４章　重積分

１節　重積分

A

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⑴ $$\int_0^3\int_0^1\left(4-y^2\right)dy\,dx=\int_0^3\left[4y-\frac{y^3}{3}\right]_0^1dx$$

$$=\int_0^3\left(4-\frac{1}{3}\right)dx$$

$$=\frac{11}{3}\int_0^3dx$$

$$=11$$

⑵ $$\int_{-1}^1\int_{-1}^0\left(x+y+1\right)dy\,dx=\int_{-1}^1\left[\left(x+1\right)y+\frac{y^2}{2}\right]_{-1}^0dx$$

$$=\int_{-1}^1\left(x+\frac{1}{2}\right)dx$$

$$=2\int_0^1\frac{1}{2}\,dx=\int_0^1dx$$

$$=1$$

⑶ $$\int_{-2}^{-1}\int_0^1\left(\sin\left(\pi y\right)+\cos\left(\pi x\right)\right)dy\,dx$$ 公式 $\boxed{26}$

$$=\int_{-2}^{-1}\left[-\frac{1}{\pi}\cos\left(\pi y\right)+y\cos\left(\pi x\right)\right]_0^1dx$$

$$=\int_{-2}^{-1}\left(\frac{2}{\pi}+\cos\left(\pi x\right)\right)dx$$

$$=\left[\frac{2}{\pi}x+\frac{1}{\pi}\sin\left(\pi x\right)\right]_{-2}^{-1}=\frac{2}{\pi}$$

⑷ $$\int_{-1}^2\int_0^1xe^{xy}\,dy\,dx$$ 公式 $\boxed{27}$

$$=\int_{-1}^2\left[e^{xy}\right]_0^1dx=\int_{-1}^2\left(e^x-1\right)dx$$

$$=\left[e^x-x\right]_{-1}^2=e^2-e^{-1}-3$$

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⑴ $\int_{-2}^{0}\int_{0}^{x+2} dy\,dx$

$$=\int_{-2}^{0}\Big[\,y\,\Big]_{0}^{x+2} dx=\int_{-2}^{0}\left(x+2\right)dx$$

$$=\left[\frac{\left(x+2\right)^{2}}{2}\right]_{-2}^{0}=2$$

⑵ $\int_{0}^{1}\int_{0}^{1-x}\left(x^{2}+y^{2}\right)dy\,dx$

$$=\int_{0}^{1}\left[x^{2}y+\frac{y^{3}}{3}\right]_{0}^{1-x} dx=\int_{0}^{1}\left(x^{2}\left(1-x\right)+\frac{\left(1-x\right)^{3}}{3}\right)dx$$

$$=\int_{0}^{1}\left(x^{2}-x^{3}+\frac{\left(1-x\right)^{3}}{3}\right)dx=\left[\frac{x^{3}}{3}-\frac{x^{4}}{4}-\frac{\left(1-x\right)^{4}}{12}\right]_{0}^{1}$$

$$=\left(\frac{1}{3}-\frac{1}{4}\right)-\left(-\frac{1}{12}\right)$$

$$=\frac{1}{6}$$

⑶ $\int_{1}^{2}\int_{x}^{2x}\frac{x}{y}\,dy\,dx$　公式$\boxed{25}$

$$=\int_{1}^{2}x\Big[\log\left|y\right|\Big]_{x}^{2x} dx=\int_{1}^{2}x\left(\log\left|2x\right|-\log\left|x\right|\right)dx$$

$=\left(\log 2\right)\int_{1}^{2}x\,dx$　公式$\boxed{6}$

$$=\log 2\cdot\left[\frac{x^{2}}{2}\right]_{1}^{2}=\frac{3}{2}\log 2$$

⑷ $\int_{0}^{\pi}\int_{0}^{\sin x}2y\,dy\,dx$

$$=\int_{0}^{\pi}\Big[\,y^{2}\,\Big]_{0}^{\sin x} dx=\int_{0}^{\pi}\sin^{2}x\,dx$$

$=\frac{1}{2}\int_{0}^{\pi}\left(1-\cos 2x\right)dx$　公式$\boxed{10}$

$$=\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_{0}^{\pi}$$

$$=\frac{\pi}{2}$$

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(1) $\int_0^1 \int_{y^2}^{y} dx\, dy$

$$= \int_0^1 \left[x \right]_{y^2}^{y} dy = \int_0^1 \left(y - y^2 \right) dy$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \frac{1}{6}$$

(2) $\int_{\pi}^{2\pi} \int_0^{\pi} \left(\sin x + \cos y \right) dx\, dy$ 公式 $\boxed{26}$

$$= \int_{\pi}^{2\pi} \left[-\cos x + x \cos y \right]_0^{\pi} dy$$

$$= \int_{\pi}^{2\pi} \left(\pi \cos y + 2y \right) dy$$

$$= \left[\pi \sin y + 2y \right]_{\pi}^{2\pi} = 2\pi$$

(3) $\int_0^1 \int_0^{\pi} y \cos\left(xy \right) dx\, dy$ 公式 $\boxed{26}$

$$= \int_0^1 \left[\sin\left(xy \right) \right]_0^{\pi} dy = \int_0^1 \sin\left(\pi y \right) dy$$

$$= \left[-\frac{1}{\pi} \cos\left(\pi y \right) \right]_0^1 = \frac{2}{\pi}$$

(4) $\int_0^1 \int_0^1 e^{x+y} dx\, dy$ 公式 $\boxed{27}$

$$= \int_0^1 e^x\, dx \cdot \int_0^1 e^y\, dy$$

$$= \left[e^x \right]_0^1 \cdot \left[e^y \right]_0^1 = \left(e - 1 \right)^2$$

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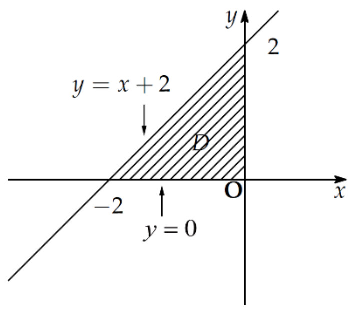
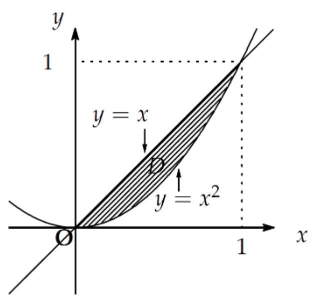
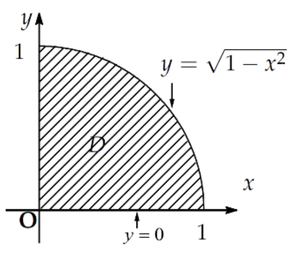
⑴ $D=\left\{(x,\ y)\,|\,-2 \leqq x \leqq 0,\quad 0 \leqq y \leqq x+2\right\}$



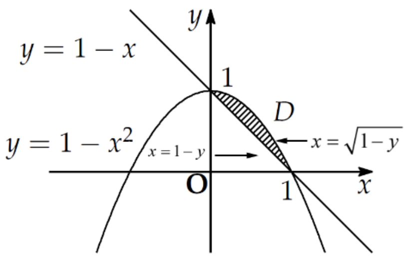
⑵ $D=\left\{(x,\ y)\,|\,0 \leqq x \leqq 1,\quad x^2 \leqq y \leqq x\right\}$



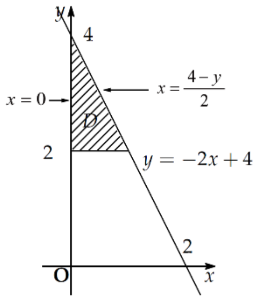
⑶ $D=\left\{(x,\ y)\,|\,0 \leqq x \leqq 1,\quad 0 \leqq y \leqq \sqrt{1-x^2}\right\}$



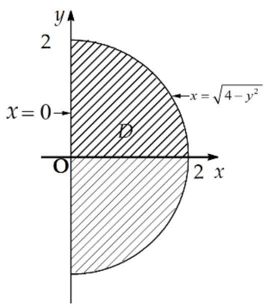
⑷ $D=\left\{(x,\ y)\,|\,0 \leqq y \leqq 1,\quad 1-y \leqq x \leqq \sqrt{1-y}\right\}$



⑸ $D=\left\{(x,\ y) \mid 2 \leqq y \leqq 4,\ \ 0 \leqq x \leqq \dfrac{4-y}{2}\right\}$



⑹ $D=\left\{(x,\ y) \mid -2 \leqq y \leqq 2,\ \ 0 \leqq x \leqq \sqrt{4-y^2}\right\}$



