新版応用数学演習 解答

3 章 ラプラス変換

1 節 ラプラス変換

120

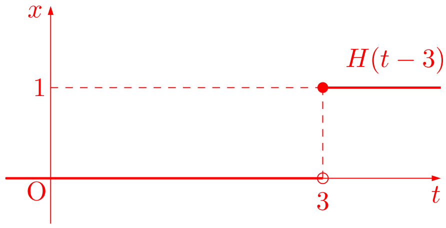
(1) $\dfrac{1}{s-1}$ (2) $\dfrac{1}{s-5}$ (3) $\dfrac{1}{s+4}$ (4) $\dfrac{1}{s+3}$

121

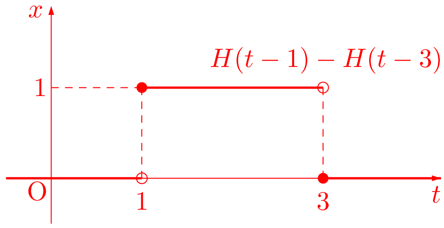
(1) $\dfrac{3}{s^2+9}$ (2) $\dfrac{s}{s^2+4}$ (3) $\dfrac{s}{s^2+3}$ (4) $\dfrac{\sqrt{2}}{s^2+2}$

122

(1)



(2)



123

(1) $\dfrac{e^{-4s}}{s}$ (2) $\dfrac{e^{-\pi s}}{s}$ (3) $\dfrac{e^{-\frac{1}{2}s}}{s}$

124

(1) e^{-s} (2) $\dfrac{1}{2}$ (3) $\dfrac{1}{3}$

125

(1) $\dfrac{3s+2}{s^3}$ (2) $-\dfrac{3}{s(s-3)}$ (3) $\dfrac{2}{s^2-1}$ (4) $\dfrac{8}{s^2(s^2+4)}$ (5) $\dfrac{s+1}{s^2+1}$ (6) $\dfrac{\sqrt{3}\,s-1}{s^2+1}$

126

(1) $\dfrac{2}{(s+1)^3}$ (2) $\dfrac{1}{(s-1)^2}$ (3) $\dfrac{1}{s^2+4s+5}$ (4) $\dfrac{s+1}{s^2+2s+2}$ (5) $\dfrac{3}{s^2-2s+10}$

(6) $\dfrac{s-1}{s^2-2s+5}$

127

(1) $\dfrac{2e^{-2s}}{s^3}$ (2) $\dfrac{e^{-s}}{s-1}$

128

(1) $sF(s)-1$ (2) $s^2F(s)-s+1$

129

(1) $\dfrac{1}{s(s-3)}$ (2) $\dfrac{1}{s^2(s-1)}$ (3) $\dfrac{2s+1}{s^3}$ (4) $\dfrac{s^2+2}{s^4}$ (5) $\dfrac{2}{s(s^2+4)}$ (6) $\dfrac{1}{s^2+9}$

130

(1) $\dfrac{1}{(s+1)^2}$ (2) $\dfrac{4s}{(s+4)^2}$ (3) $\dfrac{s^2-4}{(s^2+4)^2}$

131

(1) $\dfrac{2}{(s+1)^3}$ (2) $\dfrac{6}{(s+1)^4}$ (3) $\dfrac{2s(s^2-3)}{(s^2+1)^3}$

132

(1) $$(f*g)(t)=\int_0^t \sin\tau\sin(t-\tau)\,d\tau=-\frac{1}{2}\int_0^t\{\cos t-\cos(2\tau-t)\}\,d\tau=-\frac{1}{2}\left[\tau\cos t-\frac{1}{2}\sin(2\tau-t)\right]_0^t$$
$$=-\frac{1}{2}\left\{t\cos t-\frac{1}{2}\sin t-0+\frac{1}{2}\sin(-t)\right\}=-\frac{1}{2}\{t\cos t-\sin t\}$$

(2) $$(f*g)(t)=\int_0^t e^{-\tau}(t-\tau)\,d\tau=\Big[-e^{-\tau}(t-\tau)\Big]_0^t-\int_0^t e^{-\tau}\,d\tau=0+e^0t+\Big[e^{-\tau}\Big]_0^t=t+e^{-t}-e^0=e^{-t}+t-1$$

133

(1) $e^t * \sin t = \int_0^t e^\tau \sin(t-\tau)\,d\tau = \Big[e^\tau \cos(t-\tau)\Big]_0^t - \int_0^t e^\tau \cos(t-\tau)\,d\tau$

$= e^t \cos 0 - e^0 \cos t - \left\{\Big[-e^\tau \sin(t-\tau)\Big]_0^t + \int_0^t e^\tau \sin(t-\tau)\,d\tau\right\} = e^t - \cos t + 0 - e^0 \sin t - e^t * \sin t$

$\therefore\ e^t * \sin t = \dfrac{1}{2}(e^t - \cos t - \sin t)$

(2) $\mathscr{L}\left[\dfrac{1}{2}(e^t - \cos t - \sin t)\right] = \mathscr{L}[e^t * \sin t] = \mathscr{L}[e^t]\mathscr{L}[\sin t] = \dfrac{1}{(s-1)(s^2+1)}$

134

(1) 2 (2) t^3 (3) e^{3t} (4) e^{-4t} (5) $\sin 2t$ (6) $\cos 3t$

135

(1) $\dfrac{4}{s^2-4} = \dfrac{1}{s-2} - \dfrac{1}{s+2}$ より，$\mathscr{L}^{-1}\left[\dfrac{4}{s^2-4}\right](t) = e^{2t} - e^{-2t}$

(2) $\dfrac{1}{s^2-s} = \dfrac{1}{s-1} - \dfrac{1}{s}$ より，$\mathscr{L}^{-1}\left[\dfrac{1}{s^2-s}\right](t) = e^t - 1$

(3) $\dfrac{4}{s^2+2s-3} = \dfrac{4}{(s-1)(s+3)} = \dfrac{1}{s-1} - \dfrac{1}{s+3}$ より，$\mathscr{L}^{-1}\left[\dfrac{4}{s^2+2s-3}\right](t) = e^t - e^{-3t}$

136

(1) $\dfrac{s+1}{(s+2)^2} = \dfrac{s+2-1}{(s+2)^2} = \dfrac{1}{s+2} - \dfrac{1}{(s+2)^2}$ より，$\mathscr{L}^{-1}\left[\dfrac{s+1}{(s+2)^2}\right](t) = e^{-2t} - te^{-2t} = (1-t)e^{-2t}$

(2) $\dfrac{s^2+s-2}{(s-2)s^2} = \dfrac{1}{s-2} + \dfrac{1}{s^2}$ より，$\mathscr{L}^{-1}\left[\dfrac{s^2+s-2}{(s-2)s^2}\right](t) = e^{2t} + t$

(3) $\dfrac{3}{(s^2+1)(s^2+4)} = \dfrac{1}{s^2+1} - \dfrac{1}{s^2+4} = \dfrac{1}{s^2+1} - \dfrac{1}{2}\cdot\dfrac{2}{s^2+4}$ より，

$\mathscr{L}^{-1}\left[\dfrac{3}{(s^2+1)(s^2+4)}\right](t) = \sin t - \dfrac{1}{2}\sin 2t$

137

(1) $\dfrac{1}{s^2+2s+2} = \dfrac{1}{(s+1)^2+1}$ より，$\mathscr{L}^{-1}\left[\dfrac{1}{s^2+2s+2}\right](t) = e^{-t}\sin t$

(2) $\dfrac{s-2}{s^2-4s+5} = \dfrac{s-2}{(s-2)^2+1}$ より，$\mathscr{L}^{-1}\left[\dfrac{s-2}{s^2-4s+5}\right](t) = e^{2t}\cos t$

(3) $\dfrac{s+3}{s^2+2s+5} = \dfrac{s+1+2}{(s+1)^2+4} = \dfrac{s+1}{(s+1)^2+4} + \dfrac{2}{(s+1)^2+4}$ より，

$\mathscr{L}^{-1}\left[\dfrac{s+3}{s^2+2s+5}\right](t) = e^{-t}\cos 2t + e^{-t}\sin 2t = e^{-t}(\cos 2t + \sin 2t)$

(4) $\dfrac{s-4}{s^2-2s+10} = \dfrac{s-1-3}{(s-1)^2+9} = \dfrac{s-1}{(s-1)^2+9} - \dfrac{3}{(s-1)^2+9}$ より，

$\mathscr{L}^{-1}\left[\dfrac{s-4}{s^2-2s+10}\right](t) = e^t\cos 3t - e^t \sin 3t = e^t(\cos 3t - \sin 3t)$

138

(1) $\dfrac{6s}{(s^2+9)^2} = -\dfrac{d}{ds}\left(\dfrac{3}{s^2+9}\right)$ より，$\mathscr{L}^{-1}\left[\dfrac{6s}{(s^2+9)^2}\right](t) = t\sin 3t$

(2) $\dfrac{s^2-9}{(s^2+9)^2} = -\dfrac{s^2+9-2s^2}{(s^2+9)^2} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+9}\right)$ より，$\mathscr{L}^{-1}\left[\dfrac{s^2-9}{(s^2+9)^2}\right](t) = t\cos 3t$

(3) $\dfrac{s^2-2}{(s^2+2)^2} = -\dfrac{s^2+2-2s^2}{(s^2+2)^2} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+2}\right)$ より，$\mathscr{L}^{-1}\left[\dfrac{s^2-2}{(s^2+2)^2}\right](t) = t\cos\sqrt{2}\,t$

139

(1) $\dfrac{1}{e^{2t}} = e^{-2t}$ より，$\mathscr{L}\left[\dfrac{1}{e^{2t}}\right](s) = \dfrac{1}{s+2}$ (2) $\mathscr{L}\left[e^{\frac{t}{3}}\right](s) = \dfrac{1}{s-\dfrac{1}{3}} = \dfrac{3}{3s-1}$

(3) $\dfrac{1}{\sqrt{e^t}} = e^{-\frac{t}{2}}$ より，$\mathscr{L}\left[\dfrac{1}{\sqrt{e^t}}\right](s) = \dfrac{1}{s+\dfrac{1}{2}} = \dfrac{2}{2s+1}$

(4) $\mathscr{L}\left[\cos\dfrac{t}{2}\right](s) = \dfrac{s}{s^2+\dfrac{1}{4}} = \dfrac{4s}{4s^2+1}$ (5) $\mathscr{L}\left[\sin\dfrac{t}{3}\right](s) = \dfrac{\frac{1}{3}}{s^2+\dfrac{1}{9}} = \dfrac{3}{9s^2+1}$

(6) $\mathscr{L}\left[\sin\dfrac{3t}{2}\right](s) = \dfrac{\frac{3}{2}}{s^2+\dfrac{9}{4}} = \dfrac{6}{4s^2+9}$ (7) $\mathscr{L}[1-2\sin^2 t](s) = \mathscr{L}[\cos 2t](s) = \dfrac{s}{s^2+4}$

(8) $\mathscr{L}[\sin^2 t](s) = \mathscr{L}\left[\dfrac{1-\cos 2t}{2}\right](s) = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right) = \dfrac{2}{s(s^2+4)}$

(9) $\mathscr{L}\left[2\cos^2\dfrac{t}{3} - 1\right](s) = \mathscr{L}\left[\cos\dfrac{2t}{3}\right](s) = \dfrac{s}{s^2+\dfrac{4}{9}} = \dfrac{9s}{9s^2+4}$

140

(1) $\dfrac{e^{-4s}}{s}$ (2) $\dfrac{1}{3}$ (3) $\dfrac{1}{3}e^{-2s}$ (4) 2 (5) $2e^{-s}$ (6) $3e^{-2s}$

141

(1) $-\dfrac{(4s+1)(s-1)}{s^3}$ (2) $\dfrac{(s-3)(s+2)}{(s+6)s^2}$

(3) $\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^2 = e^t + e^{-t} + 2$ より

