１章　ベクトル解析

３節　ベクトル場

A

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⑴　$f_x = 2xy - y^2$，$f_y = x^2 - 2xy + z^2$，$f_z = 2yz$　より

$\nabla f = (2xy - y^2,\ x^2 - 2xy + z^2,\ 2yz)$，P での勾配は $\nabla f = (1,\ 3,\ 4)$

⑵　$f_x = 6xy$，$f_y = 3x^2 + 2yz^3$，$f_z = 3y^2z^2$　より

$\nabla f = (6xy,\ 3x^2 + 2yz^3,\ 3z^2y^2)$，P での勾配は $\nabla f = (6,\ 9,\ 12)$

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⑴　$f_x = y$，$f_y = x$，$f_z = -1$　より　勾配　$\nabla f = (y,\ x,\ -1)$

⑵　$(x,\ y,\ z) = (1,\ 2,\ 1)$ のとき $\nabla f = (2,\ 1,\ -1)$ で，その大きさ

$|\nabla f| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$　より　$\boldsymbol{n} = \dfrac{\pm 1}{\sqrt{6}}(2,\ 1,\ -1)$

⑶　⑵より $(2,\ 1,\ -1)$ は平面 H の法線ベクトルで，H が $\mathrm{P}(1,\ 2,\ 1)$ を通ることから H は

$2(x-1) + 1(y-2) - 1(z-1) = 0$　つまり　$2x + y - z = 3$

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⑴　$\nabla f = (2xy - y^2,\ x^2 - 2xy + z^2,\ 2yz)$ より　$\mathrm{P}(1,\ 1,\ 1)$ での勾配は　$\nabla f = (1,\ 0,\ 2)$ であり，

$\boldsymbol{e}_1$，$\boldsymbol{e}_2$，$\boldsymbol{e}_3$ 方向の方向微分係数は，それぞれ

$(\nabla f)\cdot \boldsymbol{e}_1 = (1,\ 0,\ 2)\cdot(1,\ 0,\ 0) = 1$

$(\nabla f)\cdot \boldsymbol{e}_2 = (1,\ 0,\ 2)\cdot(0,\ 1,\ 0) = 0$

$(\nabla f)\cdot \boldsymbol{e}_3 = (1,\ 0,\ 2)\cdot(0,\ 0,\ 1) = 2$

⑵　⑴と同様にして

$$(\nabla f)\cdot \boldsymbol{e} = (1,\ 0,\ 2)\cdot \frac{1}{3}(2,\ -1,\ -2)$$

$$= \frac{1}{3}(-2 + 0 - 4) = -\frac{2}{3}$$

37　$\varphi(f) = f^{-2}$　なので

$$\nabla\left(\frac{1}{f^2}\right) = \varphi'(f)(\nabla f) = -2f^{-3}\cdot(2xy - y^2,\ x^2 - 2xy + z^2,\ 2yz)$$　(p.14 $\boxed{2}$ 5)

$$= \frac{-2(2xy - y^2,\ x^2 - 2xy + z^2,\ 2yz)}{(x^2y - xy^2 + yz^2)^3}$$

38 $\nabla(fg) = (1,\ 1,\ -1)(xy-z) + (x+y-z)(y,\ x,\ -1)$ 　(p.14 $\boxed{2}$ 3)

$= (xy-z,\ \ xy-z,\ \ -xy+z) + (xy+y^2-yz,\ \ x^2+xy-xz,\ \ -x-y+z)$

$= (2xy+y^2-yz-z,\ \ x^2+2xy-xz-z,\ \ -xy-x-y+2z)$

$\nabla\left(\dfrac{f}{g}\right) = \dfrac{1}{(xy-z)^2}\left\{(1,\ 1,\ -1)(xy-z) - (x+y-z)(y,\ x,\ -1)\right\}$ 　(p.14 $\boxed{2}$ 4)

$= \dfrac{1}{(xy-z)^2}(-y^2+yz-z,\ \ -x^2+xz-z,\ \ x-xy+y)$

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(1) $\nabla\cdot\boldsymbol{f} = 4xz - x\cdot 2yz + 3y\cdot 2z$ より　P では　$\nabla\cdot\boldsymbol{f} = 4-2+6=8$

(2) $\nabla\cdot\boldsymbol{f} = 2xy+0+2y$ より　P では　$\nabla\cdot\boldsymbol{f} = 4+4=8$

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(1) $\nabla\cdot(\varphi\boldsymbol{f}) = (\nabla\varphi)\cdot\boldsymbol{f} + \varphi(\nabla\cdot\boldsymbol{f})$

$= (1,\ 1,\ 1)(x^2+y^2,\ \ y^2+z^2,\ \ z^2+x^2) + (x+y+z)(2x+2y+2z)$

$= x^2+y^2+y^2+z^2+z^2+x^2+2(x^2+y^2+z^2+2xy+2yz+2zx)$

$= 4(x^2+y^2+z^2+xy+yz+zx)$

(2) $\varphi\boldsymbol{f} = (x^3+xy^2+x^2y+y^3+zx^2+y^2z,$

$xy^2+xz^2+y^3+yz^2+y^2z+z^3,$

$xz^2+x^3+yz^2+x^2y+z^3+x^2z)$

より

$\nabla\cdot(\varphi\boldsymbol{f}) = 3x^2+y^2+2xy+2xz$

$+2xy+3y^2+z^2+2yz$

$+2xz+2yz+3z^2+x^2$

$= 4(x^2+y^2+z^2+xy+yz+zx)$

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⑴ $\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix} = (0-y^2,\ -(z^2-0),\ 0-x^2) = (-y^2,\ -z^2,\ -x^2)$

より $(x,\ y,\ z) = (1,\ 2,\ 1)$ では $\nabla \times \boldsymbol{f} = (-4,\ -1,\ -1)$