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(1)

$$\int_0^{\pi}\int_x^{\pi}\frac{\cos y}{y}\,dy\,dx$$

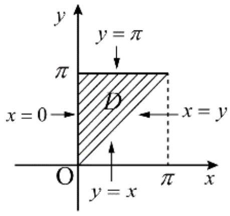
$$=\int_0^{\pi}\frac{\cos y}{y}\left(\int_0^{y}dx\right)dy$$

$$=\int_0^{\pi}\frac{\cos y}{y}\cdot y\,dy=\int_0^{\pi}\cos y\,dy$$

$$=\Big[\sin y\Big]_0^{\pi}\qquad \text{公式}\boxed{26}$$

$$=\sin\pi-\sin 0=0$$

D は $\begin{cases}0\leqq x\leqq\pi\\ x\leqq y\leqq\pi\end{cases}$ より下図



D は次のようにも表せる。

$\begin{cases}0\leqq x\leqq y\\ 0\leqq y\leqq\pi\end{cases}$

(2)

$$\int_0^{1}\int_x^{1}y^2\sin(\pi xy)\,dy\,dx$$

$$=\int_0^{1}\left(\int_0^{y}y^2\sin(\pi xy)\,dx\right)dy$$

$$=\int_0^{1}\left[-\frac{y}{\pi}\cos(\pi xy)\right]_0^{y}dy$$

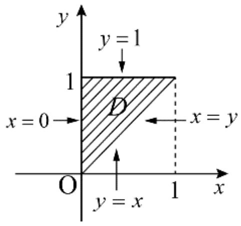
$\pi y^2=t$
とする置換積分
$2\pi y=\dfrac{dt}{dy}$

$$=\frac{1}{\pi}\int_0^{\pi}\left(y-y\cos\left(\pi y^2\right)\right)dy$$

$$=\frac{1}{\pi}\left[\frac{y^2}{2}-\frac{1}{2\pi}\sin\left(\pi y^2\right)\right]_0^{1}$$

$$=\frac{1}{\pi}\left(\frac{1}{2}-\frac{1}{2\pi}\sin(\pi)\right)=\frac{1}{2\pi}$$

D は $\begin{cases}0\leqq x\leqq 1\\ x\leqq y\leqq 1\end{cases}$ より下図



D は次のようにも表せる。

$\begin{cases}0\leqq x\leqq y\\ 0\leqq y\leqq 1\end{cases}$

(3)

$$\int_0^{1}\int_y^{1}x^2e^{xy}\,dx\,dy$$

$xy=t$
とする置換積分
$x=\dfrac{dt}{dy}$

$$=\int_0^{1}\left(\int_0^{x}x^2e^{xy}dy\right)dx$$

$$=\int_0^{1}\Big[xe^{xy}\Big]_0^{x}dx$$

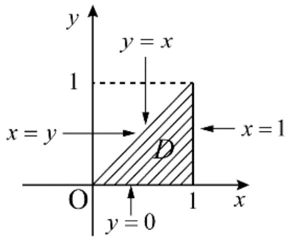
$x^2=t$
とする置換積分
$2x=\dfrac{dt}{dx}$

$$=\int_0^{1}\left(xe^{x^2}-x\right)dx$$

$$=\left[\frac{e^{x^2}}{2}-\frac{x^2}{2}\right]_0^{1}$$

$$=\frac{e}{2}-\frac{1}{2}-\frac{1}{2}=\frac{e}{2}-1$$

D は $\begin{cases}y\leqq x\leqq 1\\ 0\leqq y\leqq 1\end{cases}$ より下図



D は次のようにも表せる。

$\begin{cases}0\leqq x\leqq 1\\ 0\leqq y\leqq x\end{cases}$

(4)

$$\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} dx\, dy$$

$$= \int_0^1 e^{x^2} \left(\int_0^{2x} dy \right) dx$$

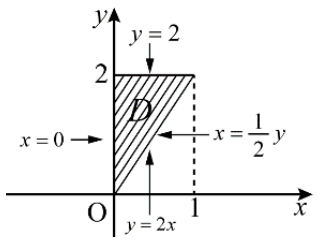
$$= \int_0^1 2xe^{x^2}\ dx$$

$$= \left[e^{x^2} \right]_0^1$$

$$= e - 1$$

$x^2 = t$
とする置換積分
$2x = \dfrac{dt}{dx}$

D は $\begin{cases} \dfrac{y}{2} \leqq x \leqq 1 \\ 0 \leqq y \leqq 2 \end{cases}$ より下図



D は次のようにも表せる。

$\begin{cases} 0 \leqq x \leqq 1 \\ 2x \leqq y \leqq 2 \end{cases}$

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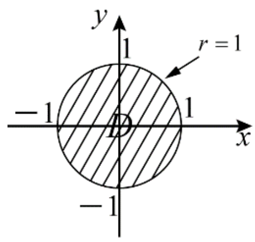
(1) $\displaystyle\iint_D \sqrt{x^2 + y^2}\ dx\, dy$

$$= \int_0^1 \int_0^{2\pi} r \cdot r\ dr\ d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 r^2\ dr$$

$$= 2\pi \left[\frac{r^3}{3} \right]_0^1 = \frac{2}{3}\pi$$

D は $\begin{cases} 0 \leqq r \leqq 1 \\ 0 \leqq \theta \leqq 2\pi \end{cases}$



(2) $\displaystyle\iint_D x\ dx\, dy$

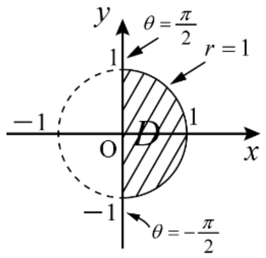
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r\cos\theta \cdot r\ dr\ d\theta$$

$$= \int_{-\frac{\pi}{2}}^{2\pi} \cos\theta\ d\theta \cdot \int_0^1 r^2\ dr$$

$$= \left[\sin\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \left[\frac{r^3}{3} \right]_0^1$$

$$= \frac{2}{3}$$

D は $\begin{cases} 0 \leqq r \leqq 1 \\ -\dfrac{\pi}{2} \leqq \theta \leqq \dfrac{\pi}{2} \end{cases}$



(3) $\begin{cases} u = x - y \\ v = x + y \end{cases}$ とおくと D は

uv 平面の開領域 $D' = \left\{ (u,\ v) \middle| \begin{array}{l} 0 \leqq u \leqq \pi \\ 0 \leqq v \leqq \pi \end{array} \right\}$ にうつりヤコビ行列は

$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ より

$\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ でヤコビアンの絶対値 $\left| J(u,\ v) \right| = \frac{1}{2}$ を用いると

$$
\begin{aligned}
\text{与式} &= \frac{1}{2}\int_0^{\pi}\int_0^{\pi} \sin v \, du \, dv \\
&= \frac{1}{2}\int_0^{\pi} \sin v \left[u \right]_0^{\pi} dv \\
&= \frac{\pi}{2}\left[-\cos v \right]_0^{\pi} = \pi
\end{aligned}
$$

B

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(1) $\iint_D 6xy^2 \, dx \, dy$

$$
\begin{aligned}
&= \int_2^4 \int_1^2 6xy^2 \, dy \, dx \\
&= \int_2^4 2x \, dx \cdot \int_1^2 3y^2 \, dy \\
&= \left[x^2 \right]_2^4 \cdot \left[y^3 \right]_1^2 \\
&= (16-4)\cdot(8-1) = 84
\end{aligned}
$$

(2) $\iint_D \frac{1}{(2x+3y)} \, dx \, dy$

$$
\begin{aligned}
&= \int_0^1 \int_1^2 (2x+3y)^{-2} \, dy \, dx \\
&= \int_0^1 \left[-\frac{1}{3}(2x+3y)^{-1} \right]_1^2 dx \\
&= \frac{1}{3}\int_0^1 \left(\frac{1}{2x+3} - \frac{1}{2x+6} \right) dx \qquad \text{公式}\boxed{25} \\
&= \frac{1}{6}\left[\log\left|(2x+3)\right| - \log\left|(2x+6)\right| \right]_0^1 \\
&= \frac{1}{6}\left(\log 5 - \log 8 - \log 3 + \log 6 \right) \qquad \text{公式}\boxed{6} \\
&= \frac{1}{6}\log\left(\frac{5}{4} \right)
\end{aligned}
$$