$\mathscr{L}\left[\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^2\right](s) = \dfrac{1}{s-1} + \dfrac{1}{s+1} + \dfrac{2}{s} = \dfrac{s^2+s+s^2-s+2s^2-2}{(s-1)(s+1)s} = \dfrac{2(2s^2-1)}{(s-1)(s+1)s}$

(4) $2\sin\left(2t-\dfrac{\pi}{3}\right) = 2\left\{\sin 2t\cos\left(-\dfrac{\pi}{3}\right) + \cos 2t\sin\left(-\dfrac{\pi}{3}\right)\right\} = \sin 2t - \sqrt{3}\cos 2t$ より

$\mathscr{L}\left[2\sin\left(2t-\dfrac{\pi}{3}\right)\right](s) = \dfrac{2}{s^2+4} - \dfrac{\sqrt{3}\,s}{s^2+4} = \dfrac{2-\sqrt{3}\,s}{s^2+4}$

(5) $\dfrac{2}{(s-3)^2+4}$

(6) $\mathscr{L}\left[e^{-t}\cos\dfrac{t}{2}\right](s) = \dfrac{s+1}{(s+1)^2+\left(\dfrac{1}{2}\right)^2} = \dfrac{s+1}{s^2+2s+\dfrac{5}{4}} = \dfrac{4(s+1)}{4s^2+8s+5}$

(7) $\mathscr{L}[H(t-\pi)\sin t](s) = \mathscr{L}[-H(t-\pi)\sin(t-\pi)](s) = -\dfrac{e^{-\pi s}}{s^2+1}$

(8) $\mathscr{L}\left[H\left(t-\dfrac{\pi}{2}\right)\cos t\right](s) = \mathscr{L}\left[-H\left(t-\dfrac{\pi}{2}\right)\sin\left(t-\dfrac{\pi}{2}\right)\right](s) = -\dfrac{e^{-\frac{\pi}{2}s}}{s^2+1}$

(9) $\mathscr{L}[e^t H(t-1)](s) = \mathscr{L}[ee^{t-1}H(t-1)](s) = \dfrac{e^{1-s}}{s-1}$

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(1) $\mathscr{L}\left[\displaystyle\int_0^t te^{2t}\,dt\right](s) = \dfrac{1}{s}\mathscr{L}[te^{2t}](s) = \dfrac{1}{s}\left\{-\dfrac{d}{ds}\left(\mathscr{L}[e^{2t}](s)\right)\right\} = \dfrac{1}{s}\left\{-\dfrac{d}{ds}\left(\dfrac{1}{s-2}\right)\right\} = \dfrac{1}{s(s-2)^2}$

(2) $\mathscr{L}\left[\displaystyle\int_0^t e^t\sin 2t\,dt\right](s) = \dfrac{1}{s}\mathscr{L}[e^t\sin 2t](s) = \dfrac{1}{s}\cdot\dfrac{2}{(s-1)^2+4} = \dfrac{2}{s(s^2-2s+5)}$

(3) $\mathscr{L}\left[\displaystyle\int_0^t e^t\cos t\,dt\right](s) = \dfrac{1}{s}\mathscr{L}[e^t\cos t](s) = \dfrac{1}{s}\cdot\dfrac{s-1}{(s-1)^2+1} = \dfrac{s-1}{s(s^2-2s+2)}$

(4) $\mathscr{L}\left[\displaystyle\int_0^t (e^t+e^{-t})\,dt\right](s) = \dfrac{1}{s}\mathscr{L}[e^t+e^{-t}](s) = \dfrac{1}{s}\left(\dfrac{1}{s-1} + \dfrac{1}{s+1}\right) = \dfrac{1}{s}\cdot\dfrac{s+1+s-1}{s^2-1}$

$= \dfrac{2}{s^2-1}$

(5) $\mathscr{L}\left[\int_0^t t\sin t\,dt\right](s) = \dfrac{1}{s}\mathscr{L}[t\sin t](s) = \dfrac{1}{s}\left\{-\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right)\right\} = \dfrac{1}{s}\cdot\dfrac{2s}{(s^2+1)^2} = \dfrac{2}{(s^2+1)^2}$

(6) $\mathscr{L}[te^{-t}\sin t](s) = -\dfrac{d}{ds}\left\{\mathscr{L}[e^{-t}\sin t](s)\right\} = -\dfrac{d}{ds}\left\{\dfrac{1}{(s+1)^2+1}\right\} = \dfrac{2(s+1)}{(s^2+2s+2)^2}$

(7) $\mathscr{L}[te^t\cos t](s) = -\dfrac{d}{ds}\left\{\mathscr{L}[e^t\cos t](s)\right\} = -\dfrac{d}{ds}\left\{\dfrac{s-1}{(s-1)^2+1}\right\} = -\dfrac{s^2-2s+2-(s-1)(2s-2)}{(s^2-2s+2)^2}$
$= -\dfrac{s^2-2s+2-2s^2+4s-2}{(s^2-2s+2)^2} = \dfrac{s(s-2)}{(s^2-2s+2)^2}$

(8) $\mathscr{L}[t\cos^2 t](s) = -\dfrac{d}{ds}\left\{\mathscr{L}[\cos^2 t](s)\right\} = -\dfrac{d}{ds}\left\{\mathscr{L}\left[\dfrac{1+\cos 2t}{2}\right](s)\right\} = -\dfrac{1}{2}\dfrac{d}{ds}\left\{\dfrac{1}{s}+\dfrac{s}{s^2+4}\right\}$
$= -\dfrac{1}{2}\left\{-\dfrac{1}{s^2}+\dfrac{s^2+4-s\cdot 2s}{(s^2+4)^2}\right\} = \dfrac{1}{2}\cdot\dfrac{s^4+8s^2+16+s^4-4s^2}{s^2(s^2+4)^2} = \dfrac{s^4+2s^2+8}{s^2(s^2+4)^2}$

(9) $\mathscr{L}\Big[t(e^t * t)\Big](s) = -\dfrac{d}{ds}\left\{\mathscr{L}[e^t * t](s)\right\} = -\dfrac{d}{ds}\left\{\dfrac{1}{s-1}\cdot\dfrac{1}{s^2}\right\} = \dfrac{3s^2-2s}{(s-1)^2 s^4} = \dfrac{3s-2}{(s-1)^2 s^3}$

143

(1) $\mathscr{L}[\sinh\omega t](s) = \mathscr{L}\left[\dfrac{e^{\omega t}-e^{-\omega t}}{2}\right](s) = \dfrac{1}{2}\left\{\dfrac{1}{s-\omega}-\dfrac{1}{s+\omega}\right\} = \dfrac{1}{2}\cdot\dfrac{s+\omega-s+\omega}{s^2-\omega^2} = \dfrac{\omega}{s^2-\omega^2}$

(2) $\mathscr{L}[\cosh\omega t](s) = \mathscr{L}\left[\dfrac{e^{\omega t}+e^{-\omega t}}{2}\right](s) = \dfrac{1}{2}\left\{\dfrac{1}{s-\omega}+\dfrac{1}{s+\omega}\right\} = \dfrac{1}{2}\cdot\dfrac{s+\omega+s-\omega}{s^2-\omega^2} = \dfrac{s}{s^2-\omega^2}$

(3) $\mathscr{L}[t\sinh\omega t](s) = -\dfrac{d}{ds}\left\{\mathscr{L}[\sinh\omega t](s)\right\} = -\dfrac{d}{ds}\left\{\dfrac{\omega}{s^2-\omega^2}\right\} = \dfrac{2\omega s}{(s^2-\omega^2)^2}$

(4) $\mathscr{L}[t\cosh\omega t](s) = -\dfrac{d}{ds}\left\{\mathscr{L}[\cosh\omega t](s)\right\} = -\dfrac{d}{ds}\left\{\dfrac{s}{s^2-\omega^2}\right\} = -\dfrac{s^2-\omega^2-s\cdot 2s}{(s^2-\omega^2)^2} = \dfrac{s^2+\omega s}{(s^2-\omega^2)^2}$

144

(1) $\mathscr{L}\left[\int_0^t f(\tau)\,d\tau\right](s) = \mathscr{L}[f(t)*1](s) = \mathscr{L}[f(t)](s)\cdot\mathscr{L}[1](s) = \dfrac{1}{s}F(s)$

(2) (1) より，$\mathscr{L}\left[\int_0^t \tau f(\tau)\,d\tau\right](s) = \dfrac{1}{s}\mathscr{L}[tf(t)](s) = -\dfrac{F'(s)}{s}$

145

(1) $\mathscr{L}^{-1}\left[\dfrac{s+3}{s^2+9}\right](t) = \mathscr{L}^{-1}\left[\dfrac{s}{s^2+9}+\dfrac{3}{s^2+9}\right](t) = \cos 3t + \sin 3t$

(2) $\mathscr{L}^{-1}\left[\dfrac{2s+1}{s^2+4}\right](t) = \mathscr{L}^{-1}\left[\dfrac{2s}{s^2+4}+\dfrac{1}{2}\cdot\dfrac{2}{s^2+4}\right](t) = 2\cos 2t + \dfrac{1}{2}\sin 2t$

(3) $\mathscr{L}^{-1}\left[\dfrac{s+1}{(s+2)^2}\right](t) = \mathscr{L}^{-1}\left[\dfrac{(s+2)-1}{(s+2)^2}\right](t) = \mathscr{L}^{-1}\left[\dfrac{1}{s+2}-\dfrac{1}{(s+2)^2}\right](t) = e^{-2t}-te^{-2t}$
$= (1-t)e^{-2t}$

(4) $\mathscr{L}^{-1}\left[\dfrac{s^2-2s+3}{(s-1)^3}\right](t) = \mathscr{L}^{-1}\left[\dfrac{(s-1)^2+2}{(s-1)^3}\right](t) = \mathscr{L}^{-1}\left[\dfrac{1}{s-1}+\dfrac{2}{(s-1)^3}\right](t) = (t^2+1)e^t$

(5) $\mathscr{L}^{-1}\left[\dfrac{2s+7}{s^2+7s-18}\right](t) = \mathscr{L}^{-1}\left[\dfrac{2s+7}{(s-2)(s+9)}\right](t) = \mathscr{L}^{-1}\left[\dfrac{1}{s-2}+\dfrac{1}{s+9}\right](t) = e^{2t}+e^{-9t}$

(6) $\mathscr{L}^{-1}\left[\dfrac{3s}{2s^2+s-1}\right](t) = \mathscr{L}^{-1}\left[\dfrac{3s}{(s+1)(2s-1)}\right](t) = \mathscr{L}^{-1}\left[\dfrac{1}{s+1}+\dfrac{1}{2s-1}\right](t)$
$= \mathscr{L}^{-1}\left[\dfrac{1}{s+1}+\dfrac{1}{2}\cdot\dfrac{1}{s-\frac{1}{2}}\right](t) = e^{-t}+\dfrac{1}{2}e^{\frac{t}{2}}$

(7) $\mathscr{L}^{-1}\left[\dfrac{s^2+s+2}{s^3-2s^2+4s-8}\right](t) = \mathscr{L}^{-1}\left[\dfrac{s^2+s+2}{(s-2)(s^2+4)}\right](t) = \mathscr{L}^{-1}\left[\dfrac{1}{s-2}+\dfrac{1}{s^2+4}\right](t)$
$= \mathscr{L}^{-1}\left[\dfrac{1}{s-2}+\dfrac{1}{2}\cdot\dfrac{2}{s^2+4}\right](t) = e^{2t}+\dfrac{1}{2}\sin 2t$

(8) $\mathscr{L}^{-1}\left[\dfrac{s^2+4s+2}{s^3+3s^2+2s}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s^2+4s+2}{s(s+1)(s+2)}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s+1}-\dfrac{1}{s+2}+\dfrac{1}{s}\right](t)$
$=e^{-t}-e^{-2t}+1$

