4 章 フーリエ解析

1 節 フーリエ級数

P. 213 練習 $\boxed{1}$

$f(x+T)=f(x)$, $g(x+T)=g(x)$ より $af(x+T)+bg(x+T)=af(x)+bg(x)$ が成り立つから，
$af(x)+bg(x)$ も周期 T の周期関数である。

P. 213 練習 $\boxed{2}$

$f(x+T)=f(x)$, $g(x+T)=g(x)$ より $f(x+T)\cdot g(x+T)=f(x)\cdot g(x)$ が成り立つから，
$f(x)\cdot g(x)$ も周期 T の周期関数である。

P. 213 練習 $\boxed{3}$

$f(x+2\pi)=C=f(x)$ であるから，周期 2π の周期関数でもある。

P. 215 練習 $\boxed{4}$

$$\int_{-\pi}^{\pi}\cos nx\cos kx\,dx=\frac{1}{2}\int_{-\pi}^{\pi}\left(\cos(n+k)x+\cos(n-k)x\right)dx \qquad \leftarrow \boxed{10}(2)$$

$$=\begin{cases}0 & (n\neq k)\\ \dfrac{1}{2}\int_{-\pi}^{\pi}\left(\cos(2n)x+1\right)dx \overset{[3]}{\underset{\downarrow}{=}} \dfrac{1}{2}\int_{-\pi}^{\pi}dx=\left[x\right]_0^{\pi}=\pi & (n=k)\end{cases}$$

$$=\pi\delta_{nk}$$

P. 217 練習 $\boxed{5}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right),\quad g(x)=\frac{c_0}{2}+\sum_{n=1}^{\infty}\left(c_n\cos nx+d_n\sin nx\right),$$

$f(x)+g(x)=\dfrac{\alpha_0}{2}+\displaystyle\sum_{n=1}^{\infty}\left(\alpha_n\cos nx+\beta_n\sin nx\right)$ と表せたとすると，フーリエ係数の定義から

$$\alpha_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(f(x)+g(x)\right)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx+\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\cos nx\,dx$$
$$=a_n+c_n$$

同様に $\beta_n=b_n+d_n$ となる。よって

$$\begin{aligned}f(x)+g(x)&=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty}\left(\alpha_n\cos nx+\beta_n\sin nx\right)\\&=\frac{a_0+c_0}{2}+\sum_{n=1}^{\infty}\left\{\left(a_n+c_n\right)\cos nx+\left(b_n+d_n\right)\sin nx\right\}\\&=\frac{a_n}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)+\frac{c_0}{2}+\sum_{n=1}^{\infty}\left(c_n\cos nx+d_n\sin nx\right)\end{aligned}$$

これは $f(x)+g(x)$ のフーリエ級数が $f(x)$ のフーリエ級数と $g(x)$ のフーリエ級数の和であることを示している。

P. 218 練習 $\boxed{6}$

(1)　$f(x)$ は奇関数と考えることができるから $a_n = 0$，　$f(x)\sin nx$ は偶関数なので，

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{2}{n\pi}\left(-\cos n\pi + \cos 0\right) = \frac{2}{n\pi}\left(1-(-1)^n\right)$$

$$f(x) \sim \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1-(-1)^n\right)\sin nx = \sum_{n=1}^{\infty}\frac{2}{(2n-1)\pi}\cdot 2\sin(2n-1) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)}{2n-1}$$

(2)

［1］$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi}\cos nx\, dx$$

（i）　$n = 0$ のとき　$a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} 1\, dx = 1$

（ii）　$n \neq 0$ のとき　$a_n = \dfrac{1}{\pi}\left[\dfrac{1}{n}\sin nx\right]_0^{\pi} = 0$　$(n = 1,\ 2,\ \cdots)$

［2］$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$
$$= \frac{1}{\pi}\int_0^{\pi}\sin nx\, dx = \frac{1}{\pi}\left[\frac{-1}{n}\cos nx\right]_0^{\pi}$$
$$= \frac{-1}{n\pi}\left(\cos n\pi - 1\right) = \frac{1}{n\pi}\left\{1-(-1)^n\right\} \qquad (n = 1,\ 2,\ \cdots)$$

［1］［2］より　$f(x) \sim \dfrac{1}{2} + \displaystyle\sum_{n=1}^{\infty}\frac{1}{n\pi}\left\{1-(-1)^n\right\}\sin nx$

