２章　積分法

２節　定積分の応用（p.66～83）

練習 1

$$\int_0^2 \left| x^2-1 \right| dx = \int_0^1 \left(1-x^2\right) dx + \int_1^2 \left(x^2-s\right) dx = \left[x-\frac{1}{3}x^3 \right]_0^1 + \left[\frac{1}{3}x^3 - x \right]_1^2$$

$$= \cancel{1} - \frac{1}{3} + \frac{8}{3} - \cancel{2} - \frac{1}{3} + \cancel{1} = 2$$

練習 2

⑴　$x=\dfrac{1}{2}$　となるのは　$t=1$，　$x=1$　となるのは　$t=\sqrt{2}$　のときであり，　$\dfrac{dx}{dt}=t$　だから，

$$\int_1^{\sqrt{2}} \left| t \cdot t \right| dt = \int_1^{\sqrt{2}} t^2 \ dt = \left[\frac{1}{3}t^3 \right]_1^{\sqrt{2}} = \frac{2\sqrt{2}-1}{3}$$

⑵　$\dfrac{dx}{dt}=-a\sin t$　だから，

$$\int_0^{\pi} \left| a\sin t \cdot \left(-a\sin t\right) \right| dt = a^2 \int_0^{\pi} \sin^2 t \ dt = a^2 \int_0^{\pi} \frac{1-\cos 2t}{2} \ dt = \frac{a^2}{2} \left[t - \frac{1}{2}\sin 2t \right]_0^{\pi}$$

$$= \frac{a^2}{2} \left\{ \left(\pi - \cancel{\frac{1}{2}\sin 2\pi} \right) - \left(0 - \cancel{\frac{1}{2}\sin 0} \right) \right\} = \frac{\pi}{2}a^2$$

⑶　1

練習 3

⑴　$\dfrac{1}{2}\displaystyle\int_0^{2\pi} a^2 \ d\theta = \dfrac{a^2}{2}\displaystyle\int_0^{2\pi} d\theta = \dfrac{a^2}{2}\Big[\theta\Big]_0^{2\pi} = \pi a^2$

⑵　$\dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \dfrac{1}{\cos^2\theta} \ d\theta = \dfrac{1}{2}\Big[\tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \dfrac{1}{2}\left\{1-\left(-1\right)\right\} = 1$

⑶　$\dfrac{\pi}{8}$

⑷　$\dfrac{1}{3}$

練習 4

⑴　$y' = 1$ より，　$L = \int_0^1 \sqrt{1+1}\ dx = \sqrt{2}\int_0^1 dx = \sqrt{2}\,\left| x \right|_0^1 = \sqrt{2}$

⑵　$y' = 2x$　より，

$$L = \int_0^1 \sqrt{1+4x^2}\ dx = 2\int_0^1 \sqrt{\frac{1}{4}+x^2}\ dx = \cancel{2}\left[\frac{1}{\cancel{2}}\left\{x\sqrt{\frac{1}{4}+x^2}+\frac{1}{4}\log\left|x+\sqrt{\frac{1}{4}+x^2}\right|\right\}\right]_0^1$$

$$= \left\{\sqrt{\frac{5}{4}}+\frac{1}{4}\log\left|1+\sqrt{\frac{5}{4}}\right|\right\} - \left\{\cancel{0}+\frac{1}{4}\log\left|0+\sqrt{\frac{1}{4}}\right|\right\}$$

$$= \frac{\sqrt{5}}{2}+\frac{1}{4}\log\left(1+\frac{\sqrt{5}}{2}\right)-\frac{1}{4}\log\frac{1}{2} = \frac{\sqrt{5}}{2}+\frac{1}{4}\left\{\log\frac{2+\sqrt{5}}{2}+\log 2\right\}$$

$$= \frac{\sqrt{5}}{2}+\frac{1}{4}\log\left(2+\sqrt{5}\right)$$

⑶　$y' = \dfrac{e^x - e^{-x}}{2}$ より，$\left(y'\right)^2 = \dfrac{\left(e^x\right)^2 - 2e^x e^{-x} + \left(e^{-x}\right)^2}{4} = \dfrac{\left(e^x\right)^2 - 2 + \left(e^{-x}\right)^2}{4}$　だから，