⑵ $\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2 & yz^2 & zx^2 \end{vmatrix} = (-2yz,\ -2xz,\ -2xy)$ より $(x,\ y,\ z) = (1,\ 2,\ 1)$ では

$\nabla \times \boldsymbol{f} = (-4,\ -2,\ -4)$

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⑴ $\varphi\boldsymbol{g}$ の回転　p.15 $\boxed{7}$ 3 より

$\nabla \times (\varphi\boldsymbol{g}) = (\nabla\varphi) \times \boldsymbol{g} + \varphi\,(\nabla \times \boldsymbol{g})$

$$= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ y & z & x \\ yz & zx & xy \end{vmatrix} + \varphi \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= (zxy - zx^2,\ -(xy^2 - xyz),\ yzx - yz^2)$$

$$\quad + (xy + yz + zx)(x - x,\ -y + y,\ z - z)$$

$$= (xyz - zx^2,\ xyz - xy^2,\ xyz - yz^2)$$

⑵ $\boldsymbol{f} + \boldsymbol{g}$ の回転　p.15 $\boxed{7}$ 1 より

$\nabla \times (\boldsymbol{f} + \boldsymbol{g}) = \nabla \times \boldsymbol{f} + \nabla \times \boldsymbol{g}$

$$= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & yx & zy \end{vmatrix} + \boldsymbol{0}$$　（⑴より）

$$= (z - 0,\ -(0 - x),\ y - 0) = (z,\ x,\ y)$$

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⑴ $\dfrac{ds}{dt}=|\boldsymbol{r}'(t)|=\sqrt{1^2+2^2+1^2}=\sqrt{6}$　より

$$\int_C f(x,\ y,\ z)\,ds=\int_0^1(t\cdot 2t+2t\cdot t)\cdot\sqrt{6}\,dt$$

$$=\int_0^1 4\sqrt{6}\,t^2\,dt=4\sqrt{6}\left[\frac{1}{3}t^3\right]_0^1=\frac{4}{3}\sqrt{6}$$

曲線 C の長さは

$$\int_C 1\,ds=\int_0^1\frac{ds}{dt}\,dt=\sqrt{6}\int_0^1 dt=\sqrt{6}$$

⑵ $\dfrac{ds}{dt}=|\boldsymbol{r}'(t)|=\sqrt{(-\sin t)^2+\cos^2 t+\sqrt{3}^{\,2}}=2$　より

$$\int_C f(x,\ y,\ z)\,ds=\int_0^1(\cos t+\sin t+\sqrt{3}t)\cdot 2\,dt$$

$$=2\left[\sin t-\cos t+\frac{\sqrt{3}}{2}t^2\right]_0^{\pi}$$

$$=2\left\{\left(1+\frac{\sqrt{3}}{2}\pi^2\right)-(-1)\right\}=4+\sqrt{3}\pi^2$$

曲線の長さは

$$\int_C 1\,ds=\int_0^{\pi}\frac{ds}{dt}\,dt=\int_0^{\pi}2dt=2\left[\ t\ \right]_0^{\pi}=2\pi$$

⑶ $\dfrac{ds}{dt}=|\boldsymbol{r}'(t)|=\sqrt{(6t^2)^2+(6t)^2+3^2}=\sqrt{36t^4+36t^2+9}$

$=3\sqrt{4t^4+4t^2+1}=3(2t^2+1)$ より

$$\int_C f(x,\ y,\ z)\,ds=\int_0^1 xyz\frac{ds}{dt}\,dt=\int_0^1 2t^3\cdot 3t^2\cdot 3t\cdot 3(2t^2+1)\,dt$$

$$=54\int_0^1 2t^8+t^6\,dt=54\left[\frac{2}{9}t^9+\frac{1}{7}t^7\right]_0^1=54\cdot\frac{14+9}{9\times 7}=\frac{138}{7}$$

一方，曲線 C の長さは

$$\int_C 1\,ds=\int_0^1\frac{ds}{dt}\,dt=\int_0^1|\boldsymbol{r}'(t)|\,dt$$

$$=\int_0^1 3(2t^2+1)\,dt=3\left[\frac{2}{3}t^3+t\right]_0^1=3\left(\frac{2}{3}+1\right)=5$$

(4) $\dfrac{ds}{dt}=|\boldsymbol{r}'(t)|=\sqrt{(2t)^2+(1-t^2)^2+(1+t^2)^2}$

$$=\sqrt{4t^2+1-2t^2+t^4+1+2t^2+t^4}$$

$$=\sqrt{2t^4+4t^2+2}=\sqrt{2}(t^2+1)\quad \text{より}$$

$$\int_C f(x,\ y,\ z)\,ds=\int_0^1(x+y+z)\frac{ds}{dt}dt=\int_0^1\left(t^2+t-\frac{t^3}{3}+t+\frac{t^3}{3}\right)\sqrt{2}(t^2+1)\,dt$$

$$=\int_0^1\sqrt{2}(t^2+2t)(t^2+1)\,dt=\sqrt{2}\int_0^1(t^4+2t^3+t^2+2t)\,dt$$

$$=\sqrt{2}\left[\frac{1}{5}t^5+\frac{2}{4}t^4+\frac{1}{3}t^3+t^2\right]_0^1=\frac{61\sqrt{2}}{30}$$

曲線 C の長さは

$$\int_C 1\,ds=\int_0^1\frac{ds}{dt}dt=\int_0^1\sqrt{2}(t^2+1)\,dt=\sqrt{2}\left[\frac{1}{3}t^3+t\right]_0^1=\frac{4}{3}\sqrt{2}$$

