新版応用数学 解答

3 章 ラプラス変換

P.134〜162 練習

練習 1

$$\int_0^T e^{-st}t^2\,dt = \left[-\frac{1}{s}e^{-st}t^2\right]_0^T + 2\int_0^T \frac{1}{s}e^{-st}t\,dt = -\frac{1}{s}e^{-sT}T^2 + 2\left\{\left[-\frac{1}{s^2}e^{-st}t\right]_0^T + \int_0^T \frac{1}{s^2}e^{-st}\,dt\right\}$$

$$= -\frac{1}{s}e^{-sT}T^2 - \frac{2}{s^2}e^{-sT}T + 2\left[-\frac{e^{-st}}{s^3}\right]_0^T = -\frac{1}{s}e^{-sT}T^2 - \frac{2}{s^2}e^{-sT}T - \frac{2}{s^3}e^{-sT} + \frac{2}{s^3}$$

$$\therefore\ \mathscr{L}[t^2](s) = \lim_{T\to\infty}\int_0^T e^{-st}t^2\,dt = \frac{2}{s^3}$$

練習 2

(1) $\mathscr{L}\left[e^{2t}\right](s) = \dfrac{1}{s-2}$　(2) $\mathscr{L}\left[e^{3t}\right](s) = \dfrac{1}{s-3}$　(3) $\mathscr{L}\left[e^{-t}\right](s) = \dfrac{1}{s+1}$

練習 3

(1) $\mathscr{L}\left[\sin t\right](s) = \dfrac{1}{s^2+1}$　(2) $\mathscr{L}\left[\cos t\right](s) = \dfrac{s}{s^2+1}$　(3) $\mathscr{L}\left[\sin 2t\right](s) = \dfrac{1}{s^2+4}$

練習 4

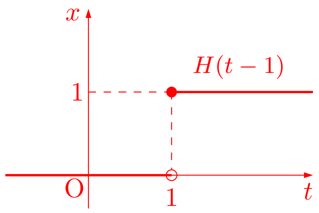
ラプラス変換が存在するための十分条件を満たすならば，正の定数 M，λ があって，$s > \lambda$ なる s について，$\left|\mathscr{L}[f(t)](s)\right| < \dfrac{M}{s-\lambda}$ が成り立つ。すると，$\displaystyle\lim_{s\to\infty}\left|\mathscr{L}[f(t)](s)\right| < \lim_{s\to\infty}\frac{M}{s-\lambda} = 0$ より，$\displaystyle\lim_{s\to\infty}\mathscr{L}[f(t)](s) = 0$

練習 5

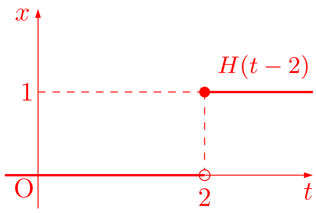
$f(t)$ が，ラプラス変換が存在するための十分条件を満たすならば，$tf(t)$ もそうである。すると同じ論法で，$t^2f(t) = t\cdot tf(t)$ もラプラス変換が存在するための十分条件を満たす。

練習 6

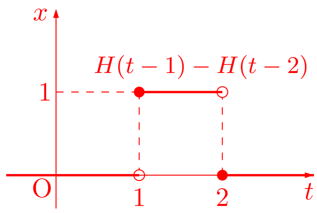
(1)



(2)



(3)



練習 7

(1) $\mathscr{L}\left[H(t-1)\right](s) = \dfrac{e^{-s}}{s}$　(2) $\mathscr{L}\left[H(t-2)\right](s) = \dfrac{e^{-2s}}{s}$　(3) $\mathscr{L}\left[H(t)\right](s) = \dfrac{1}{s}$

練習 8

(1) $\mathscr{L}\left[\delta(t-2)\right](s) = e^{-2s}$　(2) $\mathscr{L}\left[\delta(t-3)\right](s) = e^{-3s}$

(3) デルタ関数の性質より，$2\delta(2t) = 2\cdot\dfrac{1}{2}\delta(t) = \delta(t)$ だから，$\mathscr{L}\left[2\delta(2t)\right](s) = \mathscr{L}\left[\delta(t)\right](s) = 1$

練習 9

(1) $\mathscr{L}\left[2t^2+t\right](s) = \dfrac{4}{s^3} + \dfrac{1}{s^2}$　(2) $\mathscr{L}\left[e^{3t}-e^{-t}\right](s) = \dfrac{1}{s-3} - \dfrac{1}{s+1}$