$$L = \int_0^1 \sqrt{1+\frac{\left(e^x\right)^2 - 2 + \left(e^{-x}\right)^2}{4}}\ dx = \frac{1}{2}\int_0^1 \sqrt{4+\left(e^x\right)^2 - 2 + \left(e^{-x}\right)^2}\ dx$$

$$= \frac{1}{2}\int_0^1 \sqrt{\left(e^x\right)^2 + 2 + \left(e^{-x}\right)^2}\ dx = \frac{1}{2}\int_0^1 \sqrt{\left(e^x\right)^2 + 2e^x e^{-x} + \left(e^{-x}\right)^2}\ dx$$

$$= \frac{1}{2}\int_0^1 \sqrt{\left(e^x + e^{-x}\right)^2}\ dx = \frac{1}{2}\int_0^1 \left(e^x + e^{-x}\right) dx = \frac{1}{2}\left[e^x - e^{-x}\right]_0^1$$

$$= \int \left\{\left(e - e^{-1}\right) - \left(\cancel{e^0} - \cancel{e^0}\right)\right\} = \frac{e - e^{-1}}{2}$$

練習 5

⑴　証明

$$\left(\sinh x\right)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x$$

⑵　証明

$$\left(\cosh x\right)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} = \sinh x$$

練習 6

⑴ $\dfrac{dx}{dt}=-a\sin t$, $\dfrac{dy}{dt}=a\cos t$ より，

$$\int_0^{2\pi}\sqrt{\left(-a\sin t\right)^2+\left(a\cos t\right)^2}\ dt=\int_0^{2\pi}\sqrt{a^2\left(\sin^2 t+\cos^2 t\right)}\ dt=a\int_0^{2\pi}dt=a\Big[\ t\ \Big]_0^{2\pi}=2\pi a$$

⑵ $\dfrac{dx}{dt}=t$, $\dfrac{dy}{dt}=t^2$ より，求める長さを L とすると，

$$L=\int_0^1\sqrt{t^2+t^4}\ dt=\int_0^1 t\sqrt{1+t^2}\ dt$$

$u=1+t^2$ とおくと，$\dfrac{du}{dt}=2t$ より，$\dfrac{1}{2}du=tdt$，

t	$0\ \to\ 1$
u	$1\ \to\ 2$

$$\therefore L=\int_1^2\sqrt{u}\cdot\frac{1}{2}\,dt=\frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{3}u^{\frac{3}{2}}\right]_1^2=\frac{2\sqrt{2}-1}{3}$$

⑶ $\sqrt{2}\left(e^{2\pi}-1\right)$

練習 7

⑴ $r=a$, $r'=0$ より，$\displaystyle\int_0^{2\pi}\sqrt{a^2+0^2}\ d\theta=a\int_0^{2\pi}d\theta=1\Big[\ \theta\ \Big]_0^{2\pi}=2\pi a$

⑵ $r=\dfrac{1}{\cos\theta}$, $r'=-\dfrac{-\sin\theta}{\cos^2\theta}=\dfrac{\sin\theta}{\cos^2\theta}$ より，

$$\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\sqrt{\frac{1}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^4\theta}}\ d\theta=\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\sqrt{\frac{\cos^2\theta+\sin^2\theta}{\cos^4\theta}}\ d\theta=\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\frac{1}{\cos^2\theta}\ d\theta$$

$$=\Big[\ \tan\theta\ \Big]_{\frac{\pi}{4}}^{-\frac{\pi}{4}}=\tan\frac{\pi}{4}-\tan\left(-\frac{\pi}{4}\right)=2$$

⑶ $2\left(1-e^{-\pi}\right)$

練習 8

x 軸に垂直な平面による断面積は $\pi\left(\dfrac{1-x^2}{2}\right)^2=\dfrac{\pi}{4}\left(1-2x^2+x^4\right)$ だから，

$$V=\frac{\pi}{4}\int_0^1\left(1-2x^2+x^4\right)dx=\frac{\pi}{4}\left[\ x-\frac{2}{3}x^3+\frac{1}{5}x^5\ \right]_0^1=\frac{\pi}{4}\left(1-\frac{2}{3}+\frac{1}{5}\right)=\frac{\pi}{\cancel{4}}\cdot\frac{\cancel{8}^{2}}{15}=\frac{2\pi}{15}$$

