^3=2^{\frac{3}{2}\times 3}=2^{\frac{9}{2}}$

$16^{\frac{3}{4}}=\left(2^4\right)^{\frac{3}{4}}=2^3$

$-\dfrac{4}{3}<0<\dfrac{3}{2}<3<\dfrac{9}{2}$ でかつ

底 2 が 1 より大きいので

$2^{-\frac{4}{3}}<2^0<2^{\frac{3}{2}}<2^3<2^{\frac{9}{2}}$

よって，$4^{-\frac{2}{3}}<1<8^{\frac{1}{2}}<16^{\frac{3}{4}}<\left(2\sqrt{2}\right)^3$

(2) $\left(\sqrt{2}\right)^6=2^3=8,\ \left(\sqrt[3]{3}\right)^6=3^2=9$

$\therefore\ \sqrt{2}<\sqrt[3]{3}$ ……①

$\left(\sqrt{2}\right)^{10}=2^5=32,\ \left(\sqrt[5]{5}\right)^{10}=5^2=25$

$\therefore\ \sqrt[5]{5}<\sqrt{2}$ ……②

よって，①，② より $\sqrt[5]{5}<\sqrt{2}<\sqrt[3]{3}$

4．

(1) $5^x=25\sqrt{5}$

$5^x=5^2\times 5^{\frac{1}{2}}$

$5^x=5^{\frac{5}{2}}\quad\therefore\ x=\dfrac{5}{2}$

(2) $16\cdot 8^x=1$

$2^4\times\left(2^3\right)^x=2^0$

$2^{4+3x}=2^0$

$4+3x=0\quad\therefore\ x=-\dfrac{4}{3}$

(3) $\left(\dfrac{1}{9}\right)^x=3^{x+3}$

$\left(3^{-2}\right)^x=3^{x+3}$

$3^{-2x}=3^{x+3}$

$-2x=x+3\quad\therefore\ x=-1$

5．

(1) $3^{3x-1}<\sqrt{3}$

$3^{3x-1}<3^{\frac{1}{2}}$

底 3 は 1 より大きいので

$3x-1<\dfrac{1}{2}$

$3x<\dfrac{3}{2}\quad\therefore\ x<\dfrac{1}{2}$

(2) $\left(\dfrac{1}{2}\right)^3<\left(\dfrac{1}{2}\right)^x<1$

$\left(\dfrac{1}{2}\right)^3<\left(\dfrac{1}{2}\right)^x<\left(\dfrac{1}{2}\right)^0$

底 $\dfrac{1}{2}$ は 1 より小さいので

$3>x>0\quad\therefore\ 0<x<3$

(3) $\left(\dfrac{1}{3}\right)^{2x-1}>27^{2-x}$

$\left(3^{-1}\right)^{2x-1}>\left(3^3\right)^{2-x}$

$3^{-2x+1}>3^{6-3x}$

底 3 は 1 より大きいので

$-2x+1>6-3x$

$\therefore\ x>5$

６．

⑴ $9^{x+\frac{1}{2}}-10\cdot 3^x+3=0$

$\left(3^2\right)^{x+\frac{1}{2}}-10\times 3^x+3=0$

$\left(3^x\right)^2\times 3^1-10\times 3^x+3=0$

3^x-t とする　$(t>0)$

$3t^2-10t+3=0$

$(3t-1)(t-3)=0$

$\therefore t=\dfrac{1}{3},\ 3$

$t=\dfrac{1}{3}$ のとき $3^x=\dfrac{1}{3}$　　$\therefore x=-1$

$t=3$ のとき $3^x=3$　　$\therefore x=1$

よって　$x=\pm 1$

⑵ $\left(\dfrac{1}{4}\right)^x-\left(\dfrac{1}{2}\right)^{x-1}-8\geqq 0$

$\left(\dfrac{1}{2}\right)^{2x}-\left(\dfrac{1}{2}\right)^x\times\left(\dfrac{1}{2}\right)^{-1}-8\geqq 0$

$\left\{\left(\dfrac{1}{2}\right)^x\right\}^2-2\times\left(\dfrac{1}{2}\right)^x-8\geqq 0$

$\left(\dfrac{1}{2}\right)^x=t$ とする $(t>0)$

$t^2-2t-8\geqq 0$

$(t-4)(t+2)\geqq 0$

$t>0$ より　$t\geqq 4$

$\left(\dfrac{1}{2}\right)^x\geqq 4$

$\left(2^{-1}\right)^x\geqq 2^2$

$2^{-x}\geqq 2^2$

底 2 は 1 より大きいので

$-x\geqq 2$

よって　$x\leqq -2$

７．

⑴ $9^x+9^{-x}=\left(3^2\right)^x+\left(3^2\right)^{-x}=\left(3^x\right)^2+\left(3^{-x}\right)^2$

$=\left(3^x+3^{-x}\right)^2-2\times 3^x\times 3^{-x}=3^2-2\times 3^{x+(-x)}=9-2\times 3^0=9-2\times 1=7$

⑵ $27^x+27^{-x}=\left(3^3\right)^x+\left(3^3\right)^{-x}=\left(3^x\right)^3+\left(3^{-x}\right)^3$

$=\left(3^x+3^{-x}\right)\left(3^{2x}-3^x\times 3^{-x}+3^{-2x}\right)=3\left(3^{2x}+3^{-2x}-3^{x+(-x)}\right)$

$=3\left(9^x+9^{-x}-3^0\right)=3\left(7-1\right)=18$