P. 217 練習 $\boxed{7}$

(1)　$a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} x\, dx = \frac{\pi}{2}$

$n \neq 0$ のとき

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} x\left(\frac{\sin nx}{n}\right)' dx = \frac{1}{\pi}\left(\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\, dx\right) \leftarrow \boxed{31}$$
$$= \frac{1}{\pi}\left(\left[x\frac{\sin nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right) = \frac{1}{n^2\pi}\left(\cos n\pi - 1\right) = \frac{1}{n^2\pi}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} x\left(-\frac{\cos nx}{n}\right)' dx = \frac{1}{\pi}\left(\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\, dx\right) \leftarrow \boxed{31}$$
$$= \frac{1}{\pi}\left(\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right) = \frac{1}{n}(-\pi)\frac{\cos n\pi}{n} = \frac{(-1)^{n+1}}{n}$$

よって　$f(x) = \dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi}\left((-1)^n - 1\right)\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right)$

(2)

［1］ $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= \dfrac{1}{\pi}\displaystyle\int_{0}^{\pi} (\pi - x)\cos nx\,dx$

(ｉ) $n = 0$ のとき

$$a_0 = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,dx$$
$$= \frac{1}{\pi}\left[\pi x - \frac{1}{2}x^2\right]_0^{\pi} = \frac{1}{\pi}\left(\pi^2 - \frac{1}{2}\pi^2\right)$$
$$= \frac{1}{2}\pi$$

(ⅱ) $n \neq 0$ のとき

$$a_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

← 31 で $f = \pi - x$　$g' = \cos nx$

$$= \frac{1}{\pi}\left\{\left[(\pi - x)\frac{1}{n}\sin nx\right]_0^{\pi} - \int_0^{\pi}(-1)\frac{1}{n}\sin nx\,dx\right\}$$
$$= \frac{1}{\pi}\cdot\frac{1}{n}\left[\frac{-1}{n}\cos nx\right]_0^{\pi}$$
$$= \frac{1}{n^2\pi}\left\{1 - (-1)^n\right\} \quad (n = 1,\ 2,\ 3,\ \cdots)$$

［2］ $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= \dfrac{1}{\pi}\displaystyle\int_0^{\pi}(\pi - x)\sin nx\,dx$

← 31 で $f = \pi - x$　$g' = \sin nx$

$$= \frac{1}{\pi}\left\{\left[(\pi - x)\frac{-1}{n}\cos nx\right]_0^{\pi} - \int_0^{\pi}(-1)\frac{-1}{n}\cos nx\,dx\right\}$$
$$= \frac{1}{\pi}\left\{-\pi\frac{-1}{n}\cdot 1 - \frac{1}{n}\left[\frac{-1}{n}\sin nx\right]_0^{\pi}\right\}$$
$$= \frac{1}{\pi} \quad (n = 1,\ 2,\ 3,\ \cdots)$$

［1］［2］より $f(x) \sim \dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}\left\{1 - (-1)^n\right\}\cos nx + \frac{1}{n}\sin nx\right]$

P. 225 練習 $\boxed{8}$

(1) $f(x)$ は奇関数と考えることができるから，フーリエ正弦級数を求めればよい。

ここで，周期 $2L = 4$ より $L = 2$ である。

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx = \left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$
$$= -\frac{2}{n\pi}(\cos n\pi - \cos 0) = \frac{2}{n\pi}\left(1 - (-1)^n\right)$$

$$f(x) \sim \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1 - (-1)^n\right)\sin\left(\frac{n\pi x}{2}\right)$$

P. 227 練習 $\boxed{9}$

(1)　$n=0$ のとき　$c_0=\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} x\,dx=0$

$n\neq 0$ のとき

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi} x\left(-\frac{1}{in}e^{-inx}\right)'dx \qquad \leftarrow \boxed{31}\ \text{で}\ f=x \quad g'=e^{in\pi x}$$

$$=\frac{1}{2\pi}\left(\left[-\frac{1}{in}xe^{-inx}\right]_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\frac{1}{in}e^{-inx}\,dx\right)$$

$$=\frac{1}{2\pi}\left(\left[-\frac{1}{in}xe^{-inx}\right]_{-\pi}^{\pi}+\left[-\frac{1}{(in)^2}e^{-inx}\right]_{-\pi}^{\pi}\right)$$

$$=\frac{1}{2\pi}\left(-\frac{1}{in}\pi e^{-in\pi}+\frac{1}{in}(-\pi)e^{in\pi}+\frac{1}{n^2}e^{-in\pi}-\frac{1}{n^2}e^{in\pi}\right)$$