練習 9

⑴　2 つの曲線の交点の x 座標は，$x^2 = \sqrt{x}$ を解いて，$x = 0$，1 だから，

$$\pi\int_0^1 \left(\sqrt{x}\right)^2 dx - \pi\int_0^1 \left(x^2\right)^2 dx = \pi\int_0^1 \left(x - x^4\right) dx = \pi\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3\pi}{10}$$

⑵　2 つの曲線の交点の x 座標は，$\sqrt{1-x^2} = 1-x^2$ を解いて，$x = 0$，± 1 であり，どちらの曲線も y 軸について線対称だから，$0 \leqq x \leqq 1$ の範囲を求めて 2 倍すればよい。よって，

$$2\left\{\pi\int_0^1 \left(\sqrt{1-x^2}\right)^2 dx - \pi\int_0^1 \left(1-x^2\right)^2 dx\right\} = 2\pi\int_0^1 \left(\cancel{1} - x^2 - \cancel{1} + 2x^2 - x^4\right) dx$$

$$= 2\pi\int_0^1 \left(x^2 - x^4\right) dx = 2\pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{4\pi}{15}$$

⑶　$$\pi\int_{-1}^1 \left(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}\right)^2 dx - \pi\int_{-1}^1 \left(e^x + 2 + e^{-x}\right) dx = \pi\left[e^x + 2x - e^{-x}\right]_{-1}^1$$

$$= \pi\left\{\left(e + 2 - e^{-1}\right) - \left(e^{-1} - 2 - e\right)\right\} = 2\pi\left(e + 2 - e^{-1}\right)$$

練習 10

⑴　$\dfrac{dx}{dt} = a\left(1 + \cos t\right)$ より，

$$\pi\int_0^{2\pi} a^2\left(1+\cos t\right)^2 \cdot \left|a(1+\cos t)1+\cos t\right| dt = \pi a^3\int_0^{2\pi}\left(1+\cos t\right)^3 dt = 8\pi a^3\int_0^{2\pi}\cos^6\frac{t}{2}\,dt$$

ここで，$u = \dfrac{t}{2}$ とおくと，

$$= 16\pi a^3\int_0^{\pi}\cos^6 u\,du = 32\pi a^3\int_0^{\frac{\pi}{2}}\cos^6 u\,du = 32\pi a^3 \cdot \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 5\pi^2 a^3$$

②　$\dfrac{4}{3}\pi a^3$

練習 11

⑴　ε を小さい正数とする。

$$\int_{-1}^0 \frac{1}{\sqrt{x+1}}\,dx = \lim_{\varepsilon\to+0}\int_{-1+\varepsilon}^0 \frac{1}{\sqrt{x+1}}\,dx = \lim_{\varepsilon\to+0}\left[2\sqrt{x+1}\right]_{-1+\varepsilon}^0 = \lim_{\varepsilon\to+0}\left(2 - 2\sqrt{\varepsilon}\right) = 2$$

⑵　ε を小さい正数とする。

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to+0}\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to+0}\left[\mathrm{Sin}^{-1}\,x\right]_0^{1-\varepsilon} = \lim_{\varepsilon\to+0}\left(\mathrm{Sin}^{-1}\left(1-\varepsilon\right) - 0\right) = \frac{\pi}{2}$$

練習 12

ε を小さい正数とする。

$$\int_{\varepsilon}^{1} \log\, dx = \left[\, x\log x \,\right]_{\varepsilon}^{1} - \int_{\varepsilon}^{1} x\cdot\frac{1}{x}\, dx = \cancel{\log 1} - \varepsilon\log\varepsilon - \left[\, x \,\right]_{\varepsilon}^{1} = -\varepsilon\log\varepsilon - 1 + \varepsilon = \frac{\log\varepsilon}{-\frac{1}{\varepsilon}} - 1 + \varepsilon$$

$$\therefore \int_{0}^{1} \log\, dx = \lim_{\varepsilon\to+0}\int_{\varepsilon}^{1} \log\, dx = \lim_{\varepsilon\to+0}\left\{ \frac{\log\varepsilon}{-\frac{1}{\varepsilon}} - 1 + \varepsilon \right\} = \lim_{\varepsilon\to+0}\left\{ \frac{\frac{1}{\varepsilon}}{\frac{1}{\varepsilon^2}} - 1 + \varepsilon \right\}$$

$$= \lim_{\varepsilon\to+0}\left\{ \varepsilon - 1 + \varepsilon \right\} = -1$$

練習 13

⑴ K を大きい正数とする。

$$\int_{1}^{\infty} \frac{1}{x^3}\, dx = \lim_{K\to\infty}\int_{1}^{K} \frac{1}{x^3}\, dx = \lim_{K\to\infty}\left[-\frac{1}{2x^2} \right]_{1}^{K} = \frac{1}{2}\lim_{K\to\infty}\left(-\frac{1}{K^2} + 1 \right) = \frac{1}{2}$$