(9) $\mathscr{L}^{-1}\left[-\dfrac{4s}{s^4-1}\right](t)=\mathscr{L}^{-1}\left[-\dfrac{4s}{(s^2+1)(s-1)(s+1)}\right](t)=\mathscr{L}^{-1}\left[\dfrac{2s}{s^2+1}-\dfrac{1}{s-1}-\dfrac{1}{s+1}\right](t)$
$=2\cos t-e^t-e^{-t}$

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(1) $\mathscr{L}^{-1}\left[\dfrac{s^2-s+2}{s^3-2s^2+2s}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s^2-s+2}{s(s^2-2s+2)}\right](t)=\mathscr{L}^{-1}\left[\dfrac{(s^2-2s+2)+s}{s(s^2-2s+2)}\right](t)$
$=\mathscr{L}^{-1}\left[\dfrac{1}{s}+\dfrac{1}{s^2-2s+2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s}+\dfrac{1}{(s-1)^2+1}\right](t)=1+e^t\sin t$

(2) $\mathscr{L}^{-1}\left[\dfrac{s^3-2s-2}{s^4+2s^3+2s^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s^3-2s-2}{s^2(s^2+2s+2)}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s+1}{s^2+2s+2}-\dfrac{1}{s^2}\right](t)$
$=\mathscr{L}^{-1}\left[\dfrac{s+1}{(s+1)^2+1}-\dfrac{1}{s^2}\right](t)=e^{-t}\cos t-t$

(3) $\mathscr{L}^{-1}\left[\dfrac{2(s-2)^2}{(s^2+4)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{2}{s^2+4}-\dfrac{8s}{(s^2+4)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{2}{s^2+4}+2\cdot\left(\dfrac{2}{s^2+4}\right)'\right](t)$
$=\sin 2t-2t\sin 2t=(1-2t)\sin 2t$

(4) $\mathscr{L}^{-1}\left[\dfrac{s^3+s^2+s-1}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s(s^2+1)+s^2-1}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s}{s^2+1}-\dfrac{1-s^2}{(s^2+1)^2}\right](t)$
$=\mathscr{L}^{-1}\left[\dfrac{s}{s^2+1}-\left(\dfrac{s}{s^2+1}\right)'\right](t)=\cos t+t\cos t=(t+1)\cos t$

(5) $\mathscr{L}^{-1}\left[\dfrac{s^3-s}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s(s^2+1)-2s}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s}{s^2+1}-\dfrac{2s}{(s^2+1)^2}\right](t)$
$=\mathscr{L}^{-1}\left[\dfrac{s}{s^2+1}+\left(\dfrac{1}{s^2+1}\right)'\right](t)=\cos t-t\sin t$

(6) $\mathscr{L}^{-1}\left[\dfrac{4s-4}{(s^2-2s+5)^2}\right](t)=\mathscr{L}^{-1}\left[-\left\{-\dfrac{2(2s-2)}{(s^2-2s+5)^2}\right\}\right](t)=\mathscr{L}^{-1}\left[-\left\{\dfrac{2}{s^2-2s+5}\right\}'\right](t)$
$=\mathscr{L}^{-1}\left[-\left\{\dfrac{2}{(s-1)^2+4}\right\}'\right](t)=te^t\sin 2t$

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(1) $2\delta(t)$ (2) $-\delta(t)$ (3) $\delta(t-3)$ (4) $\delta\left(t-\dfrac{1}{2}\right)$ (5) $H(t-2)$

(6) $\mathscr{L}^{-1}\left[\dfrac{1}{se^s}\right](t)=\mathscr{L}^{-1}\left[\dfrac{e^{-s}}{s}\right](t)=H(t-1)$

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第 2 移動定理より，$\mathscr{L}^{-1}\left[e^{-\lambda s}F(s)\right](t)=f(t-\lambda)H(t-\lambda)$ だから

(1) $F(s)=\dfrac{1}{s^2}$ の場合を考えれば，$f(t)=t$ だから，$\mathscr{L}^{-1}\left[\dfrac{e^{-\lambda s}}{s^2}\right](t)=(t-\lambda)H(t-\lambda)$

(2) $F(s)=\dfrac{1}{s-a}$ の場合を考えれば，$f(t)=e^{at}$ だから，$\mathscr{L}^{-1}\left[\dfrac{e^{-\lambda s}}{s-a}\right](t)=e^{a(t-\lambda)}H(t-\lambda)$

(3) $F(s)=\dfrac{1}{(s-a)^2}$ の場合を考えれば，$f(t)=te^{at}$ だから，$\mathscr{L}^{-1}\left[\dfrac{e^{-\lambda s}}{(s-a)^2}\right](t)=(t-\lambda)e^{a(t-\lambda)}H(t-\lambda)$

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(1) $(f*g)(0)=\displaystyle\int_0^0 f(\tau)g(0-\tau)\,d\tau=0$

(2) $\mathscr{L}\left[(f*g)'(t)\right](s)=s\mathscr{L}\left[(f*g)(t)\right](s)-(f*g)(0)=s\mathscr{L}\left[f(t)\right](s)\mathscr{L}\left[g(t)\right](s)=sF(s)G(s)$

(3) $\mathscr{L}\big[(f*g')(t)+g(0)f(t)\big](s)=\mathscr{L}\big[(f*g')(t)\big](s)+g(0)\mathscr{L}\big[f(t)\big](s)$
$=\mathscr{L}\big[f(t)\big](s)\mathscr{L}\big[g'(t)\big](s)+g(0)F(s)=F(s)\{sG(s)-g(0)\}+g(0)F(s)=sF(s)G(s)$

(4) $(f*g)'(t)=\mathscr{L}^{-1}\big[sF(s)G(s)\big](t)=(f*g')(t)+g(0)f(t)$

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(1) $(t*t)'=\mathscr{L}^{-1}\left[s\cdot\dfrac{1}{s^2}\cdot\dfrac{1}{s^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s^3}\right](t)=\dfrac{1}{2}t^2$

(2) $(t*\cos t)'=\mathscr{L}^{-1}\left[s\cdot\dfrac{1}{s^2}\cdot\dfrac{s}{s^2+1}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s^2+1}\right](t)=\sin t$

(3) $(t*e^t)'=\mathscr{L}^{-1}\left[s\cdot\dfrac{1}{s^2}\cdot\dfrac{1}{s-1}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s(s-1)}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s-1}-\dfrac{1}{s}\right](t)=e^t-1$

(4) $(t*\sin t)'=\mathscr{L}^{-1}\left[s\cdot\dfrac{1}{s^2}\cdot\dfrac{1}{s^2+1}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s(s^2+1)}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s}-\dfrac{s}{s^2+1}\right](t)=t-\cos t$

(5) $(t\cos t*\sinh t)'=\mathscr{L}^{-1}\left[s\cdot\dfrac{s^2-1}{(s^2+1)^2}\cdot\dfrac{1}{s^2-1}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[-\dfrac{1}{2}\left(\dfrac{1}{s^2+1}\right)'\right](t)$
$=\dfrac{1}{2}t\sin t$

(6) $(e^t*e^{-t})'=\mathscr{L}^{-1}\left[s\cdot\dfrac{1}{s-1}\cdot\dfrac{1}{s+1}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s}{s^2-1}\right](t)=\cosh t$

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(1) $\displaystyle\int_0^\infty\frac{1}{t}e^{-2t}\sin t\cos t\,dt=\mathscr{L}\left[\frac{1}{t}\sin t\cos t\right](2)=\frac{1}{2}\mathscr{L}\left[\frac{1}{t}\sin 2t\right](2)=\frac{1}{2}\int_2^\infty\frac{2}{s^2+4}\,ds$
$\displaystyle=\frac{1}{2}\lim_{S\to\infty}\left[\mathrm{Tan}^{-1}\frac{s}{2}\right]_2^S=\frac{1}{2}\lim_{S\to\infty}\left(\mathrm{Tan}^{-1}\frac{S}{2}-\mathrm{Tan}^{-1}1\right)=\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{8}$

(2) $\displaystyle\int_0^\infty\frac{1}{t}e^{-t}(1*\cos t)\,dt=\mathscr{L}\left[\frac{1}{t}(1*\cos t)\right](1)=\int_1^\infty\mathscr{L}\big[1*\cos t\big](s)\,ds=\int_1^\infty\frac{1}{s}\cdot\frac{s}{s^2+1}\,ds$
$\displaystyle=\int_1^\infty\frac{1}{s^2+1}\,ds=\lim_{S\to\infty}\left[\mathrm{Tan}^{-1}s\right]_1^S=\lim_{S\to\infty}\left(\mathrm{Tan}^{-1}S-\mathrm{Tan}^{-1}1\right)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

(3) $\displaystyle\int_0^\infty te^{-\frac{t}{2}}\sin t\,dt=\mathscr{L}\big[t\sin t\big]\left(\frac{1}{2}\right)=-\left(\frac{1}{s^2+1}\right)'\left(\frac{1}{2}\right)=\left.\frac{2s}{(s^2+1)^2}\right|_{s=\frac{1}{2}}=\frac{1}{\left(\frac{1}{4}+1\right)^2}=\frac{16}{25}$

(4) $\displaystyle\int_0^\infty te^{-t}(t*\cos t)\,dt=\mathscr{L}\big[t(t*\cos t)\big](1)=-\Big(\mathscr{L}\big[t*\cos t\big]\Big)'(1)=-\left(\frac{1}{s^2}\cdot\frac{s}{s^2+1}\right)'(1)=-\left(\frac{1}{s^3+s}\right)'(1)$
$\displaystyle=\left.\frac{3s^2+1}{(s^3+s)^2}\right|_{s=1}=1$

2 節 ラプラス変換の応用

以下，$X=\mathscr{L}[x](s)$ とする。

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(1) 初期条件 $x(0)=0$ の下で $x'+2x=3e^t$ をラプラス変換すると，$sX-0+2X=\dfrac{3}{s-1}$
$\therefore\ (s+2)X=\dfrac{3}{s-1}\qquad\therefore\ X=\dfrac{3}{(s-1)(s+2)}=\dfrac{1}{s-1}-\dfrac{1}{s+2}$
両辺の逆ラプラス変換を考えると，$x=e^t-e^{-2t}$

(2) 初期条件 $x(0)=2$ の下で $x'-x=-2e^{-t}$ をラプラス変換すると，$sX-2-X=-\dfrac{2}{s+1}$
$\therefore\ (s-1)X=2-\dfrac{2}{s+1}=\dfrac{2s+2-2}{s+1}=\dfrac{2s}{s+1}\qquad\therefore\ X=\dfrac{2s}{(s-1)(s+1)}=\dfrac{1}{s-1}+\dfrac{1}{s+1}$
両辺の逆ラプラス変換を考えると，$x=e^t+e^{-t}$

(3)　初期条件 $x(0)=1$ の下で $x'-x=1$ をラプラス変換すると，$sX-1-X=\dfrac{1}{s}$

$\therefore\ (s-1)X=1+\dfrac{1}{s}=\dfrac{s+1}{s}\qquad \therefore\ X=\dfrac{s+1}{(s-1)s}=\dfrac{2}{s-1}-\dfrac{1}{s}$

両辺の逆ラプラス変換を考えると，$x=2e^t-1$

(4)　初期条件 $x(0)=2$ の下で $2x'+x=3$ をラプラス変換すると，$2(sX-2)+X=\dfrac{3}{s}$

$\therefore\ (2s+1)X=4+\dfrac{3}{s}=\dfrac{4s+3}{s}\qquad \therefore\ X=\dfrac{4s+3}{s(2s+1)}=\dfrac{3}{s}-\dfrac{2}{2s+1}=\dfrac{3}{s}-\dfrac{1}{s+\dfrac{1}{2}}$

両辺の逆ラプラス変換を考えると，$x=3-e^{-\frac{t}{2}}$

(5)　初期条件 $x(0)=-1$ の下で $x'+x=t+1$ をラプラス変換すると，$sX+1+X=\dfrac{1}{s^2}+\dfrac{1}{s}$

$\therefore\ (s+1)X=-1+\dfrac{1}{s^2}+\dfrac{1}{s}=-\dfrac{s^2-s-1}{s^2}\qquad \therefore\ X=-\dfrac{s^2-s-1}{s^2(s+1)}=\dfrac{1}{s^2}-\dfrac{1}{s+1}$