ここで，$e^{\pm in\pi}=\cos(\pm n\pi)+i\sin(\pm n\pi)=(-1)^n$ であるから

$$c_n=-\frac{1}{in}(-1)^n=\frac{i}{n}(-1)^n$$

$$f(x)\sim\sum_{n=1}^{\infty}\frac{i}{n}(-1)^n e^{inx}$$

(2)　$n=0$ のとき　$c_0=\dfrac{1}{2}\displaystyle\int_{-1}^{1}x^2\,dx=\dfrac{1}{2}\left[\dfrac{1}{3}x^3\right]_{-1}^{1}=\dfrac{1}{6}\left\{1^3-(-1)^3\right\}=\dfrac{1}{3}$

$n\neq 0$ のとき

$$c_n=\frac{1}{2}\int_{-1}^{1}x^2e^{-in\pi x}\,dx \qquad \leftarrow \boxed{31}\ \text{で}\ f=x^2 \quad g'=e^{-in\pi x}$$

$$=\frac{1}{2}\left\{\left[x^2\cdot\frac{1}{-in\pi}e^{-in\pi x}\right]_{-1}^{1}-\int_{-1}^{1}2x\cdot\frac{1}{-in\pi}e^{-in\pi x}\,dx\right\}$$

$\left(e^{-in\pi}=(-1)^n,\quad e^{in\pi}=(-1)^n\quad \text{より第1項}=0\right)$　⊛

$$=\frac{1}{in\pi}\left\{\left[x\cdot\frac{1}{-in\pi}e^{-in\pi x}\right]_{-1}^{1}-\int_{1}^{1}1\cdot\frac{1}{-in\pi}e^{-in\pi x}\,dx\right\} \qquad \leftarrow \boxed{31}\ \text{で}\ f=x^2 \quad g'=e^{-in\pi x}$$

$$=\frac{1}{in\pi}\left\{\frac{1}{-in\pi}\cdot 2(-1)^n+\frac{1}{in\pi}\cdot\left[\frac{1}{-in\pi}e^{-in\pi x}\right]_{-1}^{1}\right\}$$

$$=\frac{2}{n^2\pi^2}(-1)^n$$　（⊛ と同じ理由で第２項＝0）

よって　$f(x)\sim\dfrac{1}{3}+\displaystyle\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\dfrac{2}{n^2\pi^2}(-1)^n e^{in\pi x}$

P. 229 練習 $\boxed{10}$

⑴　$\lambda > 0$ のとき $X''(x) = \lambda X(x)$ の一般解は A，B を定数として $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ ← 54
となるが，⑧から $X(0) = 0$ より $A + B = 0$ また，⑧から $X(1) = 0$ より $Ae^{\sqrt{\lambda}} + Be^{-\sqrt{\lambda}} = 0$
であるので $A = 0 = B$ となり，結局 $X(x) = 0$。

⑵　$\lambda = 0$ のとき $X''(x) = \lambda X(x) \Rightarrow X''(x) = 0$，
したがって，2 回積分して一般解は A，B を定数として $X(x) = A + Bx$ となるが，$X(0) = 0$ より $A = 0$
また，$X(1) = 0$ より $A + B = 0$ であるので $A = 0 = B$ となり，結局 $X(x) = 0$。

4 章 1 節

節末問題 P.230

1．

⑴

(ｉ)　$n = 0$ のとき

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos ax\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} dx + \int_{0}^{\pi} 3\,dx\right) = \frac{1}{\pi}(\pi + 3\pi) = 4 \quad \Rightarrow \quad \frac{a_0}{2} = 2$$

(ⅱ)　$n \neq 0$ のとき

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} \cos nx\,dx + \int_{0}^{\pi} 3\cos nx\,dx\right) = \frac{4}{\pi}\int_{0}^{\pi} \cos nx\,dx = \frac{4}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} \sin nx\,dx + \int_{0}^{\pi} 3\sin nx\,dx\right) = \frac{2}{\pi}\int_{0}^{\pi} \sin nx\,dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{n\pi}(1 - \cos n\pi) = \frac{2}{n\pi}\left(1 - (-1)^n\right)$$

$$\therefore \quad f(x) \sim 2 + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1 - (-1)^n\right)\sin nx$$