(3) $2\sin\left(t+\dfrac{\pi}{3}\right) = 2\left(\sin t\cos\dfrac{\pi}{3} + \cos t\sin\dfrac{\pi}{3}\right) = \sin t + \sqrt{3}\cos t$

$$\therefore\ \mathscr{L}\left[2\sin\left(t+\frac{\pi}{3}\right)\right](s) = \mathscr{L}\left[\sin t + \sqrt{3}\cos t\right] = \frac{1}{s^2+1} + \frac{\sqrt{3}\,s}{s^2+1}$$

練習 10

(1) $\mathscr{L}\left[e^t t^2\right](s) = \dfrac{2}{(s-1)^3}$　(2) $\mathscr{L}\left[e^{2t}\sin t\right](s) = \dfrac{1}{(s-2)^2+1}$

(3) $\mathscr{L}\left[e^t\cos t\right](s) = \dfrac{s-1}{(s-1)^2+1}$

練習 11

(1) $\mathscr{L}\big[(t-1)H(t-1)\big](s) = \dfrac{e^{-s}}{s^2}$　　(2) $\mathscr{L}\big[(t-1)^2H(t-1)\big](s) = \dfrac{2e^{-s}}{s^3}$

(3) $\mathscr{L}\big[e^{t-2}H(t-2)\big](s) = \dfrac{e^{-2s}}{s-1}$

練習 12

$$\mathscr{L}\left[\frac{\sin\lambda t}{t}\right](s) = \mathscr{L}\left[\lambda\cdot\frac{\sin\lambda t}{\lambda t}\right](s) = \lambda\cdot\frac{1}{\lambda}\mathrm{Tan}^{-1}\frac{\lambda}{s} = \mathrm{Tan}^{-1}\frac{\lambda}{s}$$

練習 13

$$\mathscr{L}\big[f'''(t)\big](s) = s^2\mathscr{L}\big[f'(t)\big](s) - sf'(+0) - f''(+0) = s^2\Big\{s\mathscr{L}\big[f(t)\big](s) - f(+0)\Big\} - sf'(+0) - f''(+0)$$
$$= s^3\mathscr{L}\big[f(t)\big](s) - s^2f(+0) - sf'(+0) - f''(+0)$$

練習 14

(1) $\mathscr{L}\left[\displaystyle\int_0^t \left(e^t + e^{-t}\right)dt\right](s) = \dfrac{1}{s}\left(\dfrac{1}{s-1} + \dfrac{1}{s+1}\right) = \dfrac{1}{\cancel{s}}\cdot\dfrac{2\cancel{s}}{(s-1)(s+1)} = \dfrac{2}{(s-1)(s+1)}$

(2) $\mathscr{L}\left[\displaystyle\int_0^t (t-1)\,dt\right](s) = \dfrac{1}{s}\left(\dfrac{1}{s^2} - \dfrac{1}{s}\right) = \dfrac{1}{s^3} - \dfrac{1}{s^2}$

(3) $\mathscr{L}\left[\displaystyle\int_0^t \cos 2t\,dt\right](s) = \dfrac{1}{\cancel{s}}\cdot\dfrac{\cancel{s}}{s^2+4} = \dfrac{1}{s^2+4}$

練習 15

(1) $\mathscr{L}\big[te^t\big](s) = -\dfrac{d}{ds}\left(\dfrac{1}{s-1}\right) = \dfrac{1}{(s-1)^2}$　　(2) $\mathscr{L}\big[t\sin t\big](s) = -\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right) = \dfrac{2s}{(s^2+1)^2}$

(3) $\mathscr{L}\big[t\cos t\big](s) = -\dfrac{d}{ds}\left(\dfrac{s}{s^2+1}\right) = -\dfrac{1\cdot(s^2+1) - s\cdot 2s}{(s^2+1)^2} = \dfrac{s^2-1}{(s^2+1)^2}$

練習 16

(1) $\mathscr{L}\big[t^2e^t\big](s) = \dfrac{d^2}{ds^2}\left(\dfrac{1}{s-1}\right) = \dfrac{d}{ds}\left\{-\dfrac{1}{(s-1)^2}\right\} = \dfrac{2}{(s-1)^3}$