両辺の逆ラプラス変換を考えると，$x=t-e^{-t}$

(6)　初期条件 $x(0)=1$ の下で $x'+x=2\cos t$ をラプラス変換すると，$sX-1+X=\dfrac{2s}{s^2+1}$

$\therefore\ (s+1)X=1+\dfrac{2s}{s^2+1}=\dfrac{s^2+2s+1}{s^2+1}=\dfrac{(s+1)^2}{s^2+1}\qquad \therefore\ X=\dfrac{s+1}{s^2+1}=\dfrac{s}{s^2+1}+\dfrac{1}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x=\cos t+\sin t$

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(1)　$C=x(0)$ として $x'+x=e^{-t}$ をラプラス変換すると，$sX-C+X=\dfrac{1}{s+1}$

$\therefore\ (s+1)X=C+\dfrac{1}{s+1}\qquad \therefore\ X=\dfrac{C}{s+1}+\dfrac{1}{(s+1)^2}$

両辺の逆ラプラス変換を考えると，$x=(t+C)e^{-t}$

(2)　$c=x(0)$ として $x'-2x=2$ をラプラス変換すると，$sX-c-2X=\dfrac{2}{s}$

$\therefore\ (s-2)X=c+\dfrac{2}{s}=\dfrac{cs+2}{s}\qquad \therefore\ X=\dfrac{cs+2}{(s-2)s}=\dfrac{c+1}{s-2}-\dfrac{1}{s}$

$C=c+1$ とおき，両辺の逆ラプラス変換を考えると，$x=Ce^{2t}-1$

(3)　$c=x(0)$ として $2x'-x=e^t$ をラプラス変換すると，$2(sX-c)-X=\dfrac{1}{s-1}$

$\therefore\ (2s-1)X=2c+\dfrac{1}{s-1}=\dfrac{2cs-2c+1}{s-1}$

$\therefore\ X=\dfrac{2cs-2c+1}{(s-1)(2s-1)}=\dfrac{1}{s-1}+\dfrac{2c-2}{2s-1}=\dfrac{1}{s-1}+\dfrac{c-1}{s-\dfrac{1}{2}}$

$C=c-1$ とおき，両辺の逆ラプラス変換を考えると，$x=e^t+Ce^{\frac{t}{2}}$

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(1)　初期条件 $x(0)=0$，$x'(0)=1$ の下で $x''+x=5e^{2t}$ をラプラス変換すると，$s^2X-1+X=\dfrac{5}{s-2}$

$\therefore\ (s^2+1)X=1+\dfrac{5}{s-2}=\dfrac{s+3}{s-2}\qquad \therefore\ X=\dfrac{s+3}{(s-2)(s^2+1)}=\dfrac{1}{s-2}-\dfrac{s+1}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x=e^{2t}-\cos t-\sin t$

(2)　初期条件 $x(0)=0$，$x'(0)=-1$ の下で $x''+x=t$ をラプラス変換すると，$s^2X+1+X=\dfrac{1}{s^2}$

$\therefore\ (s^2+1)X=\dfrac{1}{s^2}-1=\dfrac{1-s^2}{s^2}\qquad \therefore\ X=\dfrac{1-s^2}{s^2(s^2+1)}=\dfrac{1}{s^2}-\dfrac{2}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x=t-2\sin t$

(3)　初期条件 $x(0)=2$，$x'(0)=0$ の下で $x''-x=4$ をラプラス変換すると，$s^2X-2s-X=\dfrac{4}{s}$

$\therefore\ (s^2-1)X=2s+\dfrac{4}{s}=\dfrac{2s^2+4}{s}\qquad \therefore\ X=\dfrac{2s^2+4}{(s-1)(s+1)s}=\dfrac{3}{s-1}+\dfrac{3}{s+1}-\dfrac{4}{s}$

両辺の逆ラプラス変換を考えると，$x=3e^t+3e^{-t}-4$

(4) 初期条件 $x(0)=0$，$x'(0)=-8$ の下で $x''-x=4\sin t$ をラプラス変換すると，$s^2X+8-X=\dfrac{4}{s^2+1}$

$\therefore (s^2-1)X=-8+\dfrac{4}{s^2+1}=\dfrac{-8s^2-4}{s^2+1}$

$\therefore X=\dfrac{-8s^2-4}{(s^2-1)(s^2+1)}=-\dfrac{6}{s^2-1}-\dfrac{2}{s^2+1}=\dfrac{3}{s+1}-\dfrac{3}{s-1}-\dfrac{2}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x=3e^{-t}-3e^{t}-2\sin t$

(5) 初期条件 $x(0)=1$，$x'(0)=-2$ の下で $x''+x=-2e^t$ をラプラス変換すると，$s^2X-s+2+X=-\dfrac{2}{s-1}$

$\therefore (s^2+1)X=s-2-\dfrac{2}{s-1}=\dfrac{s^2-3s}{s-1}$ $\qquad \therefore X=-\dfrac{1}{s-1}+\dfrac{2s-1}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x=-e^t+2\cos t-\sin t$

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(1) 初期条件 $x(0)=1$，$x'(0)=1$ の下で $x''-2x'+x=1$ をラプラス変換すると

$s^2X-s-1-2(sX-1)+X=\dfrac{1}{s}$ $\qquad \therefore (s^2-2s+1)X=s-1+\dfrac{1}{s}=\dfrac{s^2-s+1}{s}$

$\therefore X=\dfrac{s^2-s+1}{s(s-1)^2}=\dfrac{1}{s}+\dfrac{1}{(s-1)^2}$

両辺の逆ラプラス変換を考えると，$x=1+te^t$

(2) 初期条件 $x(0)=6$，$x'(0)=-6$ の下で $x''+x'-2x=6$ をラプラス変換すると

$s^2X-6s+6+sX-6-2X=\dfrac{6}{s}$ $\qquad \therefore (s^2+s-2)X=6s+\dfrac{6}{s}=\dfrac{6s^2+6}{s}$

$\therefore X=\dfrac{6s^2+6}{s(s-1)(s+2)}=-\dfrac{3}{s}+\dfrac{4}{s-1}+\dfrac{5}{s+2}$

両辺の逆ラプラス変換を考えると，$x=-3+4e^t+5e^{-2t}$

(3) 初期条件 $x(0)=9$，$x'(0)=18$ の下で $x''-2x'-3x=9t$ をラプラス変換すると

$s^2X-9s-18-2(sX-9)-3X=\dfrac{9}{s^2}$ $\qquad \therefore (s^2-2s-3)X=9s+\dfrac{9}{s^2}=\dfrac{9(s^3+1)}{s^2}$

$\therefore X=\dfrac{9(s+1)(s^2-s+1)}{s^2(s-3)(s+1)}=\dfrac{2}{s}-\dfrac{3}{s^2}+\dfrac{7}{s-3}$

両辺の逆ラプラス変換を考えると，$x=2-3t+7e^{3t}$

(4) 初期条件 $x(0)=0$，$x'(0)=9$ の下で $x''-x'-2x=9e^{-t}$ をラプラス変換すると

$s^2X-9-sX-2X=\dfrac{9}{s+1}$ $\qquad \therefore (s^2-s-2)X=9+\dfrac{9}{s+1}=\dfrac{9s+18}{s+1}$

$\therefore X=\dfrac{9s+18}{(s-2)(s+1)^2}=\dfrac{4}{s-2}-\dfrac{4}{s+1}-\dfrac{3}{(s+1)^2}$

両辺の逆ラプラス変換を考えると，$x=4e^{2t}-4e^{-t}-3te^{-t}$

(5) 初期条件 $x(0)=0$，$x'(0)=5$ の下で $x''+3x'+2x=-10\sin t$ をラプラス変換すると

$s^2X-5+3sX+2X=-\dfrac{10}{s^2+1}$ $\qquad \therefore (s^2+3s+2)X=5-\dfrac{10}{s^2+1}=\dfrac{5s^2-5}{s^2+1}$

$\therefore X=\dfrac{5(s+1)(s-1)}{(s+1)(s+2)(s^2+1)}=-\dfrac{3}{s+2}+\dfrac{3s-1}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x=-3e^{-2t}+3\cos t-\sin t$

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(1) $\alpha=x'(0)$ とおき，境界条件 $x(0)=0$ の下でをラプラス変換すると，$s^2X-\alpha+X=\dfrac{1}{s}$

$\therefore (s^2+1)X=\alpha+\dfrac{1}{s}=\dfrac{\alpha s+1}{s}$ $\qquad \therefore X=\dfrac{\alpha s+1}{s(s^2+1)}=\dfrac{1}{s}+\dfrac{-s+\alpha}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x=1-\cos t+\alpha\sin t$

境界条件 $x\left(\dfrac{\pi}{2}\right)=0$ より，$x\left(\dfrac{\pi}{2}\right)=1-\cos\dfrac{\pi}{2}+\alpha\sin\dfrac{\pi}{2}=1+\alpha=0$ $\qquad \therefore \alpha=-1$

よって，$x=1-\cos t-\sin t$

(2) $\alpha=x'(0)$ とおき，境界条件 $x(0)=0$ の下で $x''+x=t$ をラプラス変換すると

$s^2X-\alpha+X=\dfrac{1}{s^2}$ $\qquad \therefore (s^2+1)X=\alpha+\dfrac{1}{s^2}=\dfrac{\alpha s^2+1}{s^2}$

$\therefore\ X = \dfrac{\alpha s^2+1}{s^2(s^2+1)} = \dfrac{1}{s^2} + \dfrac{\alpha-1}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x = t + (\alpha-1)\sin t$

境界条件 $x\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2} - 1$ より，$x\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2} + (\alpha-1)\sin\dfrac{\pi}{2} = \dfrac{\pi}{2} + \alpha - 1 = \dfrac{\pi}{2} - 1 \qquad \therefore\ \alpha = 0$

よって，$x = t - \sin t$

(3) $\alpha = x'(0)$ とおき，境界条件 $x(0) = 0$ の下で $x'' - x = 1$ をラプラス変換すると

$s^2X - \alpha - X = \dfrac{1}{s} \qquad \therefore\ (s^2-1)X = \alpha + \dfrac{1}{s} = \dfrac{\alpha s+1}{s}$

$\therefore\ X = \dfrac{\alpha s+1}{(s-1)(s+1)s} = \dfrac{1}{2}\left(\dfrac{\alpha+1}{s-1} + \dfrac{1-\alpha}{s+1}\right) - \dfrac{1}{s}$

両辺の逆ラプラス変換を考えると，$x = \dfrac{1}{2}\Big((\alpha+1)e^t + (1-\alpha)e^{-t}\Big) - 1$

境界条件 $x(1) = e^{-1} - 1$ より，$x(1) = -1 + \dfrac{1}{2}\Big((\alpha+1)e + (1-\alpha)e^{-1}\Big) = e^{-1} - 1$

$\therefore\ (\alpha+1)e + (1-\alpha)e^{-1} = 2e^{-1} \qquad \therefore\ (e - e^{-1})\alpha = -(e - e^{-1}) \qquad \therefore\ \alpha = -1$

よって，$x = e^{-t} - 1$

(4) $\alpha = x'(0)$ とおき，境界条件 $x(0) = 0$ の下で $x'' - 2x' + x = 2e^t$ をラプラス変換すると

$s^2X - \alpha - 2sX + X = \dfrac{2}{s-1} \qquad \therefore\ (s^2-2s+1)X = \alpha + \dfrac{2}{s-1}$

$\therefore\ X = \dfrac{\alpha}{(s-1)^2} + \dfrac{2}{(s-1)^3}$

両辺の逆ラプラス変換を考えると，$x = \alpha te^t + t^2e^t$

境界条件 $x(1) = 2e$ より，$x(1) = \alpha e + e = (\alpha+1)e = 2e \qquad \therefore\ \alpha = 1$