(2)

(i) $n=0$ のとき

$$a_0=\frac{1}{\pi}\int_0^{2\pi}x^2\cos 0x\,dx=\frac{1}{\pi}\cdot\frac{(2\pi)^3}{3}=\frac{8}{3}\pi^2 \quad\Rightarrow\quad \frac{a_0}{2}=\frac{4}{3}\pi^2$$

(ii) $n\neq 0$ のとき

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\left(\frac{\sin nx}{n}\right)'dx\overset{\boxed{31}}{=}\frac{1}{\pi}\left(\left[\frac{1}{n}x^2\sin nx\right]_0^{2\pi}-\frac{2}{n}\int_0^{2\pi}x\sin nx\,dx\right)$$

$$=\frac{2}{n\pi}\int_0^{2\pi}x^2\left(\frac{\cos nx}{n}\right)'dx=\frac{2}{n\pi}\left(\left[\frac{1}{n}x\cos nx\right]_0^{2\pi}-\frac{1}{n}\int_0^{2\pi}\cos nx\,dx\right)\leftarrow \boxed{31}\text{ で } f=x,\ g'=\sin nx$$

$$=\frac{2}{n\pi}\left[\frac{1}{n}x\cos nx\right]_0^{2\pi}=\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\left(-\frac{\cos nx}{n}\right)dx\overset{\boxed{31}}{=}\frac{1}{\pi}\left(\left[-\frac{1}{n}x^2\cos nx\right]_0^{2\pi}+\frac{2}{n}\int_0^{2\pi}x\cos nx\,dx\right)$$

$$=\frac{1}{\pi}\left(-\frac{4\pi^2}{n}+\frac{2}{n}\int_0^{2\pi}x\left(\frac{\sin nx}{n}\right)'dx\right)\overset{\boxed{31}}{=}\frac{1}{\pi}\left(-\frac{4\pi^2}{n}+\frac{2}{n}\left(\left[\frac{1}{n}x\sin nx\right]_0^{2\pi}-\frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right)\right)$$

$$=-\frac{4}{n}\pi$$

$$\therefore\quad f(x)\sim\frac{4}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx-\sum_{n=1}^{\infty}\frac{4}{n}\pi\sin nx$$

(3)

(i) $n=0$ のとき

$$a_0=\frac{2}{L}\int_0^{L}x\cos 0x\,dx=\frac{2}{L}\cdot\frac{L^2}{2}=L \quad\Rightarrow\quad \frac{a_0}{2}=\frac{L}{2}$$

(ii) $n\neq 0$ のとき，$f(x)$ は偶関数で，かつ $-L<x<L$ であるから $b_n=0$ である。

$$a_n=\frac{2}{L}\int_0^{L}x\cos\left(\frac{n\pi}{L}x\right)dx$$

$$=\frac{2}{L}\left(\left[\frac{L}{n\pi}x\sin\left(\frac{n\pi}{L}x\right)\right]_0^{L}-\frac{L}{n\pi}\int_0^{L}\sin\left(\frac{n\pi}{L}x\right)dx\right)\quad\leftarrow \boxed{31}\text{ で } f=x,\ g'=\cos\left(\frac{n\pi}{L}x\right)$$

$$=\frac{2}{L}\left(\frac{L}{n\pi}\right)^2\left[\cos\left(\frac{n\pi}{L}x\right)\right]_0^{L}=\frac{2}{L}\left(\frac{L}{n\pi}\right)^2(\cos n\pi-1)$$

$$=\frac{2L}{n^2\pi^2}\left((-1)^n-1\right)$$

$$\therefore\quad f(x)\sim\frac{L}{2}+\sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2}\left((-1)^n-1\right)\cos\left(\frac{n\pi}{L}x\right)$$

(4)

(i) $n=0$ のとき

$$a_0=\frac{1}{L}\int_{-L}^{L}e^x\cos 0x\,dx=\frac{1}{L}\Big[e^x\Big]_{-L}^{L}=\frac{e^L-e^{-L}}{L}\quad\Rightarrow\quad\frac{a_0}{2}=\frac{e^L-e^{-L}}{2L}$$