(3) $\iint_D x\cos^2 y\,dx\,dy$

$$= \int_{-2}^{3} x\,dx \cdot \int_0^{\frac{\pi}{2}} \cos^2 y\,dy$$

$$= \int_{-2}^{3} x\,dx \cdot \int_0^{\frac{\pi}{2}} \frac{1+\cos 2y}{2}\,dy \qquad \text{公式}\boxed{10}$$

$$= \left[\frac{x^2}{2}\right]_{-2}^{3} \cdot \frac{1}{2}\left[y+\frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$$

$$= \frac{9-4}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5}{8}\pi$$

(4) $\iint_D e^{\frac{x}{y}}\,dx\,dy$

$\frac{x}{y}=t$

とする置換積分

$\frac{1}{y}=\frac{dt}{dx}$

$$= \int_1^2 \int_y^{y^3} e^{\frac{x}{y}}\,dx\,dy$$

$$= \int_1^2 \left[ye^{\frac{x}{y}}\right]_y^{y^3} dy$$

$y^2=u$

とする置換積分

$2y=\frac{du}{dy}$

$$= \int_1^2 \left(ye^{y^2} - ey\right)dy$$

$$= \left[\frac{1}{2}e^{y^2} - \frac{e}{2}y^2\right]_1^2$$

$$= \frac{e^4}{2} - 2e$$

(5) $\iint_D 4xy\,dx\,dy$

$$= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy\,dy\,dx$$

$$= \int_0^1 \left[2xy^2\right]_{x^3}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(2x^2 - 2x^7\right)dx$$

$$= \left[\frac{2}{3}x^2 - \frac{2}{8}x^8\right]_0^1$$

$$= \frac{2}{3} - \frac{2}{8} = \frac{5}{12}$$

(6) $\iint_D dx\,dy$

$$= \int_1^3 \int_{-\frac{y}{2}+\frac{3}{2}}^{2y-1} dx\,dy$$

$$= \int_1^3 \left(2y - 1 + \frac{y}{2} - \frac{3}{2} \right) dy$$

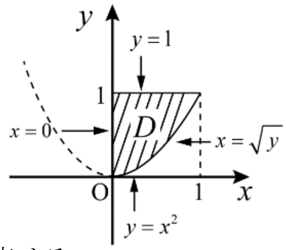
$$= \int_1^3 \left(\frac{5}{2}y - \frac{5}{2} \right) dy = \frac{5}{2}\int_1^3 (y-1)\,dy$$

$$= \frac{5}{2}\left[\frac{1}{2}(y-1)^2 \right]_1^3$$

$$= 5$$

(7) $\iint_D x^3 e^{y^3}\,dx\,dy$

D は $\begin{cases} 0 \leqq x \leqq 1 \\ x^2 \leqq y \leqq 1 \end{cases}$ より下図



$$= \int_0^1 \int_{x^2}^1 x^3 e^{y^3}\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{y}} x^3 e^{y^3}\,dx\,dy$$

$$= \int_0^1 \left[\frac{1}{4} x^4 e^{y^3} \right]_0^{\sqrt{y}} dy$$

$\left(y^3 = t \right.$ とする置換積分 $\left. 3t^2 = \dfrac{dt}{dy} \right) \longrightarrow$

$$= \int_0^1 \frac{1}{4} y^2 e^{y^3}\,dy$$

$$= \left[\frac{1}{12} e^{y^3} \right]_0^1$$

$$= \frac{1}{12}(e-1)$$

D は次のようにも表せる。

$$\begin{cases} 0 \leqq x \leqq \sqrt{y} \\ 0 \leqq y \leqq 1 \end{cases}$$

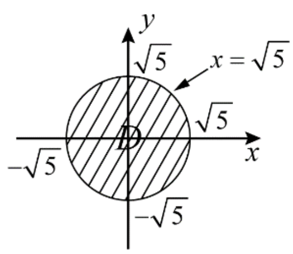
125

(1) $\iint_D \sqrt{9 - x^2 - y^2}\,dx\,dy$

D は $\begin{cases} 0 \leqq r \leqq \sqrt{5} \\ 0 \leqq \theta \leqq 2\pi \end{cases}$

$$= \int_0^{2\pi} \int_0^{\sqrt{5}} \sqrt{9 - r^2} \cdot r\,dr\,d\theta$$

$$= 2\pi \int_0^{\sqrt{5}} \left(9 - r^2 \right)^{\frac{1}{2}} r\,dr$$



$\left(9 - r^3 = t \right.$ とする置換積分 $\left. -2r = \dfrac{dt}{dr} \right) \longrightarrow$

$$= -\pi \int_0^{\sqrt{5}} \left(9 - r^2 \right)^{\frac{1}{2}} (-2r)\,dr = -\pi \left[\frac{2}{3} \left(9 - r^2 \right)^{\frac{3}{2}} \right]_0^{\sqrt{5}}$$

$$= -\frac{2}{3}\pi \left((9-5)^{\frac{3}{2}} - (9-0)^{\frac{3}{2}} \right) = -\frac{2}{3}\pi \left(4^{\frac{3}{2}} - 9^{\frac{3}{2}} \right)$$

$$= -\frac{2}{3}\pi \left(2^3 - 3^3 \right) = \frac{38}{3}\pi$$

(2)

$$\iint_D \cos\left(x^2+y^2\right)dx\,dy$$

$$=\int_0^{\frac{\pi}{2}}\int_0^1 \cos\left(r^2\right)\cdot r\,dr\,d\theta$$

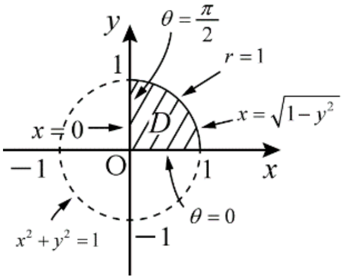
$$=\int_0^{\frac{\pi}{2}}d\theta\cdot\int_0^1 r\cos\left(r^2\right)dr$$

$r^2=t$
とする置換積分
$2r=\dfrac{dt}{dr}$ →

$$=\frac{\pi}{2}\left[\frac{1}{2}\sin\left(r^2\right)\right]_0^1$$

$$=\frac{\pi}{4}\sin 1$$

D は与式より下図



極座標表示で D は

$$\begin{cases} 0\leqq r\leqq 1 \\ 0\leqq\theta\leqq\dfrac{\pi}{2}\end{cases}$$

(3)

$$\iint_D \frac{2}{1+\sqrt{x^2+y^2}}\,dx\,dy$$

$$=\int_\pi^{\frac{3}{2}\pi}\int_0^1\frac{2}{1+r}\cdot r\,dr\,d\theta$$

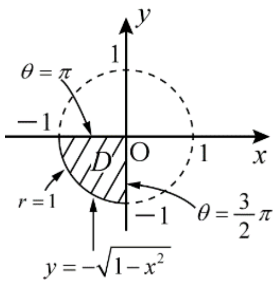
$$=\int_\pi^{\frac{3}{2}\pi}d\theta\cdot\int_0^1\frac{2r}{1+r}\,dr$$

$$=\frac{\pi}{2}\int_0^1 2\left(1-\frac{1}{1+r}\right)dr$$

$$=\pi\left[r-\log\left(1+r\right)\right]_0^1$$

$$=\pi\left(1-\log 2\right)$$

D は与式より下図



極座標表示で D は

$$\begin{cases} 0\leqq r\leqq 1 \\ \pi\leqq\theta\leqq\dfrac{3}{2}\pi\end{cases}$$

(4)

$$\iint_D e^{-\left(x^2+y^2\right)}\,dx\,dy$$

$$=\int_0^{\frac{\pi}{2}}\int_0^1 e^{-r^2}\cdot r\,dr\,d\theta$$

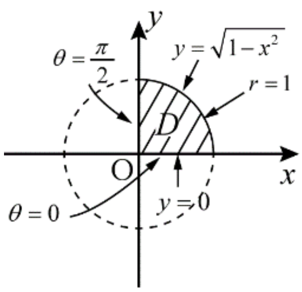
$r^2=t$
とする置換積分
$2r=\dfrac{dt}{dr}$ →

$$=\frac{\pi}{2}\int_0^1 re^{-r^2}\,dr$$

$$=\frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^1$$

$$=\frac{\pi}{4}\left(1-e^{-1}\right)$$

D は与式より下図

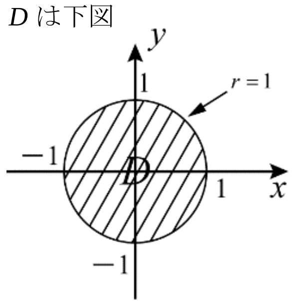


極座標表示で D は

$$\begin{cases} 0\leqq r\leqq 1 \\ 0\leqq\theta\leqq\dfrac{\pi}{2}\end{cases}$$

(5) $\displaystyle\iint_D \log\left(x^2+y^2+1\right) dx\, dy$

与式より D は下図



$$= \int_0^{2\pi}\int_0^1 \log\left(r^2+1\right)\cdot r\, dr\, d\theta$$

$$= 2\pi\int_0^1 r\log\left(r^2+1\right) dr$$

$$= \pi\int_0^1 \left(2r\right)\log\left(r^2+1\right) dr$$

（ ここで， $t=r^2+1$ と置換 ）

$$= \int_1^2 \log t\, dt$$

$$= \pi\Big[\, t\log t - t \,\Big]_0^2$$

$$= \pi\left(2\log 2 - 1\right)$$

極座標表示で D は

$$\begin{cases} 0 \leqq r \leqq 1 \\ 0 \leqq \theta \leqq 2\pi \end{cases}$$

(6) $\displaystyle\iint_D \frac{1}{1+x^2+y^2}\, dx\, dy$ （D の図は⑸と同じ）

$$= \int_0^{2\pi}\int_0^1 \frac{1}{1+r^2}\cdot r\, dr\, d\theta$$

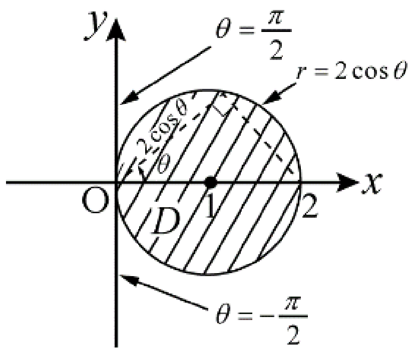
$$= 2\pi\int_0^1 \frac{r}{1+r^2}\, dr$$

$$= \pi\Big[\, \log\left(1+r^2\right) \Big]_0^1$$

$$= \pi\log 2$$

⑺　領域 D は極座標変換で，$x^2-2x+y^2=r^2-2r\cos\theta \leqq 0$

$r>0$ のとき　$r \leqq 2\cos\theta$　より領域



$D'=\left\{(r,\ \theta)\,\middle|\,0 \leqq r \leqq 2\cos\theta,\ -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\right\}$ に写るので，