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(1) 曲線 C の長さは

$\boldsymbol{f}=(t^3\cdot t^2,\ t^2,\ t\cdot t^3)$, $\boldsymbol{r}'(t)=(1,\ 3t^2,\ 2t)$ より

$$\int_C \boldsymbol{f}\cdot d\boldsymbol{r}=\int_0^1\boldsymbol{f}\cdot\boldsymbol{r}'(t)\,dt$$

$$=\int_0^1(t^5+3t^4+2t^5)\,dt$$

$$=\left[\frac{1}{2}t^6+\frac{3}{5}t^5\right]_0^1=\frac{11}{10}$$

(2) $\boldsymbol{f}=(2\cos t,\ 2\sin t,\ t^2)$, $\boldsymbol{r}'(t)=(-2\sin t,\ 2\cos t,\ 1)$ より

$$\int_C \boldsymbol{f}\cdot d\boldsymbol{r}=\int_0^1\boldsymbol{f}\cdot\boldsymbol{r}'(t)\,dt$$

$$=\int_0^{2\pi}(-4\sin^2 t+4\cos^2 t+t^2)\,dt$$

$$=\int_0^{2\pi}(4\cos 2t+t^2)\,dt=\left[2\sin 2t+\frac{t^2}{3}\right]_0^{2\pi}=\frac{8}{3}\pi^3$$

(3) $\boldsymbol{f}=(\cos t,\ -\sin t,\ 1)$, $\boldsymbol{r}'(t)=(-\sin t,\ \cos t,\ 2)$ より

$$\int_C \boldsymbol{f}\cdot d\boldsymbol{r}=\int_0^{\pi}(-\cos t\sin t-\sin t\cos t+2)\,dt$$

$$=\int_0^{\pi}(-\sin 2t+2)\,dt=\left[\frac{1}{2}\cos 2t+2t\right]_0^{\pi}$$

$$=\left(\frac{1}{2}+2\pi\right)-\frac{1}{2}=2\pi$$

⑷　$\boldsymbol{f}=(t^4,\ t-t^2-t^4,\ t^2-t),\ \boldsymbol{r}'(t)=(1,\ 2t,\ 2t+4t^3)$ より

$$\int_C \boldsymbol{f}\cdot d\boldsymbol{r}=\int_0^1\{t^4+2t(t-t^2-t^4)+2t(t^2-t)+4t^3(t^2-t)\}\,dt$$

$$=\int_0^1(t^4-2t^5+4t^5-4t^4)\,dt=\int_0^1(2t^5-3t^4)\,dt$$

$$=\left[\frac{t^6}{3}-\frac{3}{5}t^5\right]_0^1=\frac{-4}{15}$$

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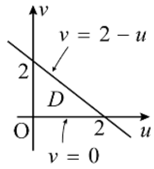
⑴　面積分は



$$\iint_D f(x,\ y,\ z)\,|\boldsymbol{r}_u\times\boldsymbol{r}_v|\,du\,dv=\iint_D(xy+z)\cdot|(1,\ 0,\ -1)\times(0,\ 1,\ -1)|\,du\,dv$$

$$=\iint_D(uv+2-u-v)\times\sqrt{1^2+1^2+1^2}\,du\,dv \qquad ㋐$$

$$=\int_{u=0}^{u=2}\sqrt{3}\int_{v=0}^{v=2-u}(u-1)v+2-u\ \,dv\,du$$

$$=\int_{u=0}^{u=2}\sqrt{3}\left[(u-1)\cdot\frac{v^2}{2}+(2-u)v\right]_{v=0}^{v=2-u}du$$

$$=\sqrt{3}\int_0^2\left\{(u-1)\frac{(-u+2)^2}{2}+(-u+2)^2\right\}du$$

$$=\sqrt{3}\int_0^2(u-2)^2\left(\frac{u-1}{2}+1\right)du$$

$$=\frac{\sqrt{3}}{2}\int_0^2(u-2)^2(u+1)\,du \quad (t=u-2\text{とおく})$$

$$=\frac{\sqrt{3}}{2}\int_{-2}^0 t^2(t+3)\,dt=\frac{\sqrt{3}}{2}\left[\frac{1}{4}t^4+t^3\right]_{-2}^0=2\sqrt{3}$$

面積は

$$\int_S 1\,dS=\iint_D|\boldsymbol{r}_u\times\boldsymbol{r}_v|\,du\,dv$$

$$=\int_{u=0}^{u=2}\int_{v=0}^{v=2-u}\sqrt{3}\,dv\,du$$

$$=\sqrt{3}\int_{u=0}^{u=2}(2-u)\,du=\sqrt{3}\left[2u-\frac{u^2}{2}\right]_0^2=2\sqrt{3}$$

（注）曲面 S が $z=\varphi(x,\ y)$ の形なので p.15 ㊍ を用いてよい。（→46⑴詳解）

⑵　面積分は

$$\iint_D f(x,\ y,\ z)\,|\boldsymbol{r}_u\times\boldsymbol{r}_v|\,du\,dv=\iint_D(u^2+2v+2-2u-2v-1)\times|(1,\ 0,\ -2)\times(0,\ 1,\ -2)|\,du\,dv$$

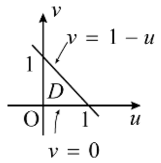
$$=3\int_{u=0}^{u=1}\int_{v=0}^{v=1-u}(u-1)^2\,dv\,du \qquad ㋐$$

$$=3\int_0^1(u-1)^2(1-u)\,du$$

$$=-3\left[\frac{1}{4}(u-1)^4\right]_0^1=\frac{3}{4}$$

面積は

$$\int_S 1\,dS=\iint_D|\boldsymbol{r}_u\times\boldsymbol{r}_v|\,du\,dv$$



$$=\int_{u=0}^{u=1}\int_{v=0}^{v=1-u}3\,dv\,du$$

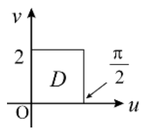
$$=3\int_0^1 1-u\,du=3\left[u-\frac{1}{2}u^2\right]_0^1=\frac{3}{2}$$

