６章　三角関数　解答

３節　三角関数の加法定理

練習１

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos 75^\circ = \cos(30^\circ + \cos 45^\circ) = \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\tan 105^\circ = \tan(60^\circ + 45^\circ) = \frac{\tan 60^\circ + \tan 45^\circ}{1 - \tan 60^\circ \tan 45^\circ}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}\times 1} = \frac{(\sqrt{3}+1)(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \frac{3+2\sqrt{3}+1}{-2} = -2-\sqrt{3}$$

練習２

$\sin^2\alpha + \cos^2\alpha = 1$ より

$$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(\frac{2}{7}\right)^2 = \frac{45}{49}$$

α は第１象限の角だから $\cos\alpha > 0 \quad \therefore \cos\alpha = \dfrac{3\sqrt{5}}{7}$

$$\sin^2\beta = 1 - \cos^2\beta = 1 - \left(-\frac{2}{3}\right)^2 = \frac{5}{9}$$

β は第２象限の角だから $\sin\beta = \dfrac{\sqrt{5}}{3}$

⑴　$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \dfrac{2}{7} \times \left(-\dfrac{2}{3}\right) - \dfrac{3\sqrt{5}}{7} \times \dfrac{\sqrt{5}}{3} = -\dfrac{11}{21}$

⑵　$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \dfrac{3\sqrt{5}}{7} \times \left(-\dfrac{2}{3}\right) - \dfrac{2}{7} \times \dfrac{\sqrt{5}}{3} = -\dfrac{8\sqrt{5}}{21}$

練習３

$\sin^2\alpha + \cos^2\theta = 1$ より

$$\sin^2\alpha = 1 - \cos^2\alpha = 1 - \left(-\frac{\sqrt{3}}{3}\right)^2 = \frac{2}{3}$$

$\pi < \alpha < \dfrac{3}{2}\pi$ より $\sin\alpha < 0 \quad \therefore \sin\alpha = -\dfrac{\sqrt{6}}{3}$

⑴　$\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \times \left(-\dfrac{\sqrt{6}}{3}\right) \times \left(-\dfrac{\sqrt{3}}{3}\right) = \dfrac{2\sqrt{2}}{3}$

⑵　$\cos 2\alpha = 2\cos^2\alpha - 1 = 2 \times \left(-\dfrac{\sqrt{3}}{3}\right)^2 - 1 = -\dfrac{1}{3}$

⑶　$\tan 2\alpha = \dfrac{\sin 2\alpha}{\cos 2\alpha} = \dfrac{\frac{2\sqrt{2}}{3}}{\left(-\frac{1}{3}\right)} = -2\sqrt{2}$

練習４

⑴ $\sin^2\frac{3}{8}\pi=\sin^2\left(\frac{3}{4}\pi\right)=\frac{1-\cos\frac{3}{4}\pi}{2}=\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}=\frac{2+\sqrt{2}}{4}$

$\sin\frac{3}{8}\pi>0$ より $\sin\frac{3}{8}\pi=\frac{\sqrt{2+\sqrt{2}}}{2}$

⑵ $\cos^2\frac{5}{8}\pi=\cos^2\left(\frac{5}{4}\pi\right)=\frac{1-\cos\frac{5}{4}\pi}{2}=\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}=\frac{2-\sqrt{2}}{4}$

$\cos\frac{5}{8}\pi<0$ より $\cos\frac{5}{8}\pi=-\frac{\sqrt{2-\sqrt{2}}}{2}$

⑶ $\tan^2\frac{7}{8}\pi=\frac{1-\cos\frac{7}{4}\pi}{1+\cos\frac{7}{4}\pi}=\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}=\frac{\sqrt{2}-1}{\sqrt{2}+1}$

$\tan\frac{7}{8}\pi<0$ だから

$\tan\frac{7}{8}\pi=-\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}=\sqrt{3-2\sqrt{2}}$

$=-\left(\sqrt{2}-1\right)=1-\sqrt{2}$

練習５

⑴ $\sin^2\frac{\alpha}{2}=\frac{1-\cos\alpha}{2}=\frac{1-\frac{2}{3}}{2}=\frac{1}{6}$

$\pi<\alpha<2\pi$ より $\frac{\pi}{2}<\frac{\alpha}{2}<\pi$

このとき $\sin\frac{\alpha}{2}>0$　　$\therefore\sin\frac{\alpha}{2}=\frac{\sqrt{6}}{6}$

⑵ $\cos^2\frac{\alpha}{2}=\frac{1+\cos\alpha}{2}=\frac{1+\frac{2}{3}}{2}=\frac{5}{6}$