(ii) $n\neq 0$ のとき

$$a_n=\frac{1}{L}\int_{-L}^{L}e^x\cos\left(\frac{n\pi}{L}x\right)dx\quad\cdots ⊛$$

$$=\frac{1}{L}\left\{\left[e^x\frac{L}{n\pi}\sin\frac{n\pi}{L}x\right]_{-L}^{L}-\int_{-L}^{L}e^x\frac{L}{n\pi}\sin\frac{n\pi}{L}x\,dx\right\}\quad\leftarrow\ \boxed{31}\ \text{で}\ f=e^x,\ g'=\cos\left(\frac{n\pi}{L}x\right)$$

$$=-\frac{1}{n\pi}\left\{\left[e^x\frac{-L}{n\pi}\cos\frac{n\pi}{L}x\right]_{-L}^{L}-\int_{-L}^{L}e^x\frac{-L}{n\pi}\cos\frac{n\pi}{L}x\,dx\right\}\quad\leftarrow\ \boxed{31}\ \text{で}\ f=e^x,\ g'=\sin\left(\frac{n\pi}{L}x\right)$$

$$=-\frac{1}{n\pi}\left\{\left(e^L\frac{-L}{n\pi}\cos n\pi-e^{-L}\frac{-L}{n\pi}\cos(-n\pi)\right)+\frac{L}{n\pi}\int_{-L}^{L}e^x\cos\frac{n\pi}{L}x\,dx\right\}$$

$$\left(\cos(-n\pi)=\cos n\pi=(-1)^n\right)$$

$$=-\frac{1}{n\pi}\left\{\frac{-L(-1)^n}{n\pi}\left(e^L-e^{-L}\right)+\frac{L}{n\pi}\cdot La_n\right\}\quad(⊛\text{ より})$$

$$=\frac{L(-1)^n}{(n\pi)^2}\left(e^L-e^{-L}\right)-\frac{L^2}{(n\pi)^2}a_n$$

第 2 項を左辺へ移項して

$$\left(1+\frac{L^2}{(n\pi)^2}\right)a_n=\frac{L(-1)^n}{(n\pi)^2}\left(e^L-e^{-L}\right)$$

$$\left((n\pi)^2+L^2\right)a_n=L(-1)^n\left(e^L-e^{-L}\right)\quad\text{より}\quad a_n=\frac{L(-1)^n\left(e^L-e^{-L}\right)}{(n\pi)^2+L^2}$$

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} e^x \sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{1}{L}\left\{\left[e^x \frac{-L}{n\pi}\cos\frac{n\pi}{L}x\right]_{-L}^{L} - \int_{-L}^{L} e^x \frac{-L}{n\pi}\cos\frac{n\pi}{L}x\,dx\right\} \quad \leftarrow \boxed{31} \text{ で } f = e^x \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad g' = \sin\left(\frac{n\pi}{L}x\right) \\
&= \frac{1}{L}\left\{e^L \frac{-L}{n\pi}(-1)^n - e^{-L}\frac{-L}{n\pi}(-1)^n + \frac{L}{n\pi} - \int_{-L}^{L} e^x \cos\frac{n\pi}{L}x\,dx\right\} \\
&= \frac{1}{L}\left\{\frac{L(-1)^{n+1}}{n\pi}\left(e^L - e^{-L}\right) + \frac{L}{n\pi}\cdot La_n\right\} \qquad (\text{※ より}) \\
&= \frac{(-1)^{n+1}}{n\pi}\left(e^L - e^{-L}\right) + \frac{L}{n\pi}\cdot\frac{L(-1)^n\left(e^L - e^{-L}\right)}{(n\pi)^2 + L^2} \\
&= \frac{\left(e^L - e^{-L}\right)(-1)^n}{n\pi}\left\{-1 + L^2\cdot\frac{1}{(n\pi)^2 + L^2}\right\} \\
&= \frac{\left(e^L - e^{-L}\right)(-1)^n}{n\pi}\cdot\frac{-(n\pi)^2 - L^2 + L^2}{(n\pi)^2 + L^2} \\
&= \frac{\left(e^L - e^{-L}\right)(-1)^{n+1}\,n\pi}{(n\pi)^2 + L^2}
\end{aligned}
$$