$$\iint_D (x+y)\,dx\,dy$$

$$=\int_{-\frac{\pi}{2}}^{2\pi}\int_0^{2\cos\theta} r(\cos\theta+\sin\theta)r\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$$

$$=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos\theta+\sin\theta)\cos^3\theta\,d\theta$$

$$=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^4\theta+\sin\theta\cos^3\theta)\,d\theta$$

（$\sin\theta\cos^3\theta$ は奇関数なので $\int_{-a}^{a}=0$）

$$=\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$

（$\cos^4\theta$ は偶関数なので $\int_{-a}^{a}=2\int_0^a$　）

$$=\frac{16}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$

公式 $\boxed{32}$ （Wallis の公式）から

$$=\frac{16}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$=\pi$$

(1) $\begin{cases} u = x - y \\ v = x + y \end{cases}$ とおく。

このとき，領域 D は $D' = \{(u,\ v) | 0 \leqq u \leqq \pi,\ 0 \leqq v \leqq \pi\}$ に写り，

またヤコビ行列は $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ より $\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$ でヤコビアンの絶対値は

$\left| J(u \quad v) \right| = | \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{vmatrix} | = | \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{2}\left(-\frac{1}{2} \right) | = \frac{1}{2}$ となるので

$$\iint_D (x-y)\sin(x+y)\,dx\,dy$$
$$= \frac{1}{2}\int_0^{\pi}\int_0^{\pi} u \sin v\,du\,dv$$
$$= \frac{1}{2}\int_0^{\pi} u\,du \cdot \int_0^{\pi} \sin v\,dv$$
$$= \frac{1}{2} \cdot \left[\frac{u^2}{2} \right]_0^{\pi} \cdot \left[-\cos v \right]_0^{\pi}$$
$$= \frac{1}{2} \cdot \frac{\pi^2}{2} \cdot 2$$
$$= \frac{\pi^2}{2}$$

(2) $x^2 + xy + y^2 = 6$ は $\left(x + \frac{1}{2}y \right)^2 + \frac{3}{4}y^2 = 6$

よって $\begin{cases} u = x + \frac{1}{2}y \\ v = \frac{\sqrt{3}}{2}y \end{cases}$ とおくと

与式の曲線内の閉領域 D は uv 平面の閉領域 $D' = \{(u,\ v) | u^2 + v^2 \leqq 6\}$ にうつる。

ヤコビアンの絶対値は $\frac{2}{\sqrt{3}}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ より $\left| J(u \quad v) \right| = | \begin{vmatrix} 1 & \frac{-1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{3}} \end{vmatrix} | = \frac{2}{\sqrt{3}}$

よって求める面積 S は

$$\frac{2}{\sqrt{3}}\iint_{D'} du\,dv = \frac{2}{\sqrt{3}} \cdot \pi\sqrt{6}^2 = 4\sqrt{3}\pi$$

127　領域 D は極座標変換で，領域

$$D' = \left\{ (r,\ \theta) \,\middle|\, 0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq \frac{\pi}{2} \right\}$$ に写るので，

$$D'_{\varepsilon} = \left\{ (r,\ \theta) \,\middle|\, 0 < \varepsilon \leqq r \leqq 1,\ 0 \leqq \theta \leqq \frac{\pi}{2} \right\}$$ を考える。

$$\begin{aligned}
\iint_D \mathrm{Tan}^{-1}\left(\frac{y}{x}\right) dx\,dy &= \lim_{\varepsilon \to +0} \int_0^{\frac{\pi}{2}} \int_{\varepsilon}^{1} \mathrm{Tan}^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) \cdot r\,dr\,d\theta \\
&= \lim_{\varepsilon \to +0} \int_0^{\frac{\pi}{2}} \int_{\varepsilon}^{1} \theta \cdot r\,dr\,d\theta \\
&= \lim_{\varepsilon \to +0} \int_0^{\frac{\pi}{2}} \theta\,d\theta \cdot \int_{\varepsilon}^{1} r\,dr \\
&= \lim_{\varepsilon \to +0} \left[\frac{\theta^2}{2}\right]_0^{\frac{\pi}{2}} \cdot \left[\frac{r^2}{2}\right]_{\varepsilon}^{1} \\
&= \lim_{\varepsilon \to +0} \frac{\pi^2}{8} \cdot \frac{1^2 - \varepsilon^2}{2} \\
&= \frac{\pi^2}{16}
\end{aligned}$$

128　領域 D は極座標変換で，領域

$$D' = \left\{ (r,\ \theta) \middle| 0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq \frac{\pi}{2} \right\}$$ に写るので，

$$D'_\varepsilon = \left\{ (r,\ \theta) \middle| 0 < \varepsilon \leqq r \leqq 1,\ 0 \leqq \theta \leqq \frac{\pi}{2} \right\}$$ を考える。

$$\begin{aligned}
\iint_D \log\sqrt{x^2+y^2}\ dx\,dy &= \lim_{\varepsilon\to+0}\int_0^{\frac{\pi}{2}}\int_\varepsilon^1 \log\sqrt{r}\cdot r\ dr\,d\theta \\
&= \lim_{\varepsilon\to+0}\int_0^{\frac{\pi}{2}} d\theta\cdot\int_\varepsilon^1 \frac{1}{2}\log r\ dr \\
&= \lim_{\varepsilon\to+0}\frac{\pi}{4}\int_\varepsilon^1 r\log dr
\end{aligned}$$

$$\left(\begin{aligned}
\int r\log r\ dr &= \int\left(\frac{1}{2}r^2\right)'\log r\ dr \\
&= \frac{1}{2}r^2\log r - \int\left(\frac{1}{2}r^2\right)(\log r)'\ dr \quad \text{公式}\boxed{31}
\end{aligned}\right)$$

$$\begin{aligned}
&= \lim_{\varepsilon\to+0}\frac{\pi}{4}\left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right]_\varepsilon^1 \\
&= \lim_{\varepsilon\to+0}\frac{\pi}{4}\left(\frac{1}{2}\log 1 - \frac{1}{4} - \frac{\varepsilon^2}{2}\log\varepsilon + \frac{\varepsilon^2}{4}\right)
\end{aligned}$$

ここで $\displaystyle\lim_{\varepsilon\to+0}\varepsilon^2\log\varepsilon = 0$ であるから

$$= -\frac{\pi}{16}$$

$$\left(\begin{aligned}
\lim_{\varepsilon\to+0}\varepsilon^2\log\varepsilon &= \lim_{\varepsilon\to+0}\frac{\log\varepsilon}{\varepsilon^{-2}}\left(\frac{-\infty}{\infty}\text{の不定形}\right) \\
&= \lim_{\varepsilon\to+0}\frac{\frac{1}{\varepsilon}}{-2\varepsilon^{-2}}\left(\text{ロピタルの定理}\right) \\
&= \lim_{\varepsilon\to+0}\frac{\varepsilon^2}{-2} = 0
\end{aligned}\right)$$

４章　重積分

２節　重積分の応用

A

129

⑴　$V=\int_0^1\int_0^1 (x+y)\,dx\,dy$

$$=\int_0^1\left[xy+\frac{y^2}{2}\right]_0^1 dx=\int_0^1\left(x+\frac{1}{2}\right)dx$$

$$=\left[\frac{x^2+x}{2}\right]_0^1=1$$

⑵　$V=\int_0^1\int_0^1 x^2\,dx\,dy$

$$=\int_0^1 dy\cdot\int_0^1 x^2\,dx$$

$$=1\cdot\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}$$

⑶　$V=\int_0^1\int_0^{x^2} (x+y)\,dx\,dy$

$$=\int_0^1\left[xy+\frac{y^2}{2}\right]_0^{x^2} dx=\int_0^1\left(x^3+\frac{x^4}{2}\right)dx$$

$$=\left[\frac{x^2}{4}+\frac{x^5}{10}\right]_0^1=\frac{1}{4}+\frac{1}{10}$$

$$=\frac{7}{20}$$

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⑴　曲面 $z=x^2+y^2$ と平面 $z=1$ との交わりは $x^2+y^2=1$ となる。