$\frac{\pi}{2}<\frac{\alpha}{2}<\pi$ より $\cos\frac{\alpha^2}{2}<0$

$\therefore\cos\frac{\alpha}{2}=-\frac{\sqrt{30}}{6}$

⑶ $\tan^2\frac{\alpha}{2}=\frac{1-\sin\alpha}{1+\cos\alpha}=\frac{1-\frac{2}{3}}{1+\frac{2}{3}}=\frac{1}{5}$

$\frac{\pi}{2}<\frac{\alpha}{2}<\pi$ より $\tan\frac{\alpha}{2}<0$

$\therefore\tan\frac{\alpha}{2}=-\frac{\sqrt{5}}{5}$

練習 6

(1) $\tan\theta = \dfrac{2\tan\dfrac{\theta}{2}}{1-\tan^2\dfrac{\theta}{2}} = \dfrac{2t}{1-t^2}$

(2) $1+\tan^2\dfrac{\theta}{2} = \dfrac{1}{\cos^2\dfrac{\theta}{2}}$ より $1+t^2 = \dfrac{1}{\cos^2\dfrac{\theta}{2}}$

$\cos^2\dfrac{\theta}{2} = \dfrac{1}{1+t^2}$

$\cos\theta = 2\cos^2\dfrac{\theta}{2} - 1 = 2\cdot\dfrac{1}{1+t^2} - 1 = \dfrac{1-t^2}{1+t^2}$

(3) $\sin\theta = \cos\theta\tan\theta$

$= \dfrac{1-t^2}{1+t^2}\cdot\dfrac{2t}{1-t^2}$

$= \dfrac{2t}{1-t^2}$

練習 7

(1) $\sin\theta + \sqrt{3}\cos\theta = \sqrt{1^2+\sqrt{3}^2}\sin\left(\theta+\dfrac{\pi}{3}\right) = 2\sin\left(\theta+\dfrac{\pi}{3}\right)$

(2) $\sin\theta - \cos\theta = \sqrt{1^2+(-1)^2}\sin\left(\theta-\dfrac{\pi}{4}\right) = \sqrt{2}\sin\left(\theta-\dfrac{\pi}{4}\right)$

練習 8

(1) $\sin\theta\cos5\theta = \dfrac{1}{2}\{\sin(\theta+5\theta)+\sin(\theta-5\theta)\}$

$= \dfrac{1}{2}(\sin6\theta - \sin4\theta)$

(2) $\cos4\theta\cos2\theta = \dfrac{1}{2}\{\cos(4\theta+2\theta)+\cos(4\theta-2\theta)\}$

$= \dfrac{1}{2}(\cos6\theta + 2\theta)$

練習 9

(1) $\sin6\theta - \sin2\theta = 2\cos\dfrac{6\theta+2\theta}{2}\sin\dfrac{6\theta-2\theta}{2} = 2\cos4\theta\sin2\theta$

(2) $\cos\theta + \cos7\theta = 2\cos\dfrac{\theta+7\theta}{2}\cos\dfrac{\theta-7\theta}{2} = 2\cos4\theta\cos3\theta$

練習 10

⑴　$\sin 75^\circ \sin 15^\circ$

$= -\frac{1}{2}\{\cos(75^\circ+15^\circ)-\cos(75^\circ-15^\circ)\} = -\frac{1}{2}(\cos 90^\circ - \cos 60^\circ) = -\frac{1}{2}\times\left(0-\frac{1}{2}\right) = \frac{1}{4}$

⑵　$\cos 10^\circ + \cos 110^\circ + \cos 130^\circ$

$= 2\cos\frac{110^\circ+10^\circ}{2}\cos\frac{110^\circ-10^\circ}{2} + \cos 130^\circ = 2\cos 60^\circ \cos 50^\circ + \cos 130^\circ = \cos 50^\circ + \cos 130^\circ$

$= 2\cos\frac{50^\circ+130^\circ}{2}\cos\frac{130^\circ-50^\circ}{2} = 2\cos 90^\circ \cos 40^\circ = 0$

練習 11

⑴

$y=\sin\left(\theta+\frac{5}{12}\pi\right)\cos\left(\theta+\frac{\pi}{12}\right)$

$=\frac{1}{2}\left\{\sin\left(2\theta+\frac{\pi}{2}\right)+\sin\frac{4}{12}\pi\right\}$