（注）曲面 S が　$z=\varphi(x,\ y)$ の形なので p.15 ㊉ を用いて解いてよい。（→46 ⑵ 詳解）

⑶　面積分は

$$\iint_D f(x,\ y,\ z)\,|\boldsymbol{r}_u\times\boldsymbol{r}_v|\,du\,dv=\iint_D 2\cos u\cdot 2\sin u\cdot v\cdot|(-2\sin u,\ 2\cos u,\ 0)\times(0,\ 0,\ 1)|\,du\,dv$$

$$=\iint_D 2v\sin 2u\,|(2\cos u,\ 2\sin u,\ 0)|\,du\,dv$$



$$=\int_{u=0}^{u=\frac{\pi}{2}}\int_{v=0}^{v=2}2v\sin 2u\cdot 2\,du\,dv$$

$$=\int_{u=0}^{u=\frac{\pi}{2}}2\sin 2u\left[\ v^2\ \right]_{v=0}^{v=2}du$$

$$=8\left[\frac{-1}{2}\cos 2u\right]_0^{\frac{\pi}{2}}=8$$

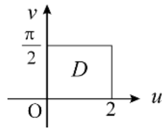
面積は

$$\int_S 1\,dS=\iint_D|\boldsymbol{r}_u\times\boldsymbol{r}_v|\,du\,dv$$

$$=\int_{u=0}^{u=\frac{\pi}{2}}\int_{v=0}^{v=2}2\,du\,dv$$

$$=\int_{u=0}^{u=\frac{\pi}{2}}2\left[\ v\ \right]_{v=0}^{v=2}du=2\pi$$

⑷　面積分は



$$\iint_D f(x,\ y,\ z)\,|\boldsymbol{r}_u \times \boldsymbol{r}_v|\,du\,dv = \iint_D u\cos v\cdot u\sin v\cdot u$$

$$\cdot\,|(\cos v,\ \sin v,\ 1)\times(-u\sin v,\ u\cos v,\ 0)|\,du\,dv$$

$$= \iint_D u^3\cdot\frac{1}{2}\sin 2v\cdot\sqrt{2u^2}\ du\,dv$$

$$= \int_{u=0}^{u=1}\int_{v=0}^{v=\frac{\pi}{2}} \frac{\sqrt{2}}{2}u^4\sin 2v\,du\,dv$$

$$= \int_{u=0}^{u=1}\frac{\sqrt{2}}{2}u^4\left[\ -\frac{1}{2}\cos 2v\ \right]_{v=0}^{v=\frac{\pi}{2}}du$$

$$= \frac{\sqrt{2}}{2}\left[\ \frac{1}{5}u^5\ \right]_0^1 = \frac{\sqrt{2}}{10}$$

面積は

$$\int_S 1\,dS = \iint_D |\boldsymbol{r}_u\times\boldsymbol{r}_v|\,du\,dv = \int_{u=0}^{u=2}\int_{v=0}^{v=\frac{\pi}{2}}\sqrt{2u^2}\ du\,dv$$

$$= \int_{u=0}^{u=2}\sqrt{2}\,u\left[\ v\ \right]_0^{\frac{\pi}{2}}du = \sqrt{2}\cdot\frac{\pi}{2}\int_{u=0}^{u=2}u\,du$$

$$= \frac{\sqrt{2}\pi}{2}\cdot\left[\ \frac{1}{2}u^2\ \right]_0^2 = \sqrt{2}\pi$$

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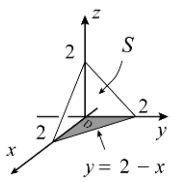
⑴　曲面 S の方程式が　$z = \varphi(x,\ y) = 2 - x - y$　なので p.15 ㊟ 式より

面積分は



$$\iint_D f(x,\ y,\ z)\sqrt{{\varphi_x}^2+{\varphi_y}^2+1}\ dx\,dy$$

$$= \iint_D (xy+2-x-y)\times\sqrt{(-1)^2+(-1)^2+1}\ dx\,dy$$

（このあとは 45⑴ ㋐ 以降と全く同じ計算）

$$= \int_{x=0}^{x=2}\sqrt{3}\int_{y=0}^{y=2-x}\{(x-1)y+2-x\}\,dy\,dx = 2\sqrt{3}$$

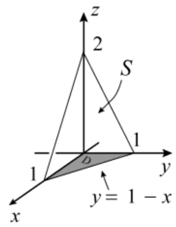
面積は p.15 ㊟ より

$$\iint_D 1\cdot\sqrt{{\varphi_x}^2+{\varphi_y}^2+1}\ dx\,dy = \int_{x=0}^{x=2}\int_{y=0}^{y=2-x}\sqrt{3}\ dx\,dy = 2\sqrt{3}$$

（これも 45⑴と同じ計算による）

⑵　曲面 S の方程式が　$z=\varphi(x,\ y)=2-2x-2y$ の形なので p.15 ㊟ より

面積分は



$$\iint_D f(x,\ y,\ z)\sqrt{{\varphi_x}^2+{\varphi_y}^2+1}\ dx\,dy$$

$$=\iint_D (x^2+2y+2-2x-2y-1)\times\sqrt{(-2)^2+(-2)^2+1}\ dx\,dy$$

$$=3\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x-1)^2\,dx\,dy=\frac{3}{4}$$

（45⑵ ㋐ と同じ計算による）

面積は

$$\iint_D 1\cdot\sqrt{{\varphi_x}^2+{\varphi_y}^2+1}\ dx\,dy=\iint_D\sqrt{(-2)^2+(-2)^2+1}\ dx\,dy$$

$$=\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}3\,dx\,dy=\frac{3}{2}$$