よって，$x = (t^2+t)e^t$

(5) $\alpha = x'(0)$ とおき，境界条件 $x(0) = 0$ の下で $x'' + 4x' + 4x = e^{-2t}$ をラプラス変換すると

$s^2X - \alpha + 4sX + 4X = \dfrac{1}{s+2} \qquad \therefore\ (s^2+4s+4)X = \alpha + \dfrac{1}{s+2}$

$\therefore\ X = \dfrac{\alpha}{(s+2)^2} + \dfrac{1}{(s+2)^3}$

両辺の逆ラプラス変換を考えると，$x = \alpha te^{-2t} + \dfrac{1}{2}t^2e^{-2t}$

境界条件 $x(1) = e^{-2}$ より，$x(1) = \alpha e^{-2} + \dfrac{1}{2}e^{-2} = e^{-2} \qquad \therefore\ \alpha = \dfrac{1}{2}$

よって，$x = \dfrac{1}{2}(t^2+t)e^{-2t}$

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(1) $q' + q = 1$

(2) $Q = \mathscr{L}[q](s)$ とし，初期条件 $q(0) = 2$ の下で (1) の微分方程式をラプラス変換すると

$sQ - 2 + Q = \dfrac{1}{s} \qquad \therefore\ (s+1)Q = 2 + \dfrac{1}{s} = \dfrac{2s+1}{s} \qquad \therefore\ Q = \dfrac{1}{s} + \dfrac{1}{s+1}$

両辺の逆ラプラス変換を考えると，$q = 1 + e^{-t}$

(3) (2) より，不等式 $1 + e^{-t} < \dfrac{3}{2}$ を解けばよい。

両辺の対数をとると，$\log e^{-t} < \log\dfrac{1}{2} = \log 2^{-1} \qquad \therefore\ -t < -\log 2 \qquad \therefore\ t > \log 2$

158

(1) $x'' = -5x - 2x'$

(2) 初期条件 $x(0) = 1$，$x'(0) = 1$ の下で (1) の微分方程式をラプラス変換すると

$s^2X - s - 1 = -5X - 2(sX - 1) \qquad \therefore\ (s^2+2s+5)X = s + 3$

$\therefore\ X = \dfrac{s+3}{(s+1)^2+4} = \dfrac{s+1}{(s+1)^2+4} + \dfrac{2}{(s+1)^2+4}$

両辺の逆ラプラス変換を考えると，$x = e^{-t}(\cos 2t + \sin 2t)$

(3) (2) の解の形より，減衰振動

159

(1) $\dfrac{d^4x}{dt^4} = -1$

(2) 境界条件 $x(0)=0$, $x(1)=0$, $x'(0)=0$, $x'(1)=0$ の下で (1) の微分方程式を解けばよい。$c_2 = x''(0)$, $c_3 = x'''(0)$ とする。境界条件 $x(0)=0$, $x'(0)=0$ の下で (1) の微分方程式をラプラス変換すると

$s^4X - c_2 s - c_3 = -\dfrac{1}{s} \qquad \therefore\ X = \dfrac{c_2}{s^3} + \dfrac{c_3}{s^4} - \dfrac{1}{s^5}$

両辺の逆ラプラス変換を考えると，$x = \dfrac{c_2}{2}t^2 + \dfrac{c_3}{6}t^3 - \dfrac{1}{24}t^4 \qquad \therefore\ x' = c_2 t + \dfrac{c_3}{2}t^2 - \dfrac{1}{6}t^3$

境界条件 $x(1)=0$, $x'(1)=0$ より

$x(1) = \dfrac{c_2}{2} + \dfrac{c_3}{6} - \dfrac{1}{24} = 0 \qquad \therefore\ 12c_2 + 4c_3 = 1 \quad \cdots$ ①

$x'(1) = c_2 + \dfrac{c_3}{2} - \dfrac{1}{6} = 0 \qquad \therefore\ 12c_2 + 6c_3 = 2 \quad \cdots$ ②

①，②を連立して，$c_2 = -\dfrac{1}{12}$，$c_3 = \dfrac{1}{2}$

$\therefore\ x = -\dfrac{1}{24}t^2 + \dfrac{1}{12}t^3 - \dfrac{1}{24}t^4 = -\dfrac{1}{24}t^2(1-2t+t^2) \qquad \therefore\ x = -\dfrac{1}{24}t^2(1-t)^2$

(3) $t = \dfrac{1}{2}$ における $x(t)$ の値を求めればよい。

$x\left(\dfrac{1}{2}\right) = -\dfrac{1}{24}\left(\dfrac{1}{2}\right)^2\left(1-\dfrac{1}{2}\right)^2 = -\dfrac{1}{384}$

160

$F(s) = \mathscr{L}[f(t)](s)$ とする。

(1) 積分方程式を $f(t) * \cos t = \sin t$ と書き直し，両辺をラプラス変換すると

$F(s)\cdot\dfrac{s}{s^2+1} = \dfrac{1}{s^2+1} \qquad \therefore\ F(s) = \dfrac{1}{s} \qquad \therefore\ f(t) = 1$

(2) 積分方程式を $f(t) * \sin t = t^3$ と書き直し，両辺をラプラス変換すると

$F(s)\cdot\dfrac{1}{s^2+1} = \dfrac{6}{s^4} \qquad \therefore\ F(s) = \dfrac{6s^2+6}{s^4} = \dfrac{6}{s^2} + \dfrac{6}{s^4} \qquad \therefore\ f(t) = 6t + t^3$

(3) 積分方程式を $f(t) * \cos t = t^2$ と書き直し，両辺をラプラス変換すると

$F(s)\cdot\dfrac{s}{s^2+1} = \dfrac{2}{s^3} \qquad \therefore\ F(s) = \dfrac{2s^2+2}{s^4} = \dfrac{2}{s^2} + \dfrac{1}{3}\cdot\dfrac{3!}{s^4} \qquad \therefore\ f(t) = 2t + \dfrac{1}{3}t^3$

(4) 積分方程式を $f(t) * e^t = t^2$ と書き直し，両辺をラプラス変換すると

$F(s)\cdot\dfrac{1}{s-1} = \dfrac{2}{s^3} \qquad \therefore\ F(s) = \dfrac{2s-2}{s^3} = \dfrac{2}{s^2} - \dfrac{2}{s^3} \qquad \therefore\ f(t) = 2t - t^2$

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(1) 初期条件 $x(0)=1$ の下で $x' + x = t$ をラプラス変換すると，$sX - 1 + X = \dfrac{1}{s^2}$

$\therefore\ (s+1)X = 1 + \dfrac{1}{s^2} = \dfrac{s^2+1}{s^2} \qquad \therefore\ X = \dfrac{s^2+1}{(s+1)s^2} = \dfrac{2}{s+1} + \dfrac{1}{s^2} - \dfrac{1}{s}$

両辺の逆ラプラス変換を考えると，$x = 2e^{-t} + t - 1$

(2) 初期条件 $x(0)=9$ の下で $x' + 2x = 9te^t$ をラプラス変換すると，$sX - 9 + 2X = \dfrac{9}{(s-1)^2}$

$\therefore\ (s+2)X = 9 + \dfrac{9}{(s-1)^2} = \dfrac{9s^2-18s+18}{(s-1)^2}$

$\therefore\ X = \dfrac{9s^2-18s+18}{(s+2)(s-1)^2} = \dfrac{3}{(s-1)^2} - \dfrac{1}{s-1} + \dfrac{10}{s+2}$

両辺の逆ラプラス変換を考えると，$x = (3t-1)e^t + 10e^{-2t}$

(3) 初期条件 $x(0)=0$, $x'(0)=2$ の下で $x'' - 4x' + 4x = -4$ をラプラス変換すると

$s^2X - 2 - 4sX + 4X = -\dfrac{4}{s} \qquad \therefore\ (s^2-4s+4)X = 2 - \dfrac{4}{s} = \dfrac{2(s-2)}{s}$

$\therefore\ X = \dfrac{2}{(s-2)s} = \dfrac{1}{s-2} - \dfrac{1}{s}$

両辺の逆ラプラス変換を考えると，$x = e^{2t} - 1$

(4) 初期条件 $x(0)=1$，$x'(0)=-1$ の下で $x''+2x'+x=e^{-t}\cos t$ をラプラス変換すると

$$s^2X-s+1+2sX-2+X=\frac{s+1}{(s+1)^2+1} \qquad \therefore\ (s^2+2s+1)X=s+1+\frac{s+1}{s^2+2s+2}$$

$$\therefore\ X=\frac{1}{s+1}+\frac{1}{(s+1)(s^2+2s+2)}=\frac{2}{s+1}-\frac{s+1}{(s+1)^2+1}$$

両辺の逆ラプラス変換を考えると，$x=e^{-t}(2-\cos t)$

(5) 初期条件 $x(0)=0$，$x'(0)=0$ の下で $x''+4x=4\cos 2t$ をラプラス変換すると，$s^2X+4X=\dfrac{4s}{s^2+4}$

$$\therefore\ (s^2+4)X=\frac{4s}{s^2+4} \qquad \therefore\ X=\frac{4s}{(s^2+4)^2}=-\left(\frac{2}{s^2+4}\right)'$$

両辺の逆ラプラス変換を考えると，$x=t\sin 2t$

(6) 初期条件 $x(0)=0$，$x'(0)=1$ の下で $x''+2x'+2x=2e^{-t}(\cos t-\sin t)$ をラプラス変換すると

$$s^2X-1+2sX+2X=2\left\{\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$\therefore\ (s^2+2s+2)X=\frac{2(s+1)}{(s+1)^2+1}+1-\frac{2}{(s+1)^2+1}=\frac{2(s+1)}{(s+1)^2+1}+\frac{(s+1)^2-1}{(s+1)^2+1}$$

$$\therefore\ X=\frac{2(s+1)}{\{(s+1)^2+1\}^2}+\frac{(s+1)^2-1}{\{(s+1)^2+1\}^2}$$

両辺の逆ラプラス変換を考えると，$x=te^{-t}(\sin t+\cos t)$

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(1) $\alpha=x'(0)$ とおき，境界条件 $x(0)=0$ の下で $x''-x=2e^t$ をラプラス変換すると

$$s^2X-\alpha-X=\frac{2}{s-1} \qquad \therefore\ (s^2-1)X=\alpha+\frac{2}{s-1}=\frac{\alpha s-\alpha+2}{s-1}$$

$$\therefore\ X=\frac{\alpha s-\alpha+2}{(s-1)^2(s+1)}=\frac{1}{(s-1)^2}+\frac{\alpha-1}{2}\cdot\frac{1}{s-1}-\frac{\alpha-1}{2}\cdot\frac{1}{s+1}$$

両辺の逆ラプラス変換を考えると，$x=te^t+\dfrac{\alpha-1}{2}e^t-\dfrac{\alpha-1}{2}e^{-t}$

境界条件 $x(1)=2e-e^{-1}$ より，$x(1)=e+\dfrac{\alpha-1}{2}e-\dfrac{\alpha-1}{2}e^{-1}=\dfrac{e-e^{-1}}{2}\alpha+\dfrac{e+e^{-1}}{2}=2e-e^{-1}$

$$\therefore\ \frac{e-e^{-1}}{2}\alpha=2e-e^{-1}-\frac{e+e^{-1}}{2}=\frac{3(e-e^{-1})}{2} \qquad \therefore\ \alpha=3$$

よって，$x=(t+1)e^t-e^{-t}$

(2) $\alpha=x'(0)$ とおき，境界条件 $x(0)=0$ の下で $x''+x=2\cos t$ をラプラス変換すると

$$s^2X-\alpha+X=\frac{2s}{s^2+1} \qquad \therefore\ (s^2+1)X=\alpha+\frac{2s}{s^2+1}$$

$$\therefore\ X=\frac{\alpha}{s^2+1}+\frac{2s}{(s^2+1)^2}=\frac{\alpha}{s^2+1}-\left(\frac{1}{s^2+1}\right)'$$