$=\frac{1}{2}\left(\cos 2\theta+\frac{\sqrt{3}}{2}\right)$

$0\leqq\theta\leqq\pi$ より　$0\leqq 2\theta\leqq 2\pi$

$\therefore\ -1\leqq\cos 2\theta\leqq 1$　だから　$\frac{-1+\sqrt{3}}{4}\leqq y\leqq\frac{1+\sqrt{3}}{4}$

⑵

$y=\cos\left(\theta+\frac{5}{12}\pi\right)-\cos\left(\theta+\frac{\pi}{12}\right)$

$=-2\sin\left(\theta+\frac{\pi}{4}\right)\sin\frac{\pi}{6}$

$=-\sin\left(\theta+\frac{\pi}{4}\right)$

$0\leqq\theta\leqq\pi$ より　$\frac{\pi}{4}\leqq\theta+\frac{\pi}{4}\leqq\frac{5}{4}\pi$

$\therefore\ -\frac{\sqrt{3}}{2}\leqq\sin\left(\theta+\frac{\pi}{4}\right)\leqq 1$　だから　$-1\leqq y\leqq\frac{\sqrt{2}}{2}$

節末問題

１．　$y=3x$ と x 軸とのなす角を α，$y=\dfrac{1}{2}x$ と x 軸とのなす角を β とする。

このとき $\tan\alpha=3,\ \tan\beta=\dfrac{1}{2}$ とするとき

$$\tan\theta=\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{3-\frac{1}{2}}{1+3\times\frac{1}{2}}=1$$

$0\leqq\theta\leqq\dfrac{\pi}{2}$ より $\theta=\dfrac{\pi}{4}$

２．$\sin^2\alpha+\cos^2\alpha=1$ より $\cos^2\alpha=1-\sin^2\alpha=1-\left(\dfrac{1}{3}\right)^2=\dfrac{8}{9}$

$\dfrac{\pi}{2}<\alpha<\pi$ より $\cos\alpha<0$ $\therefore\cos\alpha=-\dfrac{2\sqrt{2}}{3}$

(1) $\sin 2\alpha=2\sin\alpha\cos\alpha=2\bullet\left(\dfrac{1}{3}\right)\bullet\left(-\dfrac{2\sqrt{2}}{3}\right)=\dfrac{-4\sqrt{2}}{9}$

(2) $\cos 2\alpha=2\cos^2\alpha-1=2\bullet\left(-\dfrac{2\sqrt{2}}{3}\right)^2-1=\dfrac{7}{9}$

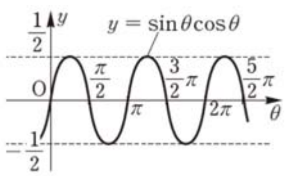
(3) $\sin^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{2}=\dfrac{1-\left(-\frac{2\sqrt{2}}{3}\right)}{2}=\dfrac{3+2\sqrt{2}}{6}$

(4) $\tan^2\dfrac{\alpha}{2}=\dfrac{1-\cos\alpha}{1+\cos\alpha}=\dfrac{1-\left(-\frac{2\sqrt{2}}{3}\right)}{1+\left(-\frac{2\sqrt{2}}{3}\right)}=\dfrac{3+2\sqrt{2}}{3-2\sqrt{2}}=\left(3+2\sqrt{2}\right)^2$

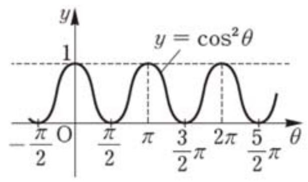
$\tan\dfrac{\alpha}{2}>0$ より $\tan\dfrac{\alpha}{2}=3+2\sqrt{2}$

３．

(1) $y=\sin\theta\cos\theta=\dfrac{1}{2}\sin 2\theta$



(2) $y=\cos^2\theta=\dfrac{1}{2}\cos 2\theta+\dfrac{1}{2}$



４．

⑴ $\sin 3\alpha = \sin\left(2\alpha + \alpha\right) = \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha$

$= 2\sin\alpha\cos^2\alpha + \left(1 - 2\sin^2\alpha\right)\sin\alpha$

$= 2\sin\alpha\left(1 - \sin^2\alpha\right) + \sin\alpha - 2\sin^3\alpha$

$= 3\sin\alpha - 4\sin^3\alpha$

よって，成り立つ。

⑵ $\cos 3\alpha = \cos\left(2\alpha + \alpha\right) = \cos 2\alpha\cos\alpha - \sin 2\alpha \sin\alpha$