（これも 45⑵と同じ計算による）

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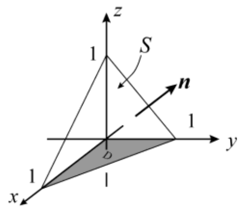
(1) $$\iint_D (3x,\ 2y,\ z)\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\,du\,dv$$



$$= \iint_D (3u,\ 2v,\ 1-u-v)\cdot\{(1,\ 0,\ -1)\times(0,\ 1,\ -1)\}\,du\,dv$$

$$((1,\ 0,\ -1)\times(0,\ 1,\ -1)=(1,\ 1,\ 1))$$

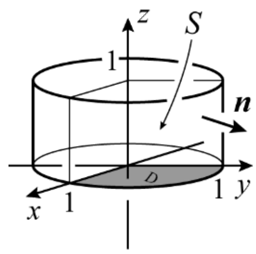
$$= \iint_D (3u+2v+1-u-v)\,du\,dv = \int_{u=0}^{u=1}\int_{v=0}^{v=1-u}(2u+v+1)\,du\,dv$$

$$= \int_{u=0}^{u=1}\left[\frac{1}{2}v^2+(2u+1)v\right]_{v=0}^{v=1-u}du = \int_0^1\left\{\frac{1}{2}(1-u)^2+(2u+1)(1-u)\right\}du$$

$$= \frac{1}{2}\int_0^1\{(1-u)^2+(4u+2)(1-u)\}\,du = \frac{1}{2}\int_0^1(1-u)(3u+3)\,du$$

$$= \frac{3}{2}\int_0^1(1-u^2)\,du = \frac{3}{2}\left[u-\frac{1}{3}u^3\right]_0^1 = 1$$

(2) $$\iint_D (3x,\ 2y,\ z)\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\,du\,dv$$



$$= \iint_D (3\cos u,\ 2\sin u,\ v)\cdot\{(-\sin u,\ \cos u,\ 0)\times(0,\ 0,\ 1)\}\,du\,dv$$

$$= \iint_D (3\cos u,\ 2\sin u,\ v)\cdot(\cos u,\ \sin u,\ 0)\,du\,dv$$

$$= \iint_D (3\cos^2 u+2\sin^2 u)\,du\,dv = \iint_D (\cos^2 u+2)\,du\,dv$$

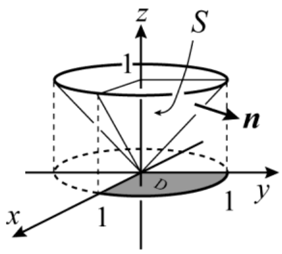
$$= \iint_D \left(\frac{1+\cos 2u}{2}+2\right)du\,dv = \int_{u=0}^{u=\frac{\pi}{2}}\int_{v=0}^{v=1}\left(\frac{5}{2}+\frac{1}{2}\cos 2u\right)dv\,du$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{5}{2}+\frac{1}{2}\cos 2u\right)\Big[\ v\ \Big]_{v=0}^{v=1}du = \left[\frac{5}{2}u+\frac{1}{4}\sin 2u\right]_0^{\frac{\pi}{2}} = \frac{5}{4}\pi$$

(3) $\displaystyle\iint_D (3x,\ 2y,\ z)\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\,du\,dv$

$(\boldsymbol{r}_u=(\cos u,\ \sin v,\ 1),\ \ \boldsymbol{r}_v=(-u\sin v,\ u\cos v,\ 0)$ より $\boldsymbol{r}_u\times\boldsymbol{r}_v=(-u\cos v,\ -u\sin v,\ u))$

$$=\iint_D (3u\cos v,\ 2u\sin v,\ u)\cdot(-u\cos v,\ -u\sin v,\ u)\,du\,dv$$



$$=\iint_D (-3u^2\cos^2 v-2u^2\sin^2 v+u^2)\,du\,dv$$

$$=\int_{u=0}^{u=1}-u^2\int_{v=0}^{v=\frac{\pi}{2}}(3\cos^2 v+2\sin^2 v-1)\,dv\,du=\int_{u=0}^{u=1}-u^2\int_{v=0}^{v=\frac{\pi}{2}}(\cos^2 v+1)\,dv\,du$$

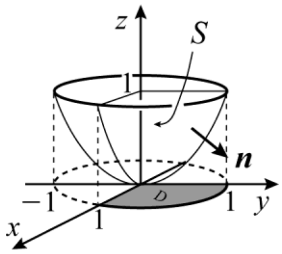
$$=\int_{u=0}^{u=1}-u^2\int_{v=0}^{v=\frac{\pi}{2}}\left(\frac{1+\cos 2v}{2}+1\right)dv\,du=\int_{u=0}^{u=1}-u^2\left[\frac{3}{2}v+\frac{1}{4}\sin 2u\right]_{v=0}^{v=\frac{\pi}{2}}du$$

$$=\frac{-3}{4}\pi\left[\frac{1}{3}u^3\right]_0^1=-\frac{\pi}{4}$$

(4) $\displaystyle\iint_D (3x,\ 2y,\ z)\cdot(\boldsymbol{r}_u\times\boldsymbol{r}_v)\,du\,dv$

$(\boldsymbol{r}_u=(\cos v,\ \sin v,\ 2u),\ \ \boldsymbol{r}_v=(-u\sin v,\ u\cos v,\ 0))$

$$=\iint_D (3u\cos v,\ 2u\sin v,\ u^2)\cdot(-2u^2\cos v,\ -2u^2\sin v,\ u)\,du\,dv$$



$$=\iint_D (-6u^3\cos^2 v-4u^3\sin^2 v+u^3)\,du\,dv$$

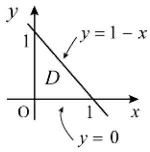
$$=\int_{u=0}^{u=1}u^3\int_{v=0}^{v=\frac{\pi}{2}}(-6\cos^2 v-4\sin^2 v+1)\,dv\,du$$