（別解）次の方法で導かれる公式①，②を用いてもよい。

$$
\begin{aligned}
\int e^{ax}\cos bx &= e^{ax}\frac{1}{b}\sin bx - \int ae^{ax}\frac{1}{b}\sin bx\,dx \leftarrow \boxed{31} \text{ で } f = e^{ax},\quad g' = \cos bx \\
&= \frac{e^{ax}}{b}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx \quad ㋐
\end{aligned}
$$

一方

$$
\begin{aligned}
\int e^{ax}\sin bx &= e^{ax}\frac{-1}{b}\cos bx - \int ae^{ax}\frac{-1}{b}\cos bx\,dx \quad \leftarrow \boxed{31} \text{ で } f = e^{ax},\quad g' = \sin bx \\
&= \frac{-e^{ax}}{b}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx \quad ㋑
\end{aligned}
$$

㋐の第 2 項に㋑を代入すると

$$
\begin{aligned}
\int e^{ax}\cos bx\,dx &= \frac{e^{ax}}{b}\sin bx \\
&\quad - \frac{a}{b}\left\{\frac{-e^{ax}}{b}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\right\} \\
&= \frac{e^{ax}}{b}\sin bx + \frac{a}{b^2}e^{ax}\cos x - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx
\end{aligned}
$$

第 3 項を左辺に移項して

$$
\frac{b^2 + a^2}{b^2}\int e^{ax}\cos bx\,dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos x
$$

よって $\displaystyle \int e^{ax}\cos bx\,dx = \frac{b}{a^2 + b^2}e^{ax}\sin bx + \frac{a}{a^2 + b^2}e^{ax}\cos bx$

$$
= \frac{e^{ax}}{a^2 + b^2}\left(a\cos bx + b\sin bx\right) \quad \text{＊公式①}
$$

同様に㋑の第 2 項に㋐を代入することによって次の公式を得る。

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)$$ ＊公式②

$a_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L}e^x\cos\left(\frac{n\pi}{L}x\right)dx$ を求めるのに

公式① $\displaystyle\int e^{ax}\cos bx=\frac{1}{a^2+b^2}\left(as^{ax}\cos bx+be^{ax}\sin bx\right)$ で $a=1$， $b=\dfrac{n\pi}{L}$ とすると，

$$a_n=\frac{1}{L}\left[\frac{1}{1+\left(\frac{n\pi}{L}\right)^2}\left(e^x\cos\left(\frac{n\pi}{L}x\right)+\frac{n\pi}{L}e^x\sin\left(\frac{n\pi}{L}x\right)\right)\right]_{-L}^{L}$$

$$=\frac{L}{L^2+\left(n\pi\right)^2}\left(e^L\cos n\pi-e^{-L}\cos\left(-n\pi\right)\right)=\frac{L}{L^2+\left(n\pi\right)^2}\left(e^L-e^{-L}\right)\left(-1\right)^n$$

同様に $b_n=\dfrac{1}{L}\displaystyle\int_{-L}^{L}e^x\sin\left(\frac{n\pi}{L}x\right)dx$ を求めるのに

公式② $\displaystyle\int e^{ax}\sin bx=\frac{1}{a^2+b^2}\left(as^{ax}\sin bx-be^{ax}\cos bx\right)$ で $a=1$， $b=\dfrac{n\pi}{L}$ とすると，

$$b_n=\frac{1}{L}\left[\frac{1}{1+\left(\frac{n\pi}{L}\right)^2}\left(e^x\sin\left(\frac{n\pi}{L}x\right)-\frac{n\pi}{L}e^x\cos\left(\frac{n\pi}{L}x\right)\right)\right]_{-L}^{L}$$

$$=\frac{L}{L^2+\left(n\pi\right)^2}\left(-\frac{n\pi}{L}e^L\cos n\pi+\frac{n\pi}{L}e^{-L}\cos\left(-n\pi\right)\right)=\frac{n\pi}{L^2+\left(n\pi\right)^2}\left(e^L-e^{-L}\right)\left(-1\right)^{n+1}$$

$$\therefore\ f\left(x\right)\sim\frac{e^L-e^{-L}}{2L}+\sum_{n=1}^{\infty}L\left(e^L-e^{-L}\right)\frac{\left(-1\right)^n}{L^2+\left(n\pi\right)^2}\cos\left(\frac{n\pi}{L}x\right)$$

$$+\sum_{n=1}^{\infty}\pi\left(e^L-e^{-L}\right)\frac{n\left(-1\right)^{n+1}}{L^2+\left(n\pi\right)^2}\sin\left(\frac{n\pi}{L}x\right)$$

2.