⑵ K を大きい正数とする。

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\, dx = \lim_{K\to\infty}\int_{-K}^{0} \frac{1}{1+x^2}\, dx = \lim_{K\to\infty}\left[\, \mathrm{Tan}^{-1}\, x \,\right]_{-K}^{0} = \lim_{K\to\infty}\left(\cancel{\mathrm{Tan}^{-1}}\, 0 - \mathrm{Tan}^{-1}\left(-K \right) \right)$$

$$= \lim_{K\to\infty} \mathrm{Tan}^{-1}\, K = \frac{\pi}{2}$$

練習 14

K を大きい正数とする。

$$\int_{0}^{K} x^2 e^{-x}\, dx = \left[-x^2 e^{-x} \right]_{0}^{K} + 2\int_{0}^{K} x e^{-x}\, dx = -K^2 e^{-K} + 0 + 2\left\{ \left[-x e^{-x} \right]_{0}^{K} + \int_{0}^{K} e^{-x}\, dx \right\}$$

$$= -K^2 e^{-K} + 2\left\{ -K e^{-K} + 0 + \left[-e^{-x} \right]_{0}^{K} \right\} = -K^2 e^{-K} - 2K e^{-K} - 2e^{-K} + 2$$

$$= -\frac{K^2 + 2K + 2}{e^K} + 2$$

$$\therefore \int_{0}^{\infty} x^2 e^{-x}\, dx = \lim_{K\to\infty}\int_{0}^{K} x^2 e^{-x}\, dx = \lim_{K\to\infty}\left\{ -\frac{K^2 + 2K + 2}{e^K} + 2 \right\} = \lim_{K\to\infty}\left\{ -\frac{2K + 2}{e^K} + 2 \right\}$$

$$= \lim_{K\to\infty}\left\{ -\frac{2}{e^K} + 2 \right\} = 2$$

節末問題（p.84）

1

⑴ $\displaystyle \int_0^4 \left|\sqrt{x}-1\right| dx = \int_0^1 \left(1-\sqrt{x}\right) dx + \int_1^4 \left(1-\sqrt{x}\right) dx = \left[x - \frac{2}{3}\sqrt{x^3} \right]_0^1 + \left[\frac{2}{3}\sqrt{x^3} - x \right]_1^4$

$$= 1 - \frac{2}{3} - 0 + \frac{16}{3} - 4 - \frac{2}{3} + 1 = 2$$

⑵ $\displaystyle \int_0^1 \mathrm{Tan}^{-1}\, x\, dx = \left[x\,\mathrm{Tan}^{-1}\, x \right]_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2}\, dx = \mathrm{Tan}^{-1}\, 1 - 0 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\, dx$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\log\left(1+x^2\right) \right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\left\{ \log 2 - \cancel{\log 1} \right\} = \frac{\pi}{4} - \frac{1}{2}\log 2$$

2

$x = a\cosh t$ より，$\displaystyle \frac{dx}{dt} = a\sinh t$ だから，求める図形の面積を S とすると，

$$S = \int_0^1 \left| b\sinh t \cdot a\sinh t \right| dt = ab\int_0^1 \sinh^2 t\, dt$$

ここで，$\displaystyle \sinh^2 t = \frac{e^{2t}+e^{-2t}-2}{4} = -\frac{1}{2}\left(1 - \frac{e^{2t}+e^{-2t}}{2}\right) = -\frac{1-\cosh 2t}{2}$ だから，

$$S = -\frac{ab}{2}\int_0^1 \left(1-\cosh 2t\right) dt = -\frac{ab}{2}\left[t - \frac{1}{2}\sinh 2t \right]_0^1$$

$$= -\frac{ab}{2}\left\{ \left(1 - \frac{1}{2}\sinh 2\right) - \left(\cancel{0} - \frac{1}{2}\cancel{\sinh 0} \right) \right\} = -\frac{ab}{4}\left(2 - \sinh 2\right)$$