㋐ 『新版微分積分Ⅰ』p.126，p.129

$$\left(\begin{array}{l}\displaystyle\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\int_0^{\frac{\pi}{2}}\sin^n x\,dx\quad (n>0)\\ \displaystyle=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\ \cdots\cdots\ \cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2} & (n\,;\ \text{偶数})\\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\ \cdots\cdots\ \cdot\dfrac{2}{3}\cdot 1 & (n\,;\ \text{奇数})\end{cases}\end{array}\right.$$

$$=\int_0^1 u^3\left(-6\cdot\frac{1}{2}\cdot\frac{\pi}{2}-4\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{\pi}{2}\right)du=-\frac{\pi}{2}$$

(1) $\varphi = 1 - x - y$ より $\varphi_x = -1,\ \varphi_y = -1$ なので p.16 ** を用いて



$$\iint_D (3x,\ 2y,\ 1 - x - y) \cdot (1,\ 1,\ 1)\, dx\, dy$$

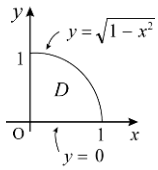
$$= \iint_D (3x + 2y + 1 - x - y)\, dx\, dy = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \{ y + (2x+1) \}\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left[\frac{1}{2} y^2 + (2x+1)\, y \right]_{y=0}^{y=1-x} dx = \int_0^1 \left\{ \frac{1}{2} (1-x)^2 + (2x+1)(1-x) \right\} dx$$

$$= \frac{1}{2} \int_0^1 \{ (1-x)^2 + (4x+2)(1-x) \}\, dx = \frac{1}{2} \int_0^1 (1-x)(3x+3)\, dx$$

$$= \frac{3}{2} \int_0^1 (1 - x^2)\, dx = \frac{3}{2} \left[x - \frac{1}{3} x^3 \right]_0^1 = 1$$

(2) $\varphi = x^2 + y^2$ より $\varphi_x = 2x,\ \varphi_y = 2y$ なので p.13 ** を用いて



$$\iint_D (3x,\ 2y,\ x^2 + y^2) \cdot (-2x,\ -2y,\ 1)\, dx\, dy$$

$$= \iint_D (-6x^2 - 4y^2 + x^2 + y^2)\, dx\, dy = \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} (-5x^2 - 3y^2)\, dy\, dx$$

$$= \int_{x=0}^{x=1} \left[-5x^2 y - y^3 \right]_{y=0}^{y=\sqrt{1-x^2}} dx = \int_{x=0}^{x=1} \left\{ -5x^2 \sqrt{1-x^2} - (1-x^2)\sqrt{1-x^2} \right\} dx$$

$$= \int_0^1 (-1 - 4x^2) \sqrt{1-x^2}\, dx$$

$$\left(\begin{array}{l} x = \sin t \quad として \quad dx = \cos t\, dt \\ \begin{array}{c|c} x & 0 \to 1 \\ \hline y & 0 \to \frac{\pi}{2} \end{array},\quad ここで \quad \cos t \geqq 0 \end{array} \right)$$

$$= \int_0^{\frac{\pi}{2}} (-1 - 4\sin^2 t) \cos t \cdot \cos t\, dt = \int_0^{\frac{\pi}{2}} (-1 - 4\sin^2 t)(1 - \sin^2 t)\, dt$$

$$= \int_0^{\frac{\pi}{2}} (-1 + \sin^2 t - 4\sin^2 t + 4\sin^4 t)\, dt = -\frac{\pi}{2} - 3 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + 4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \qquad (\ 47\ (4)\ ㋐\ )$$

$$= -\frac{\pi}{2}$$

B

49　スカラー場の面積分

C_3 は $\boldsymbol{r}=\boldsymbol{r}(t)=(2t,\ 2t,\ 0)$　$(0 \leqq t \leqq 1)$

C_4 は $\boldsymbol{r}=\boldsymbol{r}(t)=(0,\ 0,\ 3t)$　$(0 \leqq t \leqq 1)$

と表せるので　$\dfrac{ds}{dt}=|\boldsymbol{r}'(t)|=|(2,\ 2,\ 0)|=2\sqrt{2}$　より

$$\int_{C_3}(x+y+2z)\,ds=\int_0^1(2t+2t)\frac{ds}{dt}\,dt=2\sqrt{2}\Big[\ 2t^2\ \Big]_0^1=4\sqrt{2} \qquad \cdots\cdots ㋐$$

$$\int_{C_4}(x+y+2z)\,ds=\int_0^1(0+0+6t)\frac{ds}{dt}\,dt=2\sqrt{2}\Big[\ 3t^2\ \Big]_0^1=6\sqrt{2} \qquad \cdots\cdots ㋑$$

㋐，㋑ より　$\displaystyle\int_C f(x,\ y,\ z)\,ds=10\sqrt{2}$

ベクトル場の面積分

$$\int_{C_3}(y,\ z,\ xy)\cdot d\boldsymbol{r} \qquad \left(\frac{d\boldsymbol{r}}{dt}=(2,\ 2,\ 0)\right)$$

$$=\int_0^1(2t,\ 0,\ 4t^2)\cdot(2,\ 2,\ 0)\,dt=\int_0^1 4t\,dt=\Big[\ 2t^2\ \Big]_0^1=2 \qquad \cdots\cdots ㋒$$

$$\int_{C_4}(y,\ z,\ xy)\cdot d\boldsymbol{r} \qquad \left(\frac{d\boldsymbol{r}}{dt}=(0,\ 0,\ 3)\right)$$

$$=\int_0^1(0,\ 3t,\ 0)\cdot(0,\ 0,\ 3)\,dt=0 \qquad \cdots\cdots ㋓$$

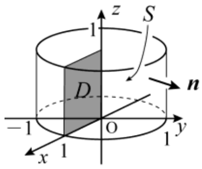
㋒，㋓ より　$\displaystyle\int_C \boldsymbol{f}(x,\ y,\ z)\cdot d\boldsymbol{r}=2$