∴求める体積 V と積分領域 D は

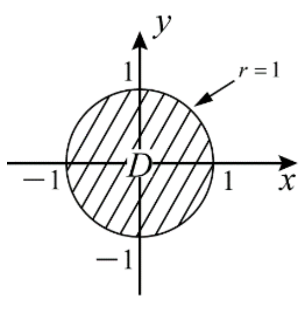
$$V=\iint_D\left(1-\left(x^2+y^2\right)\right)dx\,dy,\quad D=\left\{(x,\ y)\mid x^2+y^2\leqq 1\right\}$$

極座標に変換すれば P.42 $\boxed{2}$ より

$$V=\int_0^{2\pi}\int_0^1\left(1-r^2\right)r\,dr\,dy=2\pi\int_0^1\left(r-r^3\right)dr$$

$$=2\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1=2\pi\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$=\frac{\pi}{2}$$



(2) 曲面 $z=x^2+y^2$ と平面 $z=2x$ との交わりは $x^2+y^2=2x$ より $(x-1)^2+y^2=1$ であるから，

求める体積 V と積分領域 D は

$V=\iint_D \left(2x-\left(x^2+y^2\right)\right) dx\, dy$，　$D=\left\{(x,\ y) \mid (x-1)^2+y^2 \leqq 1\right\}$

$x^2-2x+1+y^2 \leqq 1$

$r^2-2r\cos\theta \leqq 0$

$r>0$ のとき

$r-2\cos\theta \leqq 0$ より

極座標に変換する D は

$D'=\left\{(r,\ \theta) \mid 0 \leqq r \leqq 2\cos\theta,\ -\dfrac{\pi}{2} \leqq \theta \leqq \dfrac{\pi}{2}\right\}$ に写るので

$$V=\iint_D \left(2x-\left(x^2+y^2\right)\right) dx\, dy$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\left(2r\cos\theta-r^2\right)\cdot r\, dr\, d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\left(2r^2\cos\theta-r^3\right)dr$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{2}{3}r^3\cos\theta-\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{16}{3}\cos^4\theta-4\cos^4\theta\right)d\theta$$

$$=\frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\, d\theta$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta\, d\theta$$

$$=\frac{8}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$　（公式 $\boxed{32}$）

$$=\frac{\pi}{2}$$

⑶　$x^2+y^2+z^2=2^2=4$　より　$z=\pm\sqrt{4-x^2-y^2}$　であるから，

求める体積 V と積分領域 D は

$$V=\iint_D\left\{\sqrt{4-x^2-y^2}-\left(-\sqrt{4-x^2-y^2}\right)\right\}dx\,dy$$

$$=2\iint_D\sqrt{4-x^2-y^2}\ dx\,dy$$

$$D=\left\{\left(x\quad y\right)|\,x^2+y^2\leqq 1\right\}$$

極座標に変換して

$$V=2\int_0^{2\pi}\int_0^1\left(4-r^2\right)^{\frac{1}{2}}r\,dr\,d\theta$$

$$=4\pi\int_0^1\left(4-r^2\right)^{\frac{1}{2}}\ r\,dr$$

$$=4\pi\left[-\frac{1}{3}\left(4-r^2\right)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{4}{3}\left(8-3\sqrt{3}\right)\pi$$

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⑴　$$\int_0^{+\infty}\int_0^{+\infty}e^{-\left(x^2+y^2\right)}\ dx\,dy=\int_0^{+\infty}e^{-x^2}\ dx\cdot\int_0^{+\infty}e^{-y^2}\ dy$$

$$=\frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2}$$

$$=\frac{\pi}{4}$$

⑵　極座標に変換して

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(x^2+y^2\right)e^{-\left(x^2+y^2\right)}\ dx\,dy=\int_0^{2\pi}\int_0^{+\infty}r^2e^{-r^2}\ r\,dr\,d\theta$$

$$=2\pi\int_0^{+\infty}r^2e^{-r^2}\ r\,dr$$　$\left(t=r^2\text{ と置換して }\frac{dt}{dr}=2r\right)$

$$=\pi\int_0^{+\infty}te^{-t}\ dt$$

$$=\pi\int_0^{+\infty}t\left(-e^{-t}\right)'\ dt$$

$\left(\begin{array}{l} St\left(-e^{-t}\right)'\ dt \\ =t\left(-e^{-t}\right)-S\left(t\right)'\left(-e^{-t}\right)dt\quad \text{公式}\boxed{31}\end{array}\right)$

$$=\pi\left(\left[-te^{-t}\right]_0^{+\infty}+\int_0^{+\infty}e^{-t}\ dt\right)$$

$$=\pi\left[-e^{-t}\right]_0^{+\infty}$$

$$=\pi$$

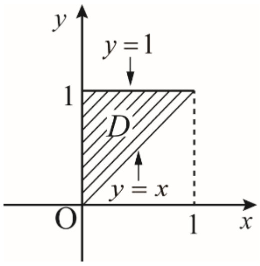
$\left(\begin{array}{l}\lim_{t\to\infty}te^{-t}=\lim_{t\to\infty}\frac{t}{e^t}\ \left(\frac{\infty}{\infty}\text{の不定形}\right) \\ \qquad=\lim_{t\to\infty}\frac{1}{e^t}\ \left(\text{ロピタルの定理}\right)\end{array}\right)$

132　質量 M は

⑴

$$M = \int_0^1 \int_x^1 (x+y)\,dy\,dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_x^1 dx = \int_0^1 \left\{ x + \frac{1}{2} - \left(x^2 + \frac{x^2}{2} \right) \right\} dx$$



$$= \int_0^1 \left(\frac{1}{2} + x - \frac{3x^2}{2} \right) dx$$

$$= \left[\frac{x}{2} + \frac{x^2}{2} - \frac{x^3}{2} \right]_0^1 = \frac{1}{2} + \frac{1}{2} - \frac{1}{5}$$

$$= \frac{1}{2}$$

重心の x 座標は次の M_x について $\dfrac{M_x}{M}$

$$M_x = \int_0^1 \int_x^1 x(x+y)\,dy\,dx$$

$$= \int_0^1 \left[x^2 y + x\frac{y^2}{2} \right]_x^1 dx = \int_0^1 \left\{ \left(x^2 + \frac{x}{2} \right) - \left(x^3 + \frac{x^3}{2} \right) \right\} dx$$

$$= \int_0^1 \left(\frac{x}{2} + x^2 - \frac{3x^3}{2} \right) dx$$

$$= \left[\frac{x^2}{4} + \frac{x^3}{3} - \frac{3x^4}{8} \right]_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{3}{8} = \frac{6+8-9}{24}$$

$$= \frac{5}{24}$$

重心の y 座標は次の M_y について $\dfrac{M_y}{M}$

$$M_y = \int_0^1 \int_x^1 y(x+y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_x^1 dx = \int_0^1 \left\{ \left(\frac{x}{2} + \frac{1}{3} \right) - \left(\frac{x^3}{2} + \frac{x^3}{3} \right) \right\} dx$$

$$= \int_0^1 \left(\frac{1}{3} + \frac{x}{2} - \frac{5x^3}{6} \right) dx$$

$$= \left[\frac{x}{3} + \frac{x^2}{4} - \frac{5x^4}{24} \right]_0^1 = \frac{8+6-5}{24}$$

$$= \frac{9}{24} = \frac{3}{8}$$

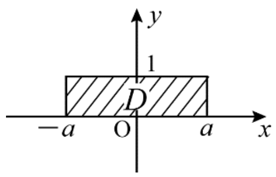
以上から重心 G の座標は

$$\mathrm{G}\left(\frac{M_x}{M},\ \frac{M_y}{M} \right) = \left(\frac{5}{24} \times \frac{8}{7},\ \frac{3}{8} \times \frac{8}{7} \right) = \left(\frac{5}{12},\ \frac{3}{4} \right)$$

(2) $I_y = \iint_D \rho x^2 \, dx \, dy = \int_{-a}^{a} \rho x^2 \int_0^1 dy \, dx$

$= \rho \left[\frac{1}{3} x^3 \right]_{-a}^{a} = \frac{\rho}{3} \left\{ a^3 - (-a)^3 \right\}$

$= \frac{2\rho a^3}{3}$



133

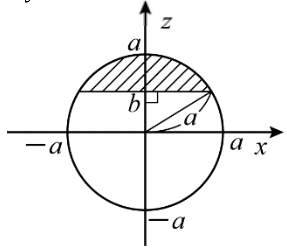
(1) $x^2 + y^2 + z^2 = a^2$ より $z = \sqrt{a^2 - x^2 - y^2}$ であるから