(1)

$$b_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{4}-\frac{x}{2}\right)\sin nx\,dx = \frac{2}{n\pi}\int_0^{\pi}\left(\frac{x}{2}-\frac{\pi}{4}\right)\left(\cos nx\right)'\,dx \leftarrow \boxed{31}\ \text{で}\ f=\frac{x}{2}-\frac{\pi}{4},\quad g'=\sin nx$$

$$= \frac{2}{n\pi}\left(\left[\left(\frac{x}{2}-\frac{\pi}{4}\right)\cos nx\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi}\cos nx\,dx\right) = \frac{2}{n\pi}\left(\frac{\pi}{4}\cos n\pi + \frac{\pi}{4}\right)$$

$$= \frac{2}{n\pi}\cdot\frac{\pi}{4}\left(\cos n\pi + 1\right) = \frac{1}{2n}\left(1+\left(-1\right)^n\right)$$

$$\therefore\quad f\left(x\right) \sim \sum_{n=1}^{\infty}\frac{1}{2n}\left(1+\left(-1\right)^n\right)\sin nx$$

(2)

$$b_n = \frac{2}{\pi}\int_0^{\pi}x^2\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}x^2\left(-\frac{\cos nx}{n}\right)'dx \overset{\boxed{31}}{=} \frac{2}{\pi}\left(\left[-\frac{1}{n}x^2\cos nx\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi}x\cos nx\,dx\right)$$

$$= \frac{2}{\pi}\left(-\frac{\pi^2}{n}\left(-1\right)^n + \frac{2}{n}\int_0^{\pi}x\left(\frac{\sin nx}{n}\right)'dx\right) \qquad \leftarrow \boxed{31}\ \text{で}\ f=x,\quad g'=\cos nx$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{n}\left(-1\right)^{n+1} + \frac{2}{n}\left(\left[\frac{1}{n}x\sin nx\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right)\right)$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{n}\left(-1\right)^{n+1} - \frac{2}{n^2}\int_0^{\pi}\sin nx\,dx\right)$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{n}\left(-1\right)^{n+1} + \frac{2}{n^3}\left[\cos nx\right]_0^{\pi}\right) = \frac{2}{\pi}\left(\frac{\pi^2}{n}\left(-1\right)^{n+1} + \frac{2}{n^3}\left(\left(-1\right)^n - 1\right)\right)$$

$$\therefore\quad f\left(x\right) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{\pi^2}{n}\left(-1\right)^{n+1} + \frac{2}{n^3}\left(\left(-1\right)^n - 1\right)\right\}\sin nx$$

3．

(1)

(ｉ)　$n=0$ のとき

$$a_0=\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos 0x\,dx=\frac{2}{\pi}\left[\frac{\pi}{4}x-\frac{x^2}{4}\right]_0^{\pi}=0$$

(ⅱ)　$n\neq 0$ のとき

$$a_n=\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos nx\,dx=\frac{2}{n\pi}\int_0^{\pi}\left(\frac{\pi}{4}-\frac{x}{2}\right)\left(\sin nx\right)'\,dx \quad \leftarrow \quad \boxed{31} \text{ で } f=\frac{\pi}{4}-\frac{x}{2},\quad g=\cos nx$$

$$=\frac{2}{n\pi}\left(\left[\left(\frac{\pi}{4}-\frac{x}{2}\right)\sin nx\right]_0^{\pi}+\frac{1}{2}\int_0^{\pi}\sin nx\,dx\right)=\frac{1}{n\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}=\frac{1}{n^2\pi}\left(1-\left(-1\right)^n\right)$$

$$\therefore\quad f\left(x\right)\sim\sum_{n=1}^{\infty}\frac{1}{n^2\pi}\left(1-\left(-1\right)^n\right)\cos nx$$

(2)

(ｉ)　$n=0$ のとき

$$a_0=\frac{2}{\pi}\int_0^{\pi}\sin 2x\cos 0x\,dx=\frac{1}{\pi}\left[-\cos 2x\right]_0^{\pi}=0$$