50　p.16 公式 ** と同様

$y = \varphi(x,\ z) = \sqrt{1-x^2}$　とおくと

$\boldsymbol{r} = (x,\ \varphi(x,\ z),\ z)$, $\boldsymbol{r}_z = (0,\ \varphi_z,\ 1)$, $\boldsymbol{r}_x = (1,\ \varphi_x,\ 0)$ なので

$\boldsymbol{r}_z \times \boldsymbol{r}_x = (-\varphi_x,\ 1,\ -\varphi_z)$ となり



$$\int_S \boldsymbol{f}\cdot d\boldsymbol{S}$$

$$= \iint_D (3x,\ 2y,\ z)\cdot(\boldsymbol{r}_z \times \boldsymbol{r}_x)\,dz\,dx = \iint_D (3x,\ 2y,\ z)\cdot(-\varphi_x,\ 1,\ -\varphi_z)\,dz\,dx$$

（ここで $\varphi_x = \dfrac{-x}{\sqrt{1-x^2}}$, $\varphi_z = 0$）

$$= \int_{z=0}^{z=1}\int_{x=0}^{x=1}\left\{3x\left(\frac{x}{\sqrt{1-x^2}}\right)+2\sqrt{1-x^2}\right\}dx\,dz = \int_{z=0}^{z=1}\int_{x=0}^{x=1}\frac{x^2+2}{\sqrt{1-x^2}}\,dx\,dz$$

$$\left(\begin{array}{l} x = \sin t \quad \text{とおくと} \quad dx = \cos t\,dt \\ \begin{array}{c|c} x & 0 \to 1 \\ \hline t & 0 \to \frac{\pi}{2} \end{array},\quad \text{ここで} \quad \cos t \geqq 0 \end{array}\right)$$

$$= \int_{z=0}^{z=1}\int_{t=0}^{t=\frac{\pi}{2}}\frac{\sin^2 t+2}{\cos t}\cdot\cos t\,dt\,dz = \int_0^1\left(\frac{1}{2}\cdot\frac{\pi}{2}+2\cdot\frac{\pi}{2}\right)dz = \frac{5}{4}\pi$$

（詳解 47 ⑷ ㋐ より）　　（注）以上は 47⑵の別解である。

51　p.16 $\boxed{13}$ より

$$\iint_D (3x,\ 2y,\ z)\cdot(\boldsymbol{r}_u \times \boldsymbol{r}_v)\,du\,dv$$

$$\begin{cases} \boldsymbol{r}_u = (-\sin u\cos v,\ \cos u\cos v,\ 0), \\ \boldsymbol{r}_v = (-\cos u\sin v,\ -\sin u\sin v,\ \cos v), \\ \boldsymbol{r}_u \times \boldsymbol{r}_v = (\cos u\cos^2 v,\ \sin u\cos^2 v,\ \sin^2 u\cos v\sin v + \cos^2 u\cos v\sin v) \end{cases}$$

$$= \iint_D (3\cos u\cos v,\ 2\sin u\cos v,\ \sin v)\cdot(\cos u\cos^2 v,\ \sin u\cos^2 v,\ \cos v\sin v)\,du\,dv$$

$$= \iint_D (3\cos^2 u\cos^3 v + 2\sin^2 u\cos^3 v + \cos v\sin^2 v)\,du\,dv$$

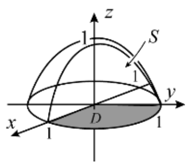
$$\begin{cases} \displaystyle\iint_D \cos^2 u\cos^3 v\,du\,dv = \int_{u=0}^{u=\frac{\pi}{2}} \cos^2 u \\ \displaystyle\cdot\left\{\int_{v=0}^{v=\frac{\pi}{2}} \cos^3 v\,dv\right\}du = \int_{v=0}^{v=\frac{\pi}{2}} \cos^2 u\cdot\frac{2}{3}\cdot 1\,du = \frac{2}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{6} \end{cases}\quad ㋐$$

（詳解 47 ⑷ ㋐）

$$\begin{cases} \displaystyle\iint_D \sin^2 u\cos^3 v\,du\,dv = \int_{u=0}^{u=\frac{\pi}{2}} \sin^2 u \\ \displaystyle\cdot\left\{\int_{v=0}^{v=\frac{\pi}{2}} \cos^3 v\,dv\right\}du = \frac{\pi}{6} \end{cases}\quad ㋑$$

（㋐ と同様　詳解 47 ⑷ ㋐を用いる）

$$\begin{cases} \displaystyle\iint_D \cos v\sin^2 v\,du\,dv \qquad \left(\sin^2 v = 1 - \cos^2 v\right) \\ \displaystyle= \int_{u=0}^{u=\frac{\pi}{2}}\int_{v=0}^{v=\frac{\pi}{2}} (\cos v - \cos^3 v)\,dv\,du \\ \displaystyle= \int_{u=0}^{u=\frac{\pi}{2}} \left(1 - \frac{2}{3}\cdot 1\right)du \qquad (\text{詳解 47 ⑷ ㋐}) \\ \displaystyle= \frac{\pi}{6} \quad ㋒ \end{cases}$$



㋐，㋑，㋒　より答は　$3\times\dfrac{\pi}{6} + 2\times\dfrac{\pi}{6} + \dfrac{\pi}{6} = \pi$

52　$x^2+y^2+z^2=1$　より　$z=\varphi(x,\ z)=\sqrt{1-x^2-y^2}=(1-x^2-y^2)^{\frac{1}{2}}$　とおくと