$z_x = -x\left(a^2 - x^2 - y^2 \right)^{-\frac{1}{2}}$

$z_y = -y\left(a^2 - x^2 - y^2 \right)^{-\frac{1}{2}}$

$\therefore \quad \sqrt{\left(z_x \right)^2 + \left(z_y \right)^2 + 1} = a\left(a^2 - x^2 - y^2 \right)^{-\frac{1}{2}}$

y 軸方向からみると



着目している曲面のふちは

半径 $\sqrt{a^2 - b^2}$ の円

従って，求める面積 S は

$$S = \iint_D a\left(a^2 - x^2 - y^2 \right)^{-\frac{1}{2}} dx \, dy$$

$$D = \left\{ \left(x, \ y \right) | \ x^2 + y^2 \leqq a^2 - b^2 \right\}$$

となる。

極座標に変換すると

$$S = \int_0^{2\pi} \int_0^{\sqrt{a^2 - b^2}} a\left(a^2 - r^2 \right)^{-\frac{1}{2}} \cdot r \, dr \, d\theta$$

$$= 2\pi a \int_0^{\sqrt{a^2 - b^2}} \left(a^2 - r^2 \right)^{-\frac{1}{2}} \cdot r \, dr$$

$$= -\pi a \int_0^{\sqrt{a^2 - b^2}} \left(a^2 - r^2 \right)^{-\frac{1}{2}} \cdot \left(-2r \right) dr$$

$\left[\begin{array}{l} a^2 - r^2 = t \text{ とする置換積分} \\ -2r = \dfrac{dt}{dr} \end{array} \right]$

$$= -\pi a \left[2\left(a^2 - r^2 \right)^{\frac{1}{2}} \right]_0^{\sqrt{a^2 - b^2}}$$

$$= -2\pi a \left(\sqrt{b^2} - \sqrt{a^2} \right) = 2a\left(a - b \right)\pi$$

⑵　$z = \mathrm{Tan}^{-1}\left(\dfrac{y}{x}\right)$ より　$z_x = \dfrac{y\left(-x^{-2}\right)}{1+\left(\dfrac{y}{x}\right)^2} = \dfrac{-y}{x^2+y^2}$　　（公式 $\boxed{28}$ ）

同様にして

$$z_y = \frac{x}{x^2+y^2}$$

$$\therefore\quad \sqrt{\left(z_x\right)^2+\left(z_y\right)^2+1} = \sqrt{\frac{y^2}{\left(x^2+y^2\right)^2}+\frac{x^2}{\left(x^2+y^2\right)^2}+1}$$

$$= \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}$$

従って，求める曲面積 S は

$$S = \iint_D \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}\,dx\,dy$$

$$D_\varepsilon = \left\{\left(x,\ y\right) \mid x^2+y^2 \leqq 1,\ x>0,\ y \geqq 0\right\}$$

十分小さい正数 ε，δについて

$$D'_\varepsilon = \left\{\left(r \quad \theta\right) \middle| \begin{array}{l} 0<\varepsilon \leqq r \leqq 1 \\ 0 \leqq \theta \leqq \dfrac{\pi}{2}-\delta \end{array}\right\}$$

となる。極座標に変換すると

$$S_\varepsilon = \int_0^{\frac{\pi}{2}-\delta}\int_{0+\varepsilon}^{1} \frac{\sqrt{r^2+1}}{\sqrt{r^2}}\cdot r\,dr\,d\theta$$

$$= \left(\frac{\pi}{2}-\delta\right)\int_{0+\delta}^{1}\sqrt{r^2+1}\,dr \quad \text{（P.20（IV）公式）}$$

$$= \left(\frac{\pi}{2}-\delta\right)\left[\frac{1}{2}r\sqrt{r^2+1}+\frac{1}{2}\log\left(r+\sqrt{r^2+1}\right)\right]_{0+\varepsilon}^{1}$$

$\varepsilon \to 0$，$\delta \to 0$ とすると

$$S = \frac{\pi}{4}\left(\sqrt{2}+\log\left(1+\sqrt{2}\right)\right)$$

４章　重積分

４章の問題

1

⑴　$$\int_0^1\int_0^2 xy^2\ dx\,dy=\int_0^1 y^2\ dy\cdot\int_0^2 x\,dx$$
$$=\left[\frac{y^3}{3}\right]_0^1\cdot\left[\frac{x^2}{2}\right]_0^2$$
$$=\frac{1}{3}\cdot\frac{4}{2}$$
$$=\frac{2}{3}$$

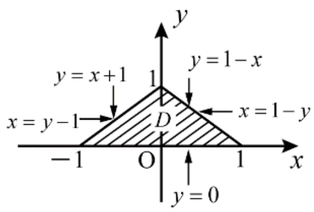
⑵　$$\int_1^2\int_0^2(2x+y)\,dy\,dx=\int_1^2\left[2xy+\frac{y^2}{2}\right]_0^2 dx$$
$$=\int_1^2(4x+2)\,dx$$
$$=\left[2x^2+2x\right]_1^2$$
$$=8$$

2

⑴　$$\int_{-1}^0\int_0^{x+1} f(x,\ y)\,dy\,dx+\int_0^1\int_0^{1-x} f(x,\ y)\,dy\,dx$$

または

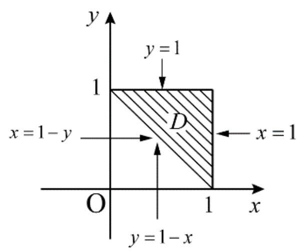
$$\int_0^1\int_{y-1}^{1-y} f(x,\ y)\,dx\,dy$$



⑵　$$\int_0^1\int_{1-x}^1 f(x,\ y)\,dx\,dy$$

または

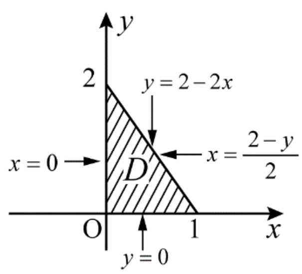
$$\int_0^1\int_{1-y}^1 f(x,\ y)\,dx\,dy$$



⑶　$$\int_0^1\int_0^{2-2x} f(x,\ y)\,dy\,dx$$

または

$$\int_0^2\int_0^{\frac{2-y}{2}} f(x,\ y)\,dx\,dy$$



3

⑴ $\int_0^2 \int_1^2 xy^2 \ dx \, dy = \int_0^2 y^2 \ dy \cdot \int_1^2 x \, dx$

$$= \left[\frac{y^3}{3} \right]_0^2 \cdot \left[\frac{x^2}{2} \right]_1^2 = \frac{8}{3} \cdot \frac{3}{2}$$

$$= 4$$

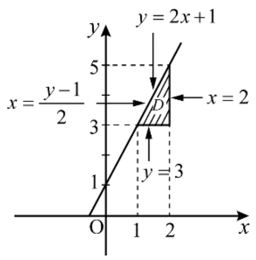
⑵ $\int_2^4 \int_1^3 (x+y) \, dx \, dy = \int_2^4 \left[\frac{x^2}{2} + xy \right]_1^3 dy = \int_2^4 \left\{ \left(\frac{9}{2} + 2y \right) - \left(\frac{1}{2} + y \right) \right\} dy$

$$= \int_2^4 (4+2y) \, dy = \left[4y + y^2 \right]_2^4$$

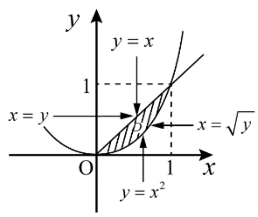
$$= 20$$

4

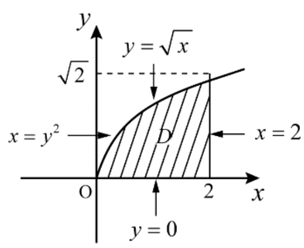
⑴ $\int_1^2 \int_3^{2x+1} f(x, \ y) \, dy \, dx = \int_3^5 \int_{\frac{y-1}{2}}^2 f(x, \ y) \, dx \, dy$



⑵ $\int_0^1 \int_y^{\sqrt{y}} f(x, \ y) \, dx \, dy = \int_0^1 \int_{x^2}^{x} f(x, \ y) \, dy \, dx$

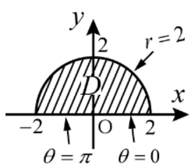


⑶ $\int_0^{\sqrt{2}} \int_{y^2}^2 f(x, \ y) \, dx \, dy = \int_0^2 \int_0^{\sqrt{x}} f(x, \ y) \, dy \, dx$

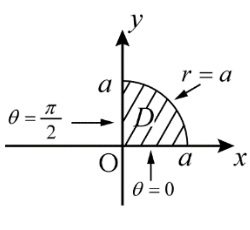


5

⑴ $\int_0^{\pi} \int_0^2 r^2 \ dr \ d\theta$



⑵ $\int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin\theta \ dr \ d\theta$