(ⅱ)　$n\neq 0$ のとき

n が偶数，すなわち $n=2k$ の場合：

$$a_{2k}=\frac{2}{\pi}\int_0^{\pi}\sin 2x\cos 2kx\,dx=\frac{1}{\pi}\int_0^{\pi}\left(\sin 2\left(k+1\right)x+\sin 2\left(1-k\right)x\right)dx=0 \qquad \leftarrow \quad \boxed{10} \text{ (2) ①}$$

n が奇数，すなわち $n=2k+1$ の場合：

$$a_{2k+1}=\frac{2}{\pi}\int_0^{\pi}\sin 2x\cos\left(2k+1\right)x\,dx=\frac{1}{\pi}\int_0^{\pi}\left(\sin\left(2k+3\right)x+\sin\left(1-2k\right)x\right)dx \quad \leftarrow \quad \boxed{10} \text{ (2) ①}$$

$$=-\frac{1}{\pi}\left[\frac{1}{2k+3}\cos\left(2k+3\right)x+\frac{1}{1-2k}\cos\left(1-2k\right)x\right]_0^{\pi}=\frac{1}{\pi}\left(\frac{2}{2k+3}+\frac{2}{1-2k}\right)$$

$$=\frac{8}{\pi}\cdot\frac{1}{\left(2k+3\right)\left(1-2k\right)}=\frac{8}{\pi}\cdot\frac{1}{2^2-\left(2k+1\right)^2}$$

$$\therefore\quad f\left(x\right)\sim\frac{8}{\pi}\sum_{k=1}^{\infty}\frac{1}{2^2-\left(2k+1\right)^2}\cos\left(2k+1\right)x$$

4．

(1)

（i） $n=0$ のとき

$$c_0=\frac{1}{2\pi}\left(\int_{-\pi}^{0}(-\pi)e^{-i0x}\,dx+\int_{0}^{\pi}(\pi)e^{-i0x}\,dx\right)=0$$

（ii） $n\neq 0$ のとき

$$c_n=\frac{1}{2\pi}\left(\int_{-\pi}^{0}(-\pi)e^{-inx}\,dx+\int_{0}^{\pi}\pi e^{-inx}\,dx\right)=\frac{1}{2}\left(\left[\frac{1}{in}e^{-inx}\right]_{-\pi}^{0}+\left[-\frac{1}{in}e^{-inx}\right]_{0}^{\pi}\right)$$

$$=\frac{1}{2}\left(\frac{1}{in}\left(1-e^{in\pi}\right)-\frac{1}{in}\left(e^{-in\pi}-1\right)\right)=\frac{1}{in}\left(1-(-1)^n\right)=\frac{i}{n}\left((-1)^n-1\right)$$

$$\left(\begin{array}{l}e^{n\pi i}=\cos n\pi+i\sin n\pi=(-1)^n\,,\\ e^{-n\pi i}=\cos(-n\pi)+i\sin(-n\pi)=\cos n\pi-i\sin n\pi=(-1)^n\qquad\leftarrow\ 47\end{array}\right)$$

$$\therefore\quad f(x)\sim\sum_{n=-\infty}^{+\infty}\frac{i}{n}\left((-1)^n-1\right)e^{inx}$$

(2)

（i） $n=0$ のとき

$$c_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}|x|e^{-i0x}\,dx=\frac{1}{2\pi}\cdot\pi\cdot\pi=\frac{\pi}{2}$$

（ii） $n\neq 0$ のとき

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}|x|e^{-inx}\,dx\overset{\overset{47}{\downarrow}}{=}\frac{1}{2\pi}\int_{-\pi}^{\pi}|x|\left(\cos nx-i\sin nx\right)dx=\frac{1}{\pi}\int_{0}^{\pi}x\cos nx\,dx$$

$$\overset{\overset{31}{\downarrow}}{=}\frac{1}{\pi}\left\{\left[x\frac{1}{n}\sin nx\right]_{0}^{\pi}-\int_{0}^{\pi}1\cdot\frac{1}{n}\sin nx\,dx\right\}$$

$$=\frac{1}{\pi}\left[\frac{1}{n}x\sin nx+\frac{1}{n^2}\cos nx\right]_{0}^{\pi}=\frac{1}{n^2\pi}\left((-1)^n-1\right)$$

$$\therefore\quad f(x)\sim\sum_{n=-\infty}^{+\infty}\frac{1}{n^2\pi}\left((-1)^n-1\right)e^{inx}$$