両辺の逆ラプラス変換を考えると，$x=\alpha\sin t+t\sin t=(t+\alpha)\sin t$

境界条件 $x\left(\dfrac{\pi}{6}\right)=\dfrac{7\pi}{12}$ より，$x\left(\dfrac{\pi}{6}\right)=\left(\dfrac{\pi}{6}+\alpha\right)\cdot\dfrac{1}{2}=\dfrac{\pi+6\alpha}{12}=\dfrac{7\pi}{12} \qquad \therefore\ \alpha=\pi$

よって，$x=(t+\pi)\sin t$

(3) $\alpha=x'(0)$ とおき，境界条件 $x(0)=0$ の下で $x''+x'=e^{-t}$ をラプラス変換すると

$$s^2X-\alpha+sX=\frac{1}{s+1} \qquad \therefore\ (s^2+s)X=\alpha+\frac{1}{s+1}=\frac{\alpha s+\alpha+1}{s+1}$$

$$\therefore\ X=\frac{\alpha s+\alpha+1}{s(s+1)^2}=\frac{\alpha+1}{s}-\frac{\alpha+1}{s+1}-\frac{1}{(s+1)^2}$$

両辺の逆ラプラス変換を考えると，$x=\alpha+1-(\alpha+1)e^{-t}-te^{-t}$

$\therefore\ x'=(\alpha+1)e^{-t}-e^{-t}+te^{-t}$

$\therefore\ x(1)=\alpha+1-(\alpha+1)e^{-1}-e^{-1}=\alpha+1-(\alpha+2)e^{-1}$，$x'(1)=(\alpha+1)e^{-1}-e^{-1}+e^{-1}=(\alpha+1)e^{-1}$

境界条件 $x'(1)-x(1)=e^{-1}$ より

$$x'(1)-x(1)=(\alpha+1)e^{-1}-\{\alpha+1-(\alpha+2)e^{-1}\}=(2\alpha+3)e^{-1}-\alpha-1=e^{-1}$$

$$\therefore\ (2e^{-1}-1)\alpha=-(2e^{-1}-1) \qquad \therefore\ \alpha=-1$$

よって，$x=-te^{-t}$

(4) $\alpha = x'(0)$ とおき，境界条件 $x(0)=0$ の下で $x''-x'=2\sin t$ をラプラス変換すると

$s^2X-\alpha-sX=\dfrac{2}{s^2+1}$ $\qquad\therefore\ (s^2-s)X=\alpha+\dfrac{2}{s^2+1}=\dfrac{\alpha s^2+\alpha+2}{s^2+1}$

$\therefore\ X=\dfrac{\alpha s^2+\alpha+2}{s(s-1)(s^2+1)}=-\dfrac{\alpha+2}{s}+\dfrac{\alpha+1}{s-1}+\dfrac{s-1}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x=-\alpha-2+(\alpha+1)e^t+\cos t-\sin t$ $\qquad\therefore\ x'=(\alpha+1)e^t-\sin t-\cos t$

$\therefore\ x(\pi)=-\alpha-2+(\alpha+1)e^{\pi}-1,\ x'(\pi)=(\alpha+1)e^{\pi}+1$

境界条件 $x'(\pi)-x(\pi)=2$ より，$x'(\pi)-x(\pi)=(\alpha+1)e^{\pi}+1+\alpha+2-(\alpha+1)e^{\pi}+1=\alpha+4=2$

$\therefore\ \alpha=-2$ $\qquad\therefore\ x=-e^t+\cos t-\sin t$

(5) $\alpha = x'(0)$ とおき，境界条件 $x(0)=0$ の下で $x''+2x'-3x=4e^t$ をラプラス変換すると

$s^2X-\alpha+2sX-3X=\dfrac{4}{s-1}$ $\qquad\therefore\ (s^2+2s-3)X=\alpha+\dfrac{4}{s-1}=\dfrac{\alpha s-\alpha+4}{s-1}$

$\therefore\ X=\dfrac{\alpha s-\alpha+4}{(s-1)^2(s+3)}=\dfrac{1}{4}\left\{\dfrac{\alpha-1}{s-1}+\dfrac{4}{(s-1)^2}-\dfrac{\alpha-1}{s+3}\right\}$

両辺の逆ラプラス変換を考えると，$x=\dfrac{1}{4}\left\{(\alpha-1)e^t+4te^t-(\alpha-1)e^{-3t}\right\}$

$\therefore\ x'=\dfrac{1}{4}\left\{(\alpha-1)e^t+4e^t+4te^t+3(\alpha-1)e^{-3t}\right\}$

$\therefore\ x(1)=\dfrac{1}{4}\left\{(\alpha-1)e+4e-(\alpha-1)e^{-3}\right\},\ x'(1)=\dfrac{1}{4}\left\{(\alpha-1)e+4e+4e+3(\alpha-1)e^{-3}\right\}$

境界条件 $x'(1)-x(1)=e+4e^{-3}$ より

$x'(1)-x(1)=\dfrac{1}{4}\left\{4e+4(\alpha-1)e^{-3}\right\}=e+(\alpha-1)e^{-3}=e+4e^{-3}$ $\qquad\therefore\ \alpha=5$ $\qquad\therefore\ x=(t+1)e^t-e^{-3t}$

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この回路の方程式は，$Rq'+\dfrac{1}{C}q=\sin t$ $\qquad\therefore\ q'+\dfrac{1}{RC}q=\dfrac{1}{R}\sin t$

$Q=\mathscr{L}[q](s)$ とし，初期条件 $q(0)=0$ の下でこの方程式をラプラス変換すると

$sQ+\dfrac{1}{RC}Q=\dfrac{1}{R}\cdot\dfrac{1}{s^2+1}$ $\qquad\therefore\ \left(s+\dfrac{1}{RC}\right)Q=\dfrac{1}{R}\cdot\dfrac{1}{s^2+1}$

$\therefore\ Q=\dfrac{C}{RCs+1}\cdot\dfrac{1}{s^2+1}=\dfrac{C}{R^2C^2+1}\left(\dfrac{R^2C^2}{RCs+1}-\dfrac{RCs}{s^2+1}+\dfrac{1}{s^2+1}\right)$

$=\dfrac{C}{R^2C^2+1}\left(\dfrac{RC}{s+\dfrac{1}{RC}}-\dfrac{RCs}{s^2+1}+\dfrac{1}{s^2+1}\right)$

両辺の逆ラプラス変換を考えると，$q=\dfrac{C}{R^2C^2+1}\left(RCe^{-\frac{t}{RC}}-RC\cos t+\sin t\right)$

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(1) 初期条件 $x(0)=1,\ x'(0)=0$ の下で $x''+8x'+20x=0$ をラプラス変換すると

$s^2X-s+8(sX-1)+20X=0$ $\qquad\therefore\ (s^2+8s+20)X=s+8$

$\therefore\ X=\dfrac{s+8}{s^2+8s+20}=\dfrac{s+4}{(s+4)^2+4}+\dfrac{2\cdot 2}{(s+4)^2+4}$

両辺の逆ラプラス変換を考えると，$x=e^{-4t}(\cos 2t+2\sin 2t)$

(2) $x'(0)=\alpha$ とし，境界条件 $x(0)=0$ の下で $x''+8x'+20x=0$ をラプラス変換すると，

$s^2X-\alpha+8sX+20X=0$ $\qquad\therefore\ (s^2+8s+20)X=\alpha$ $\qquad\therefore\ X=\dfrac{\alpha}{s^2+8s+20}=\dfrac{\alpha}{2}\cdot\dfrac{2}{(s+4)^2+4}$

両辺の逆ラプラス変換を考えると，$x=\dfrac{\alpha}{2}e^{-4t}\sin 2t$

$\therefore\ x'=\dfrac{\alpha}{2}\left(-4e^{-4t}\sin 2t+e^{-4t}\cdot 2\cos 2t\right)=-\alpha e^{-4t}\left(2\sin 2t-\cos 2t\right)$

境界条件 $x'\left(\dfrac{\pi}{4}\right)=4e^{-\pi}$ より

$x'\left(\dfrac{\pi}{4}\right)=-\alpha e^{-\pi}\left(2\sin\dfrac{\pi}{2}-\cos\dfrac{\pi}{2}\right)=-2\alpha e^{-\pi}=4e^{-\pi}$ $\qquad\therefore\ \alpha=-2$ $\qquad\therefore\ x=-e^{-4t}\sin 2t$

(3) $c_0=x(0),\ c_1=x'(0)$ とし，$x''+8x'+20x=0$ をラプラス変換すると

$s^2X-c_0s-c_1+8(sX-c_0)+20X=0$ $\qquad\therefore\ (s^2+8s+20)X=c_0s+8c_0+c_1$

$$\therefore\ X = \frac{c_0 s + 8c_0 + c_1}{s^2 + 8s + 20} = c_0 \cdot \frac{s}{(s+4)^2 + 4} + \frac{8c_0 + c_1}{2} \cdot \frac{2}{(s+4)^2 + 4}$$

両辺の逆ラプラス変換を考えると

$$x = c_0 e^{-4t} \cos 2t + \frac{8c_0 + c_1}{2} e^{-4t} \sin 2t = e^{-4t}\left(c_0 \cos 2t + \frac{8c_0 + c_1}{2} \sin 2t\right)$$

$$\therefore\ x' = -4e^{-4t}\left(c_0 \cos 2t + \frac{8c_0 + c_1}{2} \sin 2t\right) + e^{-4t}\left(-2c_0 \sin 2t + (8c_0 + c_1)\cos 2t\right)$$

$$\therefore\ x(\pi) = c_0 e^{-4\pi},\ x'(\pi) = -4c_0 e^{-4\pi} + (8c_0 + c_1)e^{-4\pi} = (4c_0 + c_1)e^{-4\pi}$$

境界条件 $x(\pi) = e^{-4\pi}$，$x'(\pi) = 2e^{-4\pi}$ より，$c_0 = 1$，$c_1 = -2$

よって，$x = e^{-4t}(\cos 2t + 3\sin 2t)$

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$c_1 = x'(0)$，$c_3 = x'''(0)$ とし，境界条件 $x(0) = 0$，$x''(0) = 0$ の下で弾性曲線方程式をラプラス変換すると，$s^4 X - c_1 s^2 - c_3 = -\dfrac{1}{EIs}$ $\qquad \therefore\ X = \dfrac{c_1}{s^2} + \dfrac{c_3}{s^4} - \dfrac{1}{EIs^5}$

両辺の逆ラプラス変換を考えると，$x = c_1 t + \dfrac{c_3}{6}t^3 - \dfrac{1}{24EI}t^4$ $\qquad \therefore\ x' = c_1 + \dfrac{c_3}{2}t^2 - \dfrac{1}{6EI}t^3$

境界条件 $x(1) = 0$，$x'(1) = 0$ より，$x(1) = c_1 + \dfrac{c_3}{6} - \dfrac{1}{24EI} = 0$，$x'(1) = c_1 + \dfrac{c_3}{2} - \dfrac{1}{6EI} = 0$

$$\therefore\ c_1 = -\frac{1}{48EI},\ c_3 = \frac{3}{8EI}$$

$$x = -\frac{1}{48EI}t + \frac{1}{16EI}t^3 - \frac{1}{24EI}t^4 = -\frac{1}{48EI}t(1 - 3t^2 + 2t^3) \qquad \therefore\ x = -\frac{1}{48EI}t(t-1)^2(2t+1)$$

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$F(s) = \mathscr{L}[f(t)](s)$ とする。

(1) 積分方程式を $f(t) * \sqrt{t} = t$ と書き直し，両辺をラプラス変換すると，$F(s) \cdot \dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}} = \dfrac{1}{s^2}$

$$\therefore\ F(s) = \frac{2}{\sqrt{\pi} \cdot s^{\frac{1}{2}}} = \frac{2}{\pi}\sqrt{\frac{\pi}{s}} \qquad \therefore\ f(t) = \frac{2}{\pi\sqrt{t}}$$