$\varphi_x=\dfrac{1}{2}(1-x^2-y^2)^{-\frac{1}{2}}\cdot(1-x^2-y^2)_x=\dfrac{1}{\sqrt{1-x^2-y^2}}$，$\varphi_y=\dfrac{-y}{\sqrt{1-x^2-y^2}}$　　p.16 $\boxed{13}$ ㊟ を用い

D は 51 と同じ，$x^2+y^2\to 1-0$　の広義積分で

$$\iint_D(3x,\ 2y,\ \sqrt{1-x^2-y^2})\cdot\left(\frac{x}{\sqrt{1-x^2-y^2}},\ \frac{y}{\sqrt{1-x^2-y^2}},\ 1\right)dx\,dy$$

$$=\iint_D\left(\frac{3x^2+2y^2}{\sqrt{1-x^2-y^2}}+\sqrt{1-x^2-y^2}\right)dx\,dy$$

$$=\iint_D\frac{2x^2+y^2+1}{\sqrt{1-x^2-y^2}}dx\,dy\qquad\left(\begin{array}{l}x=r\cos\theta\\ y=r\sin\theta\end{array}\right.$$

$$=\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1}\frac{2r^2\cos^2\theta+r^2\sin^2\theta+1}{\sqrt{1-r^2}}\ r\,dr\,d\theta\quad(r\to 1-0\text{の広義積分})$$

（『新版微分積分Ⅱ』p.137　公式）

$$=\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1}\frac{r^2\cos^2\theta+r^2+1}{\sqrt{1-r^2}}\ r\,dr\,d\theta\quad\left(t=1-r^2\text{として}\frac{dt}{dr}=-2r\right)$$

$$=\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{t=1}^{t=0}\frac{(1-t)\cos^2\theta+2-t}{\sqrt{t}}\ r\cdot\frac{1}{-2r}dt\,d\theta\quad(t\to +0\text{の広義積分})$$

$$=\int_{\theta=0}^{\theta=\frac{\pi}{2}}\frac{-1}{2}\left[\left(2t^{\frac{1}{2}}-\frac{2}{3}t^{\frac{3}{2}}\right)\cos^2\theta+4t^{\frac{1}{2}}-\frac{2}{3}t^{\frac{3}{2}}\right]_1^0 d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left(\frac{4}{3}\cos^2\theta+\frac{10}{3}\right)d\theta\qquad(\text{ 詳解 47 ⑷ ㋐ })$$

$$=\frac{1}{2}\left(\frac{4}{3}\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{10}{3}\cdot\frac{\pi}{2}\right)=\pi$$

（注）以上は 51 の別解である。

53

$$\int_C (grad\ f)\cdot d\boldsymbol{r} = \int_{t=\alpha}^{t=\beta} (\nabla f)\cdot \boldsymbol{r}'(t)\,dt$$

$$= \int_{t=\alpha}^{t=\beta} (f_x,\ f_y,\ f_z)\cdot(x'(t),\ y'(t),\ z'(t))\,dt$$

$$= \int_{t=\alpha}^{t=\beta} f_x x'(t) + f_y y'(t) + f_z z'(t)\ dt$$

$$= \int_{\alpha}^{\beta} \frac{d}{dt} f(x(t),\ y(t),\ z(t))\ dt$$

（『新版微分積分Ⅱ』p.97　合成関数の微分法（Ⅰ））

$$= \Big[\ f(x(t),\ y(t),\ z(t))\Big]_{\alpha}^{\beta}$$

$$= f(x(\beta),\ y(\beta),\ z(\beta)) - f(x(\alpha),\ y(\alpha),\ z(\alpha))$$

54　問題 44　⑵

$\boldsymbol{f} = (y,\ x,\ z^2) = (f_x,\ f_y,\ f_z)$ となるように $f(x,\ y,\ z)$ をきめる。x 成分に注目して，y を x で積分すると $f = xy + \varphi(y,\ z)$。（$\varphi(y,\ z)$ は y と z の関数）と表せる。

よって　$f_y = x + \varphi_y(y,\ z)$ となるが y 成分に注目して $\varphi_y(y,\ z) = 0$ だから $\varphi(y,\ z)$ は $\varphi(z)$ とかいてよいので　$f = xy + \varphi(z)$

したがって　$f_z = \varphi'(z) = z^2$

つまり　$\varphi(z) = \frac{1}{3}z^3 + C$　（C は定数）である。

以上により　$f = xy + \frac{1}{3}z^3 + C$

ここで　$t = 0$ のとき $(x,\ y,\ z) = (2,\ 0,\ 0)$：始点，

$t = 2\pi$ のとき $(x,\ y,\ z) = (2,\ 0,\ 2\pi)$：終点である。

53 より線積分の値は

$$f = (2,\ 0,\ 2\pi) - f(2,\ 0,\ 0) = \left(0 + \frac{1}{3}\cdot 8\pi^3\right) - (0) = \frac{8}{3}\pi^3$$

問題 44　⑶

$\boldsymbol{f} = (x,\ -y,\ 1) = (f_x,\ f_y,\ f_z)$ となるように $f(x,\ y,\ z)$ をきめる。

x 成分に注目して，$f = \frac{1}{2}x^2 + \varphi(y,\ z)$

$f_y = \varphi_y(y,\ z) = -y$ より $\varphi(y,\ z) = -\frac{1}{2}y^2 + \psi(z)$ と表せて $f = \frac{1}{2}x^2 - \frac{1}{2}y^2 + \psi(z)$

よって　$f_z = \psi'(z) = 1$，$\psi(z) = z + c$，$f = \frac{1}{2}x^2 - \frac{1}{2}y^2 + z + c$

53 より線積分の値は

$$f = (-1,\ 0,\ 2\pi) - f(1,\ 0,\ 0) = \left(\frac{1}{2} + 2\pi\right) - \frac{1}{2} = 2\pi$$