6

⑴

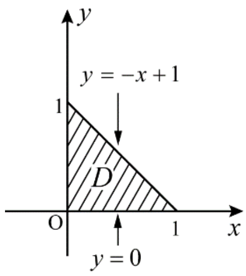
$$\iint_D (1+x+y)\,dx\,y=\int_0^1\int_0^{1-x}(1+x+y)\,dy\,dx=\int_0^1\left[y+xy+\frac{y^2}{2}\right]_0^{1-x}dx$$

$$=\int_0^1\left((1-x)+x(1-x)+\frac{(1-x)^2}{2}\right)dx$$

$$=\int_0^1\left(1-x^2+\frac{1}{2}(x-1)^2\right)dx$$

$$=\left[x-\frac{1}{3}x^3+\frac{1}{6}(x-1)^3\right]_0^1=1-\frac{1}{3}-\left(\frac{-1}{6}\right)$$

$$=\frac{5}{6}$$



⑵

$$\iint_D \frac{1}{\sqrt{1+x}}\,dx\,dy\int_0^1\int_{x^2}^{x}\frac{1}{\sqrt{1+x}}\,dy\,dx$$

$$=\int_0^1\frac{1}{\sqrt{1+x}}\left[y\right]_{x^2}^{x}dx=\int_0^1\frac{1}{\sqrt{1+x}}\left(x-x^2\right)dx$$

ここで，$t=\sqrt{1+x}$ と置換すれば，

$x=t^2-1$，$dx=2tdt$，$x:0\to 1$ のとき，$t:1\to\sqrt{2}$ であるから

$$=\int_1^{\sqrt{2}}\frac{1}{t}\left(t^2-1-\left(t^2-1\right)^2\right)2t\,dt$$

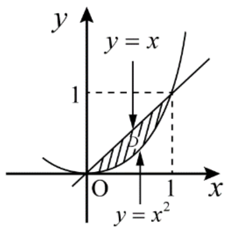
$$=2\int_1^{\sqrt{2}}\left(t^2-1-t^4+2t^2-1\right)dt$$

$$=2\int_1^{\sqrt{2}}\left(-t^4+3t^2-2\right)dt$$

$$=2\left[-\frac{1}{5}t^5+t^3-2t\right]_1^{\sqrt{2}}$$

$$=2\left(-\frac{4\sqrt{2}}{5}+2\sqrt{2}-2\sqrt{2}+\frac{1}{5}-1+2\right)$$

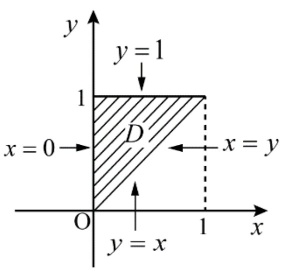
$$=2\left(-\frac{4\sqrt{2}}{5}+\frac{6}{5}\right)=\frac{12-8\sqrt{2}}{5}$$



⑶ $\iint_D x^2y\,dx\,dy=\int_0^1\int_0^y x^2y\,dx\,dy$

$$=\int_0^1\left[\frac{x^3y}{3}\right]_0^y dy=\int_0^1\frac{y^4}{3}dy$$

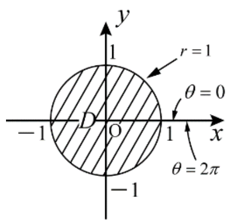
$$=\left[\frac{y^5}{15}\right]_0^1=\frac{1}{15}$$



(4) $\displaystyle \iint_D e^{-\left(x^2+y^2\right)}\,dx\,dy = \int_0^{2\pi}\int_0^1 e^{-r^2} r\,dr\,d\theta$

$\displaystyle = \int_0^{2\pi} d\theta \cdot \int_0^1 e^{-r^2} r\,dr$

$\displaystyle = 2\pi \cdot \int_0^1 e^{-r^2} r\,dr$



$r^2 = t$ とおくと

$2r = \dfrac{dt}{dr}$ $\qquad \begin{array}{c|ccc} r & 0 & \to & 1 \\ \hline t & 0 & \to & 1 \end{array}$

$\displaystyle = 2\pi \int_0^1 e^{-t} \cdot \frac{1}{2}\,dt$

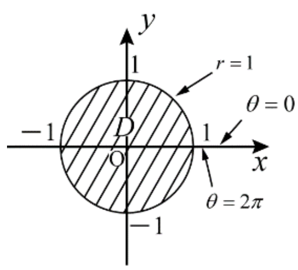
$= \pi \left[-e^{-t}\right]_0^1$

$= \pi \left(-e^{-1} + 1\right)$

(5)

$1 - r^2 = t$ とおくと

$-2r = \dfrac{dt}{dr}\left(r dr = \dfrac{1}{-2}dt\right)$ $\qquad \begin{array}{c|ccc} r & 0 & \to & 1 \\ \hline t & 1 & \to & 0 \end{array}$



$\displaystyle \iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy = \int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}}\,r^2\,dr\,d\theta$

$\displaystyle = 2\pi \int_0^1 (1-r^2)^{-\frac{1}{2}}\,dr$

$\displaystyle = 2\pi \int_1^0 t^{-\frac{1}{2}}\left(-\frac{1}{2}\right)dr$

$\displaystyle = \pi \int_0^1 t^{-\frac{1}{2}}\,dt$

$\displaystyle = \pi \left[2t^{-\frac{1}{2}}\right]_0^1$

$= 2\pi$

6 $I=\iint_D\left\{\left(x-y\right)^2+\left(x+2y\right)^2\right\}dx\,dy$

今 $-2\leqq x-y\leqq 2$，$-1\leqq x+2y\leqq 2$ なので

$\begin{cases} u=x-y \\ v=x+2y \end{cases}$ つまり $\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ と変換すると

$\frac{1}{3}=\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}$ で $\left|J\left(u\quad v\right)\right|=\left|\frac{2}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{1}{3}\right|=\frac{1}{3}$ なので

$dx\,dy=\frac{1}{3}du\,dv$ かつ $-2\leqq u\leqq 2$，$-1\leqq v\leqq 1$となる。

$$
\begin{aligned}
I &= \frac{1}{3}\int_{-2}^{2}\left(\int_{-1}^{1}\left(u^2+v^2\right)dv\right)du \\
&= \frac{4}{3}\int_{0}^{2}\left(\int_{0}^{1}\left(u^2+v^2\right)dv\right)du \\
&= \frac{4}{3}\int_{0}^{2}\left[u^2v+\frac{1}{3}v^3\right]_0^1 du \\
&= \frac{4}{3}\int_{0}^{2}\left(u^2+\frac{1}{3}\right)du \\
&= \frac{4}{3}\left[\frac{1}{3}u^3+\frac{1}{3}u\right]_0^2 \\
&= \frac{4}{9}\left[u^3+u\right]_0^2 \\
&= \frac{4}{9}\cdot\left(8+2\right) \\
&= \frac{40}{9}
\end{aligned}
$$

(7) $I = \iint_D \left(x - y \right) e^{x-y} \ dx \ dy$

ここで $\begin{cases} u = x + y \\ v = x - y \end{cases}$ つまり $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ と変換すると

$\dfrac{1}{-2} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ で $\left| J \left(u \quad v \right) \right| = \left| \dfrac{1}{2} \times \left(-\dfrac{1}{2} \right) - \dfrac{1}{2} \times \dfrac{1}{2} \right| = \dfrac{1}{2}$ なので

$dx\, dy = \dfrac{1}{2} du\, dv$ かつ $0 \leqq u \leqq 1, \ 0 \leqq v \leqq 1$ となる。

$$
\begin{aligned}
I &= \frac{1}{2} \int_0^1 \int_0^1 u e^v \ du \ dv \\
&= \frac{1}{2} \int_0^1 u \ du \cdot \int_0^1 e^v \ dv \\
&= \frac{1}{2} \left[\frac{u^2}{2} \right]_0^1 \cdot \left[e^v \right]_0^1 \\
&= \frac{1}{2} \cdot \frac{1}{2} \cdot \left(e - 1 \right) \\
&= \frac{e-1}{4}
\end{aligned}
$$

(8) $\iint_D \left(x^2 + y^2 \right) dx\, dy$

$\begin{cases} u = \dfrac{x}{a} \\ v = \dfrac{y}{b} \end{cases}$ つまり $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{1}{a} & 0 \\ 0 & \dfrac{1}{b} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ と変換すると

$ab \begin{pmatrix} \dfrac{1}{b} & 0 \\ 0 & \dfrac{1}{a} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ なので $\left| J\left(u \quad v \right) \right| = \left| ab - 0 \times 0 \right| = ab$

$\therefore \quad dx\, dy = ab\, du\, dv$ かつ

積分領域 D は $D = \left\{ \left(u,\ v \right) \mid u^2 + v^2 \leqq 1 \right\}$ となるから

$$I = \iint_D \left(\left(au \right)^2 + \left(bv \right)^2 \right) ab\, du\, dv$$
$$= \iint_D \left(a^3bu^2 + ab^3v^2 \right) au\, dv$$