$= \left(2\cos^2\alpha - 1\right)\cos\alpha - 2\sin^2\alpha\cos\alpha$

$= 2\cos^3\alpha - \cos\alpha - 2\left(1 - \cos^2\alpha\right)\cos\alpha$

$= 4\cos^2\alpha - 3\cos\alpha$

よって，成り立つ。

５．

⑴ $y = 2\sin\theta + 3\cos\theta = \sqrt{2^2 + 3^2}\,\sin\left(\theta + \alpha\right) = \sqrt{13}\,\sin\left(\theta + \alpha\right)$

三平方の定理より　$r^2 = 2^2 + 3^2 = 4 + 9 = 13$　$r > 0$　より　$r = \sqrt{13}$

$\cos\alpha = \dfrac{2}{\sqrt{13}}$，$\sin\alpha = \dfrac{3}{\sqrt{13}}$

⑵ 最大値　$\sqrt{13}$　　最小値　-3

６．

⑴ $\sin 2\theta = \sqrt{3}\sin\theta$

$2\sin\theta\cos\theta = \sqrt{3}\sin\theta$

$\sin\theta\left(2\cos\theta - \sqrt{3}\right) = 0$

$\sin\theta = 0$　または　$\cos\theta = \dfrac{\sqrt{3}}{2}$

$\therefore \theta = 0,\ \dfrac{\pi}{6},\ \pi,\ \dfrac{11}{6}\pi$

⑵ $\cos 2\theta + 5\cos\theta = 2$

$2\cos^2\theta - 1 + 5\cos\theta - 2 = 0$

$2\cos^2\theta + 5\cos\theta - 3 = 0$

$\left(2\cos\theta - 1\right)\left(\cos\theta + 3\right) = 0$

$\cos\theta = \dfrac{1}{2}$　または　$\cos\theta = -3$

$-1 \leqq \cos\theta \leqq 1$　より　$\cos\theta = \dfrac{1}{2}$

$\therefore \theta = \dfrac{\pi}{3},\ \dfrac{5}{3}\pi$

⑶ $\cos 2\theta > \sin\theta$

$1 - 2\sin^2\theta + \sin\theta$

$2\sin^2\theta + \sin\theta - 1 < 0$

$\left(2\sin\theta - 1\right)\left(\sin\theta + 1\right) < 0$

$-1 < \sin\theta < \dfrac{1}{2}$

$0 \leqq \theta < \dfrac{\pi}{6},\ \dfrac{5}{6}\pi < \theta < \dfrac{3}{2}\pi,\ \dfrac{3}{2}\pi < \theta < 2\pi$

⑷ $-\sin\theta + \sqrt{3}\cos\theta < \sqrt{2}$

$\sqrt{(-1)^2 + \sqrt{3}^{\,2}}\,\sin\left(\theta + \dfrac{2}{3}\pi\right) < \sqrt{2}$

$\sin\left(\theta + \dfrac{2}{3}\pi\right) < \dfrac{\sqrt{2}}{2}$

$\dfrac{\pi}{12} < \theta < \dfrac{19}{12}\pi$

7．$\sin\theta + \cos\theta = t$ より

$(\sin\theta + \cos\theta)^2 = t^2$

$(\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta = t^2$

$1 + 2\sin\theta\cos\theta = t^2$

⑴ $y = \sin 2\theta + 2\sin\theta + 2\cos\theta + 3 = 2\sin\theta\cos\theta + 2(\sin\theta + \cos\theta) + 3$

$= (t^2 - 1) + 2t + 3 = t^2 + 2t + 2$

⑵ $\sin\theta + \cos\theta = \sqrt{1^2 + 1^2}\sin\left(\theta + \dfrac{\pi}{4}\right) = \sqrt{2}\sin\left(\theta + \dfrac{\pi}{4}\right)$

よって $-\sqrt{2} \leqq \sin\theta + \cos\theta \leqq \sqrt{2}$

$-\sqrt{2} \leqq t \leqq \sqrt{2}$

⑶ $y = t^2 + 2t + 2 = (t^2 + 2t + 1) + 1 = (t+1)^2 + 1$

$-\sqrt{2} \leqq t \leqq \sqrt{2}$ において $t = -1$ のとき最小値 1 をとる

このとき $\sqrt{2}\sin\left(\theta + \dfrac{\pi}{4}\right) = -1$

$\sin\left(\theta + \dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$

$\therefore \theta = \pi,\ \dfrac{3\pi}{2}$

$t = \sqrt{2}$ のとき最大値 $4 + 2\sqrt{2}$ をとる

このとき $\sqrt{2}\sin\left(\theta + \dfrac{\pi}{4}\right) = \sqrt{2}$

$\sin\left(\theta + \dfrac{\pi}{4}\right) = 1$

$\therefore \theta = \dfrac{\pi}{4}$