(2) $\displaystyle\int_0^t f(u)e^{u-t}\,du = \int_0^t f(u)e^{-(t-u)}\,du$ だから，積分方程式を $f(t) * e^{-t} = e^t \sin t$ と書き直し，両辺をラプラス変換すると，$F(s) \cdot \dfrac{1}{s+1} = \dfrac{1}{(s-1)^2 + 1}$ $\qquad F(s) = \dfrac{s+1}{(s-1)^2 + 1} = \dfrac{(s-1) + 2}{(s-1)^2 + 1}$

$\therefore\ f(t) = e^t(\cos t + 2\sin t)$

3 章の問題

1

(1) $\mathscr{L}\left[(2t-1)(t+3)e^{-t}\right](s) = \mathscr{L}\left[2t^2 e^{-t} + 5te^{-t} - 3e^{-t}\right](s) = \dfrac{4}{(s+1)^3} + \dfrac{5}{(s+1)^2} - \dfrac{3}{s+1}$

(2) $\mathscr{L}\left[(t + \cos t)e^{2t}\right](s) = \mathscr{L}\left[te^{2t} + e^{2t}\cos t\right](s) = \dfrac{1}{(s-2)^2} + \dfrac{s-2}{(s-2)^2 + 1}$

(3) $\mathscr{L}\left[(\sin t - \cos t)^2\right](s) = \mathscr{L}\left[\sin^2 t + \cos^2 t - 2\sin t\cos t\right](s) = \mathscr{L}\left[1 - \sin 2t\right](s) = \dfrac{1}{s} - \dfrac{2}{s^2 + 4}$

(4) $\mathscr{L}\left[t * \cos t\right](s) = \dfrac{1}{s^2} \cdot \dfrac{s}{s^2 + 1} = \dfrac{1}{s(s^2 + 1)}$

(5) $\mathscr{L}\left[\delta(2t-1)\right](s) = \mathscr{L}\left[\dfrac{1}{2}\delta\left(t - \dfrac{1}{2}\right)\right](s) = \dfrac{1}{2}e^{-\frac{s}{2}}$

(6) $\mathscr{L}\left[(1 - e^{\frac{t}{2}})\sin\sqrt{2}\,t\right](s) = \mathscr{L}\left[\sin\sqrt{2}\,t - e^{\frac{t}{2}}\sin\sqrt{2}\,t\right](s)$

$$= \frac{\sqrt{2}}{s^2 + 2} - \frac{\sqrt{2}}{\left(s - \frac{1}{2}\right)^2 + 2} = \frac{\sqrt{2}}{s^2 + 2} - \frac{4\sqrt{2}}{(2s-1)^2 + 8}$$

2

(1) $\mathscr{L}\left[2\sin t\cos t\cos 2t\right](s)=\mathscr{L}\left[\sin 2t\cos 2t\right](s)=\mathscr{L}\left[\frac{1}{2}\sin 4t\right](s)=\frac{1}{2}\cdot\frac{4}{s^2+16}=\frac{2}{s^2+16}$

(2) $\mathscr{L}\left[\left(e^{-t}+e^{2t}\right)^3\right](s)=\mathscr{L}\left[e^{-3t}+3e^{-2t}e^{2t}+3e^{-t}e^{4t}+e^{6t}\right](s)=\mathscr{L}\left[e^{-3t}+3+3e^{3t}+e^{6t}\right](s)$
$=\frac{1}{s+3}+\frac{3}{s}+\frac{3}{s-3}+\frac{1}{s-6}$

(3) $\mathscr{L}\left[\frac{\sin 3t}{t}\right](s)=\int_s^\infty\frac{3}{s^2+9}\,ds=\lim_{S\to\infty}\left[\mathrm{Tan}^{-1}\frac{s}{3}\right]_s^S=\lim_{S\to\infty}\left(\mathrm{Tan}^{-1}\frac{S}{3}-\mathrm{Tan}^{-1}\frac{s}{3}\right)$
$=\frac{\pi}{2}-\mathrm{Tan}^{-1}\frac{s}{3}=\mathrm{Tan}^{-1}\frac{3}{s}$

(4) $\mathscr{L}\left[H(2t-1)\right](s)=\frac{1}{2}\mathscr{L}\left[H(t-1)\right]\left(\frac{s}{2}\right)=\frac{1}{2}\cdot\frac{e^{-\frac{s}{2}}}{\frac{s}{2}}=\frac{1}{s}e^{-\frac{s}{2}}$

(5) $\mathscr{L}\left[\cosh^2 t-\sinh^2 t\right](s)=\mathscr{L}\left[\left(\frac{e^t+e^{-t}}{2}\right)^2-\left(\frac{e^t-e^{-t}}{2}\right)^2\right](s)$
$=\frac{1}{4}\mathscr{L}\left[\left(e^{2t}+2+e^{-2t}\right)-\left(e^{2t}-2+e^{-2t}\right)\right](s)=\mathscr{L}[1](s)=\frac{1}{s}$

(6) $(t-\sin t)^2=t^2-2t\sin t+\sin^2 t=t^2-2t\sin t+\frac{1-\cos 2t}{2}$ より
$\mathscr{L}\left[(t-\sin t)^2\right](s)=\frac{2}{s^3}+2\left(\frac{1}{s^2+1}\right)'+\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)=\frac{2}{s^3}-\frac{4s}{(s^2+1)^2}+\frac{1}{2s}-\frac{s}{2(s^2+4)}$

(7) $\mathscr{L}\left[\int_0^t t^2e^t\,dt\right](s)=\frac{1}{s}\mathscr{L}\left[t^2e^t\right](s)=\frac{1}{s}\cdot(-1)^2\left(\frac{1}{s-1}\right)''=\frac{2}{s(s-1)^3}$

(8) $\mathscr{L}\left[\frac{1}{\sqrt{t}}\right](s)=\sqrt{\frac{\pi}{s}}$ より，$\mathscr{L}\left[\frac{1}{\sqrt{t}\,e^t}\right](s)=\mathscr{L}\left[e^{-t}\frac{1}{\sqrt{t}}\right](s)=\sqrt{\frac{\pi}{s+1}}$

3

(1) $\mathscr{L}^{-1}\left[\frac{1}{2s+1}\right](t)=\mathscr{L}^{-1}\left[\frac{1}{2}\cdot\frac{1}{s+\frac{1}{2}}\right](t)=\frac{1}{2}e^{-\frac{1}{2}t}$

(2) $\mathscr{L}^{-1}\left[\frac{1}{(2s-3)^2}\right](t)=\mathscr{L}^{-1}\left[\frac{1}{4}\cdot\frac{1}{\left(s-\frac{3}{2}\right)^2}\right](t)=\frac{1}{4}te^{\frac{3}{2}t}$

(3) $\mathscr{L}^{-1}\left[\frac{1}{s^2+3}\right](t)=\mathscr{L}^{-1}\left[\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{s^2+3}\right](t)=\frac{1}{\sqrt{3}}\sin\sqrt{3}\,t$

(4) $\mathscr{L}^{-1}\left[\frac{s^2-3}{(s^2+3)^2}\right](t)=\mathscr{L}^{-1}\left[-\left(\frac{s}{s^2+3}\right)'\right](t)=t\cos\sqrt{3}\,t$

(5) $\mathscr{L}^{-1}\left[\frac{s}{s^2-6s+12}\right](t)=\mathscr{L}^{-1}\left[\frac{s-3+3}{(s-3)^2+3}\right](t)=\mathscr{L}^{-1}\left[\frac{s-3}{(s-3)^2+3}+\sqrt{3}\cdot\frac{\sqrt{3}}{(s-3)^2+3}\right](t)$
$=e^{3t}\left(\cos\sqrt{3}\,t+\sqrt{3}\sin\sqrt{3}\,t\right)$

(6) $\mathscr{L}^{-1}\left[\frac{e^{-3s}}{s}\right](t)=H(t-3)$ (7) $\mathscr{L}^{-1}\left[\frac{1-s}{s}\right](t)=\mathscr{L}^{-1}\left[\frac{1}{s}-1\right](t)=1-\delta(t)$

(8) $\mathscr{L}^{-1}\left[\sqrt{\frac{\pi}{s}}\right](t)=\frac{1}{\sqrt{t}}$ より，$\mathscr{L}^{-1}\left[\sqrt{\frac{\pi}{s-1}}\right](t)=\frac{e^t}{\sqrt{t}}$

4

(1) $\int_0^\infty e^{-t}\sin^2 t\,dt=\mathscr{L}\left[\sin^2 t\right](1)=\mathscr{L}\left[\frac{1-\cos 2t}{2}\right](1)=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)\Big|_{s=1}=\frac{1}{2}\left(1-\frac{1}{5}\right)$
$=\frac{2}{5}$

(2) $\mathscr{L}[\sqrt{t}\,](s)=\dfrac{\sqrt{\pi}}{2\sqrt{s^3}}$ より

$$\int_0^\infty e^{-\pi t}\sqrt{t}\,dt=\mathscr{L}[\sqrt{t}\,](\pi)=\left.\frac{\sqrt{\pi}}{2\sqrt{s^3}}\right|_{s=\pi}=\frac{\sqrt{\pi}}{2\sqrt{\pi^3}}=\frac{1}{2\pi}$$

5

(1) $\mathscr{L}[t^2*\cos t](s)=\mathscr{L}[t^2](s)\cdot\mathscr{L}[\cos t](s)=\dfrac{2}{s^3}\cdot\dfrac{s}{s^2+1}=2\cdot\dfrac{1}{s^2}\cdot\dfrac{1}{s^2+1}=2\mathscr{L}[t](s)\cdot\mathscr{L}[\sin t](s)$

$=2\mathscr{L}[t*\sin t](s)$

(2) $\mathscr{L}[t^r*\cos t](s)=\mathscr{L}[t^r](s)\cdot\mathscr{L}[\cos t](s)=\dfrac{r!}{s^{r+1}}\cdot\dfrac{s}{s^2+1}=r\cdot\dfrac{(r-1)!}{s^r}\cdot\dfrac{1}{s^2+1}$

$=r\mathscr{L}[t^{r-1}](s)\cdot\mathscr{L}[\sin t](s)=r\mathscr{L}[t^{r-1}*\sin t](s)$

6

(1) 初期条件 $x(0)=0,\ x'(0)=\dfrac{5}{4}$ の下で $4x''-x=-\sin\dfrac{t}{2}$ をラプラス変換すると

$$4\left(s^2X-\frac{5}{4}\right)-X=-\frac{\frac{1}{2}}{s^2+\frac{1}{4}}\qquad\therefore\ (4s^2-1)X=5-\frac{2}{4s^2+1}=\frac{20s^2+3}{4s^2+1}$$

$$\therefore\ X=\frac{20s^2+3}{(2s-1)(2s+1)(4s^2+1)}=\frac{2}{2s-1}-\frac{2}{2s+1}+\frac{1}{4s^2+1}$$

$$=\frac{1}{s-\frac{1}{2}}-\frac{1}{s+\frac{1}{2}}+\frac{1}{2}\cdot\frac{\frac{1}{2}}{s^2+\frac{1}{4}}$$

両辺の逆ラプラス変換を考えると，$x=e^{\frac{t}{2}}-e^{-\frac{t}{2}}+\dfrac{1}{2}\sin\dfrac{t}{2}$

(2) 初期条件 $x(0)=1,\ x'(0)=-1$ の下で $2x''+x'-x=9e^{-t}$ をラプラス変換すると

$$2(s^2X-s+1)+sX-1-X=\frac{9}{s+1}\qquad\therefore\ (2s^2+s-1)X=2s-1+\frac{9}{s+1}=\frac{2s^2+s+8}{s+1}$$

$$\therefore\ X=\frac{2s^2+s+8}{(2s-1)(s+1)^2}=\frac{4}{2s-1}-\frac{1}{s+1}-\frac{3}{(s+1)^2}=\frac{2}{s-\frac{1}{2}}-\frac{1}{s+1}-\frac{3}{(s+1)^2}$$