極座標を導入して $\begin{cases} u = r\cos\theta \\ v = r\sin\theta \end{cases}$ として

$$I = 4\int_0^{\frac{\pi}{2}} \int_0^1 \left(a^3br^2\cos^2\theta + ab^3r^2\sin^2\theta \right) r\, dr\, d\theta$$
$$= 4\int_0^{\frac{\pi}{2}} \left(a^3b\cos^2\theta + ab^3\sin^2\theta \right) \left(\int_0^1 r^3 dr \right) d\theta$$
$$= 4\int_0^{\frac{\pi}{2}} \left(a^3b\cos^2\theta + ab^3\sin^2\theta \right) \cdot \frac{1}{4}\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \left(a^3b\cos^2\theta + ab^3\sin^2\theta \right) d\theta$$
$$= a^3b\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta + ab^3\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$
$$= a^3b \cdot \frac{1}{2} \cdot \frac{\pi}{2} + ab^3 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$
$$= a^3b \times \frac{\pi}{4} + ab^3 \times \frac{\pi}{4} \quad \text{（公式}\boxed{32}\text{）}$$
$$= \frac{ab}{4}\left(a^2 + b^2 \right)\pi$$

⑼ $$\iint_D \frac{x+y}{x^2+y^2}\,dx\,dy = \int_0^1 \int_0^x \frac{x+y}{x^2+y^2}\,dy\,dx$$

ここで，まず $\displaystyle\int_0^x \frac{x+y}{x^2+y^2}\,dy$ を計算する

$$\int_0^x \frac{x+y}{x^2+y^2}\,dy = \int_0^x \frac{x}{x^2+y^2}\,dy + \int_0^x \frac{y}{x^2+y^2}\,dy$$

$$= x\int \frac{1}{x^2+y^2}\,dy + \frac{1}{2}\int_0^x \frac{2y}{x^2+y^2}\,dy$$

$$= x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x + \left[\frac{1}{2}\log\left(x^2+y^2\right)\right]_0^x \quad （公式\boxed{28}\ \boxed{30}\ 教P.59）$$

$$= \tan^{-1}(1) - \tan^{-1}(0) + \frac{1}{2}\left(\log\left(2x^2\right) - \log\left(x^2\right)\right)$$

$$= \frac{\pi}{4} + \frac{1}{2}\cdot\log\left(\frac{2x^2}{x^2}\right)$$

$$= \frac{\pi}{4} + \frac{1}{2}\log 2$$

したがって

$$\int_0^1 \int_0^x \frac{x+y}{x^2+y^2}\,dy\,dx = \int_0^1 \left(\frac{\pi}{4} + \frac{1}{2}\log 2\right)dx$$

$$= \left(\frac{\pi}{4} + \frac{1}{2}\log 2\right)\int_0^1 dx$$

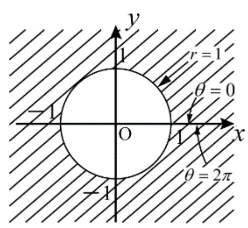
$$= \left(\frac{\pi}{4} + \frac{1}{2}\log 2\right)\Big[\,x\,\Big]_0^1$$

$$= \left(\frac{\pi}{4} + \frac{1}{2}\log 2\right)(1-0)$$

$$= \frac{\pi}{4} + \frac{1}{2}\log 2$$

⑽ 極座標に変換する

$$\iint_D \frac{1}{\left(x^2+y^2\right)^2}\,dx\,dy = \int_0^{2\pi}\int_1^{+\infty} \frac{1}{r^4}\cdot r\,dy\,d\theta$$



$$= 2\pi\int_0^{+\infty} r^{-3}\,dr$$

$$= \lim_{R\to+\infty} 2\pi\int_0^R r^{-3}\,dr$$

$$= \lim_{R\to+\infty} 2\pi\left[-\frac{1}{2r^2}\right]_1^R$$

$$= \lim_{R\to+\infty} 2\pi\left(-\frac{1}{2R^2} + \frac{1}{2}\right)$$

$$= \pi$$

7　まず $z=-a$ によって切られる

$\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}+z^2=1$ の切り口の方程式を求めると $z=-a$ を代入して

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\left(-a\right)^2=1 \quad\Longrightarrow\quad x^2+y^2=a^2-a^4=a^2\left(1-a^2\right)$$

であることから，切り口は半径 $a\sqrt{1-a^2}$ の円であることが分かる。

$\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}+z^2=1$ を z について解けば

$$z=\pm\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2}}=\pm\sqrt{\frac{a^2\left(x^2+y^2\right)}{a^2}}=\pm\sqrt{\frac{a^2-\left(x^2+y^2\right)}{a}}$$

であるから，着目している立体の上面は平面 $z=-a$，下面は $z=-\dfrac{\sqrt{a^2-\left(x^2+y^2\right)}}{a}$ なので

求める体積 V は

$$V=\iint_D\left\{-a-\left(-\frac{\sqrt{a^2-\left(x^2+y^2\right)}}{a}\right)\right\}dx\,dy$$

$$D=\left\{\left(x,\ y\right)\mid x^2+y^2\leqq a^2\left(1-a^2\right)\right\}$$

ゆえに V は　$V=\displaystyle\iint_D\left(\frac{\sqrt{a^2-\left(x^2+y^2\right)}}{a}-a\right)dx\,dy$

曲座標を導入すればより

$$V=\int_0^{2\pi}\int_0^{a\sqrt{1-a^2}}\left(\frac{\sqrt{a^2-r^2}}{a}-a\right)r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\cdot\int_0^{a\sqrt{1-a^2}}\left(\frac{1}{a}r\left(a^2-r^2\right)^{\frac{1}{2}}-ar\right)dr$$

$$=2\pi\cdot\left[-\frac{1}{3a}\left(a^2-r^2\right)^{\frac{3}{2}}-\frac{a}{2}r^2\right]_0^{a\sqrt{1-a^2}}$$

$$\left(\begin{array}{l}\left\{\left(a^2-r^2\right)^{\frac{3}{2}}\right\}'=\dfrac{3}{2}\left(a^2-r^2\right)^{\frac{1}{2}}\cdot\left(a^2-r^2\right)'=-3r\left(a^2-r^2\right)^{\frac{1}{2}}\\ \text{であることから}\quad\left\{-\dfrac{1}{3}\left(a^2-r^2\right)^{\frac{3}{2}}\right\}'=r\left(a^2-r^2\right)^{\frac{1}{2}}\end{array}\right)$$

$$=2\pi\left\{-\frac{1}{3a}\left(a^2-a^2\left(1-a^2\right)\right)^{\frac{3}{2}}-\frac{a}{2}\cdot a^2\left(1-a^2\right)+\frac{1}{3a}\left(a^2\right)^{\frac{3}{2}}\right\}$$

$$=2\pi\left\{-\frac{1}{3a}\cdot\left(a^4\right)^{\frac{3}{2}}-\frac{a^3}{2}+\frac{a^5}{2}+\frac{1}{3a}\cdot a^3\right\}$$

$$=2\pi\left\{-\frac{1}{3}a^5-\frac{1}{2}a^3+\frac{1}{2}a^5+\frac{1}{3}a^2\right\}$$

$$=2\pi\cdot\frac{a^5-3a^3+2a^2}{6}=\frac{\pi}{3}\left(a^5-3a^3+2a^2\right)$$

V を最大にする a を求めるために $g(a)=a^5-3a^3+2a^2$ とおく

$$
\begin{aligned}
g'(a) &= 5a^4-9a^2+4a \\
&= a(5a^3-9a+4) \\
&= a(a-1)(5a^2+5a-4)
\end{aligned}
$$

極値を求めるために $g'(a)=0$ になる a を求める。

$a(a-1)(5a^2+5a-4)=0$ から，

まず $a=0$，$a=1$，次に $5a^2+5a-4=0$ を解いて

$$
\begin{aligned}
a &= \frac{-5\pm\sqrt{25-4\times5\times(-4)}}{10} \\
&= \frac{-5\pm\sqrt{105}}{10}
\end{aligned}
$$

$0<a\leqq1$ が a の満たすべき条件なので，結局 $g'(a)=0$ なる a は $a=1$ と

$a=\dfrac{-5+\sqrt{105}}{10}$　$\left(\sqrt{105}\fallingdotseq10.2\right)$ となる。

増減表を書いて

a	0	$\cdots$	$\frac{-5+\sqrt{105}}{10}$	$\cdots$	1
$g'(a)$	×	$+$	0	$-$	0
$g(a)$	×	↗	**最大**	↘	0

増減表から（最大値自体を求める必要はない）V を最大にする a の値は

$a=\dfrac{-5+\sqrt{105}}{10}$ である。