3

⑴ $\displaystyle \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^4 \theta\, d\theta = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{32}$

⑵ $\displaystyle \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{2n} \theta\, d\theta = \frac{1}{2} \cdot \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4} \cdot \frac{\left(2n-1\right)\left(2n-3\right)\cdots 3 \cdot 1}{2n\left(2n-2\right)\cdots 4 \cdot 2}$

4

⑴　$y=\frac{1}{3}\sqrt{x^3}$ より，$\frac{dy}{dx}=\frac{1}{2}\sqrt{x}$　だから

$$\int_0^5\sqrt{1+\frac{1}{4}x}\ dx=\frac{1}{2}\int_0^5\sqrt{x+4}\ dx=\frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{3}\sqrt{\left(x+4\right)^3}\right]_0^5=\frac{1}{3}\left(27-8\right)=\frac{19}{3}$$

⑵　$x=4\sqrt{2}\cos t,\quad y=\sin 2t$ より，$\frac{dx}{dt}=-4\sqrt{2}\sin t$，$\frac{dy}{dt}=2\cos 2t$ だから，

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=32\sin^2 t+4\cos^2 2t=4\left\{4\cdot 2\sin^2 t+\cos^2 2t\right\}=4\left\{4\left(1-\cos 2t\right)+\cos^2 2t\right\}$$

$$=4\left\{4-4\cos 2t+\cos^2 2t\right\}=4\left(2-\cos 2t\right)^2$$　であり，

$$\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt=2\int_0^{\frac{\pi}{2}}\left(2-\cos 2t\right)dt=2\left[2t-\frac{1}{2}\sin t\right]_0^{\frac{\pi}{2}}$$

$$=2\left\{\left(\pi-\frac{1}{2}\cancel{\sin\pi}\right)-\left(\cancel{0}-\frac{1}{2}\cancel{\sin 0}\right)\right\}=2\pi$$

⑶　$r=\theta^2$　より，$\frac{dr}{d\theta}=2\theta$　だから，

$$\int_0^{\pi}\sqrt{\theta^4+4\theta^2}\ d\theta=\int_0^{\pi}\theta\sqrt{\theta^2+4}\ d\theta=\left[\frac{1}{3}\sqrt{\left(\theta^2+4\right)^3}\right]_0^{\pi}=\frac{1}{3}\left\{\sqrt{\left(\pi^2+4\right)^3}-8\right\}$$

5

$\frac{dx}{dt}\neq 0$　とすると，媒介変数表示の関数の微分より　$f'\left(x\right)=\frac{h'\left(t\right)}{g'\left(t\right)}$　であり，$x=g\left(t\right)$　とおく置換積分を考えると，

$$\int_{\alpha}^{\beta}\sqrt{\left\{g'\left(t\right)\right\}^2+\left\{h'\left(t\right)\right\}^2}\ dt=\int_{\alpha}^{\beta}\sqrt{1+\left\{\frac{h'\left(t\right)}{g'\left(t\right)}\right\}^2}\cdot g'\left(t\right)dt=\int_{\alpha}^{\beta}\sqrt{1+\left\{f'\left(x\right)\right\}^2}\ dx$$

6

$\frac{dx}{dt}=\cos t$　より，

$$V=\pi\int_0^{\frac{\pi}{2}}\left(\sin t+\cos t\right)^2\left|\cos t\right|dt=\pi\int_0^{\frac{\pi}{2}}\left(\sin^2 t+\cos^2 t+2\sin t\cos t\right)\cos t\ dt$$

$$=\pi\int_0^{\frac{\pi}{2}}\left(\cos t+2\sin t\cos^2 t\right)dt=\pi\left[\sin t-\frac{2}{3}\cos^3 t\right]_0^{\frac{\pi}{2}}$$

$$=\pi\left\{\left(\sin\frac{\pi}{2}-\frac{2}{3}\cancel{\cos^3\frac{\pi}{2}}\right)\right\}-\left(\cancel{\sin 0}-\frac{2}{3}\cos^3 0\right)=\pi\left(1+\frac{2}{3}\right)=\frac{5\pi}{3}$$