両辺の逆ラプラス変換を考えると，$x=2e^{\frac{t}{2}}-e^{-t}-3te^{-t}$

(3) 初期条件 $x(0)=1,\ x'(0)=\dfrac{1}{2}$ の下で $4x''+4x'-3x=18-9t$ をラプラス変換すると

$$4\left(s^2X-s-\frac{1}{2}\right)+4(sX-1)-3X=\frac{18}{s}-\frac{9}{s^2}$$

$$\therefore\ (4s^2+4s-3)X=4s+6+\frac{18s-9}{s^2}=2(2s+3)+\frac{9(2s-1)}{s^2}$$

$$\therefore\ X=\frac{2}{2s-1}+\frac{4}{2s+3}+\frac{3}{s^2}-\frac{2}{s}=\frac{1}{s-\frac{1}{2}}+\frac{2}{s+\frac{3}{2}}+\frac{3}{s^2}-\frac{2}{s}$$

両辺の逆ラプラス変換を考えると，$x=e^{\frac{t}{2}}+2e^{-\frac{3}{2}t}+3t-2$

7

(1) $\alpha=x'(0)$ とおき，境界条件 $x(0)=1$ の下で $4x''+12x'+9x=9$ をラプラス変換すると

$$4(s^2X-s-\alpha)+12(sX-1)+9X=\frac{9}{s}$$

$$\therefore\ (4s^2+12s+9)X=4s+4\alpha+12+\frac{9}{s}=\frac{4s^2+(4\alpha+12)s+9}{s}$$

$$\therefore\ X=\frac{4s^2+(4\alpha+12)s+9}{s(2s+3)^2}=\frac{1}{s}+\frac{4\alpha}{(2s+3)^2}=\frac{1}{s}+\frac{\alpha}{\left(s+\frac{3}{2}\right)^2}$$

両辺の逆ラプラス変換を考えると，$x=1+\alpha te^{-\frac{3}{2}t}\qquad\therefore\ x'=\alpha e^{-\frac{3}{2}t}-\dfrac{3}{2}\alpha te^{-\frac{3}{2}t}=\dfrac{1}{2}\alpha(2-3t)e^{-\frac{3}{2}t}$

境界条件 $x'(1)=e^{-2}$ より，$x'(1)=-\dfrac{1}{2}\alpha e^{-\frac{3}{2}}=e^{-2}\qquad\therefore\ \alpha=-2e^{-\frac{1}{2}}\qquad\therefore\ x=1-2te^{-\frac{3t+1}{2}}$

(2) $\alpha = x'(0)$ とおき，境界条件 $x(0) = 0$ の下で $x'' - x' = te^t$ をラプラス変換すると

$$s^2X - \alpha - sX = \frac{1}{(s-1)^2} \qquad \therefore\ (s^2 - s)X = \alpha + \frac{1}{(s-1)^2} = \frac{\alpha s^2 - 2\alpha s + \alpha + 1}{(s-1)^2}$$

$$\therefore\ X = \frac{\alpha s^2 - 2\alpha s + \alpha + 1}{s(s-1)^3} = -\frac{\alpha+1}{s} + \frac{1}{(s-1)^3} - \frac{1}{(s-1)^2} + \frac{\alpha+1}{s-1}$$

両辺の逆ラプラス変換を考えると，$x = -(\alpha+1) + \frac{1}{2}t^2e^t - te^t + (\alpha+1)e^t = -(\alpha+1) + \left(\frac{1}{2}t^2 - t + \alpha + 1\right)e^t$

$$\therefore\ x' = (t-1)e^t + \left(\frac{1}{2}t^2 - t + \alpha + 1\right)e^t = \left(\frac{1}{2}t^2 + \alpha\right)e^t$$

境界条件 $x'(1) = e$ より，$x'(1) = \left(\frac{1}{2} + \alpha\right)e = e \qquad \therefore\ \alpha = \frac{1}{2} \qquad \therefore\ x = -\frac{3}{2} + \frac{1}{2}(t^2 - 2t + 3)e^t$

(3) $\alpha = x'(0)$ とおき，境界条件 $x(0) = 1$ の下で $2x'' - x' = \sqrt{t} - \frac{1}{\sqrt{t}}$ をラプラス変換すると

$$2(s^2X - s - \alpha) - (sX - 1) = \frac{1}{2}\sqrt{\frac{\pi}{s^3}} - \sqrt{\frac{\pi}{s}}$$

$$\therefore\ (2s^2 - s)X = 2s - 1 + 2\alpha + \frac{\sqrt{\pi}}{2}\left(\frac{1}{\sqrt{s^3}} - \frac{2}{\sqrt{s}}\right) = 2s - 1 + 2\alpha - \frac{\sqrt{\pi}}{2}\cdot\frac{2s-1}{\sqrt{s^3}}$$

$$\therefore\ X = \frac{1}{s} - \frac{2\alpha}{s(2s-1)} - \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{s^3}} = \frac{1}{s} - \frac{2\alpha}{s} + \frac{4\alpha}{2s-1} - \frac{1}{2}\sqrt{\frac{\pi}{s^3}}$$

$$= \frac{1-2\alpha}{s} + \frac{2\alpha}{s - \frac{1}{2}} - \frac{1}{2}\sqrt{\frac{\pi}{s^3}}$$

両辺の逆ラプラス変換を考えると，$x = 1 - 2\alpha + 2\alpha e^{\frac{1}{2}t} - \sqrt{t} \qquad \therefore\ x' = \alpha e^{\frac{1}{2}t} - \frac{1}{2\sqrt{t}}$

境界条件 $x'(1) = \frac{\sqrt{e}-1}{2}$ より，$x'(1) = \alpha e^{\frac{1}{2}} - \frac{1}{2} = \frac{\sqrt{e}-1}{2} \qquad \therefore\ \alpha = \frac{1}{2} \qquad \therefore\ x = e^{\frac{1}{2}t} - \sqrt{t}$

8

(1) $f_1(t) = \frac{e^t}{1!}\frac{d}{dt}(te^{-t}) = e^t(e^{-t} - te^{-t}) = 1 - t$

$$f_2(t) = \frac{e^t}{2!}\frac{d^2}{dt^2}(t^2e^{-t}) = \frac{e^t}{2}\frac{d}{dt}(2te^{-t} - t^2e^{-t}) = \frac{e^t}{2}\frac{d}{dt}\left\{(2t - t^2)e^{-t}\right\} = \frac{e^t}{2}\left\{(2-2t)e^{-t} - (2t-t^2)e^{-t}\right\}$$

$$= \frac{e^t}{2}(2 - 4t + t^2)e^{-t} = \frac{1}{2}(2 - 4t + t^2)$$

$$\therefore\ f_1(t)f_2(t) = (1-t)\cdot\frac{1}{2}(2 - 4t + t^2) = \frac{1}{2}(2 - 6t + 5t^2 - t^3)$$

$$\therefore\ \mathscr{L}[f_1(t)f_2(t)](s) = \frac{1}{2}\left(\frac{2}{s} - \frac{6}{s^2} + \frac{10}{s^3} - \frac{6}{s^4}\right) = \frac{1}{s} - \frac{3}{s^2} + \frac{5}{s^3} - \frac{3}{s^4}$$

$$\therefore\ \mathscr{L}[f_1(t)f_2(t)](1) = 1 - 3 + 5 - 3 = 0$$

(2) $f_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}(t^ne^{-t}) = \frac{e^t}{n!}\sum_{r=0}^{n}{}_n\mathrm{C}_r(t^n)^{(r)}(e^{-t})^{(n-r)}$

$$= \frac{e^t}{n!}\left\{{}_n\mathrm{C}_0(t^n)^{(0)}(e^{-t})^{(n)} + \sum_{r=1}^{n}{}_n\mathrm{C}_r(t^n)^{(r)}(e^{-t})^{(n-r)}\right\}$$

$$= \frac{e^t}{n!}\left\{{}_n\mathrm{C}_0t^n(-1)^ne^{-t} + \sum_{r=1}^{n}{}_n\mathrm{C}_rn(n-1)\cdots(n-r+1)t^{n-r}(-1)^{n-r}e^{-t}\right\}$$

$$= {}_n\mathrm{C}_0\frac{1}{n!}t^n(-1)^n + \sum_{r=1}^{n}{}_n\mathrm{C}_r\frac{n(n-1)\cdots(n-r+1)}{n!}t^{n-r}(-1)^{n-r}$$

$$= {}_n\mathrm{C}_0\frac{(-1)^{n-0}}{(n-0)!}t^{n-0} + \sum_{r=1}^{n}{}_n\mathrm{C}_r\frac{(-1)^{n-r}}{(n-r)!}t^{n-r} = \sum_{r=0}^{n}{}_n\mathrm{C}_r\frac{(-1)^{n-r}}{(n-r)!}t^{n-r}$$

(3) $\displaystyle\int_0^{\infty} e^{-t} f_n(t)\,dt = \int_0^{\infty} e^{-t} \sum_{r=0}^{n} {}_n\mathrm{C}_r \frac{(-1)^{n-r}}{(n-r)!} t^{n-r}\,dt = \sum_{r=0}^{n} {}_n\mathrm{C}_r \frac{(-1)^{n-r}}{(n-r)!} \int_0^{\infty} e^{-t} t^{n-r}\,dt$

$\displaystyle = \sum_{r=0}^{n} {}_n\mathrm{C}_r \frac{(-1)^{n-r}}{(n-r)!} \mathscr{L}\left[t^{n-r}\right](1) = \sum_{r=0}^{n} \left\{ {}_n\mathrm{C}_r \frac{(-1)^{n-r}}{(n-r)!} \cdot \left.\frac{(n-r)!}{s^{n-r+1}}\right|_{s=1} \right\} = \sum_{r=0}^{n} \left\{ {}_n\mathrm{C}_r \frac{(-1)^{n-r}}{1^{n-r+1}} \right\}$

$\displaystyle = \sum_{r=0}^{n} {}_n\mathrm{C}_r (-1)^{n-r} = \sum_{r=0}^{n} {}_n\mathrm{C}_r 1^r (-1)^{n-r} = \big(1 + (-1)\big)^n = 0$

9

(1) $X = \mathscr{L}\left[x\right](s)$ とする。$x(t) = \dfrac{3}{2}\displaystyle\int_0^t x(u)\sin 2(t-u)\,du + t = \dfrac{3}{2} x(t) * \sin 2t + t$ と変形し，両辺をラプラス変換すると，$X = \dfrac{3}{2}\mathscr{L}\left[x(t) * \sin 2t\right](s) + \dfrac{1}{s^2} = \dfrac{3}{2} X \cdot \dfrac{2}{s^2+4} + \dfrac{1}{s^2} = \dfrac{3}{s^2+4} X + \dfrac{1}{s^2}$

$\therefore \left(1 - \dfrac{3}{s^2+4}\right) X = \dfrac{1}{s^2} \qquad \therefore X = \dfrac{s^2+4}{s^2(s^2+1)} = \dfrac{4}{s^2} - \dfrac{3}{s^2+1} \qquad \therefore x = 4t - 3\sin t$

(2) $\displaystyle\int_0^t x(u)\sin 2(t-u)\,du = \frac{2}{3}\left\{x(t) - t\right\} = \frac{2}{3}\left\{(4t - 3\sin t) - t\right\} = 2(t - \sin t)$ だから

$\displaystyle\int_{\pi}^{2\pi} x(u)\sin 2u\,du = \int_0^{2\pi} x(u)\sin 2u\,du - \int_0^{\pi} x(u)\sin 2u\,du$

$\displaystyle = -\int_0^{2\pi} x(u)\sin(-2u)\,du + \int_0^{\pi} x(u)\sin(-2u)\,du$

$\displaystyle = -\int_0^{2\pi} x(u)\sin(4\pi - 2u)\,du + \int_0^{\pi} x(u)\sin(2\pi - 2u)\,du$

$\displaystyle = -\int_0^{2\pi} x(u)\sin 2(2\pi - u)\,du + \int_0^{\pi} x(u)\sin 2(\pi - u)\,du$

$= -2(2\pi - \sin 2\pi) + 2(\pi - \sin\pi)$

$= -2\pi$