(2) $\mathscr{L}\big[t^3e^t\big](s) = -\dfrac{d^3}{ds^3}\left(\dfrac{1}{s-1}\right) = -\dfrac{d}{ds}\left\{\dfrac{d^2}{ds^2}\left(\dfrac{1}{s-1}\right)\right\} = -\dfrac{d}{ds}\left\{\dfrac{2}{(s-1)^3}\right\} = \dfrac{6}{(s-1)^4}$

(3) $\mathscr{L}\big[t^2\sin t\big](s) = \dfrac{d^2}{ds^2}\left(\dfrac{1}{s^2+1}\right) = \dfrac{d}{ds}\left\{-\dfrac{2s}{(s^2+1)^2}\right\} = -\dfrac{2(s^2+1)^2 - 2s\cdot 4s(s^2+1)}{(s^2+1)^4}$

$= -\dfrac{2(s^2+1) - 8s^2}{(s^2+1)^3} = \dfrac{6s^2-2}{(s^2+1)^3}$

練習 17

$$\mathscr{L}\left[\frac{\sin 2t}{t}\right](s) = \int_s^\infty \frac{2}{\sigma^2+4}\,d\sigma = \lim_{S\to\infty}\left[\mathrm{Tan}^{-1}\frac{\sigma}{2}\right]_s^S = \frac{\pi}{2} - \mathrm{Tan}^{-1}\frac{s}{2}$$

ここで，$\theta = \dfrac{\pi}{2} - \mathrm{Tan}^{-1}\dfrac{s}{2}$ とおくと，公式 $\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{1}{\tan\theta}$ より，$\dfrac{s}{2} = \dfrac{1}{\tan\theta}$ である。

すると，$\tan\theta = \dfrac{2}{s}$ より，$\theta = \mathrm{Tan}^{-1}\dfrac{2}{s}$，すなわち，$\mathscr{L}\left[\dfrac{\sin 2t}{t}\right](s) = \mathrm{Tan}^{-1}\dfrac{2}{s}$

練習 18

(1) $\cos t * \cos t = \displaystyle\int_0^t \cos(\tau)\cos(t-\tau)\,d\tau = \dfrac{1}{2}\displaystyle\int_0^t \Big\{\cos t + \cos(2\tau - t)\Big\}\,d\tau = \dfrac{1}{2}\left[\tau\cos t + \dfrac{1}{2}\sin(2\tau - t)\right]_0^t$

$= \dfrac{1}{2}\left\{t\cos t + \dfrac{1}{2}\sin t - \cancel{0} - \dfrac{1}{2}\sin(-t)\right\} = \dfrac{1}{2}(t\cos t + \sin t)$

(2) $e^t * t = \displaystyle\int_0^t (t-\tau)e^\tau\,d\tau = \Big[(t-\tau)e^\tau\Big]_0^t + \displaystyle\int_0^t e^\tau\,d\tau = 0 - te^0 + \Big[e^\tau\Big]_0^t = -t + e^t - e^0 = e^t - t - 1$

練習 19

(1) $\sin t * t = \displaystyle\int_0^t (t-\tau)\sin\tau\,d\tau = \Big[-(t-\tau)\cos\tau\Big]_0^t - \displaystyle\int_0^t \cos\tau\,d\tau = 0 + t - \Big[\sin\tau\Big]_0^t = t - \sin t$

(2) $\mathscr{L}\big[t - \sin t\big](s) = \mathscr{L}\big[\sin t * t\big](s) = \mathscr{L}\big[\sin t\big](s)\cdot\mathscr{L}\big[\,t\,\big](s) = \dfrac{1}{(s^2+1)s^2}$

練習 20

(1) $\mathscr{L}^{-1}\left[\dfrac{1}{s^2}\right](t)=t$　(2) $\mathscr{L}^{-1}\left[\dfrac{2}{s^3}\right](t)=t^2$　(3) $\mathscr{L}^{-1}\left[\dfrac{1}{s-2}\right](t)=e^{2t}$

(4) $\mathscr{L}^{-1}\left[\dfrac{1}{s+1}\right](t)=e^{-t}$　(5) $\mathscr{L}^{-1}\left[\dfrac{s}{s^2+1}\right](t)=\cos t$　(5) $\mathscr{L}^{-1}\left[\dfrac{1}{(s-1)^2}\right](t)=te^{t}$

練習 21

(1) $\dfrac{1}{s^2-3s+2}=\dfrac{1}{s-2}-\dfrac{1}{s-1}$ だから，

$$\mathscr{L}^{-1}\left[\frac{1}{s^2-3s+2}\right](t)=\mathscr{L}^{-1}\left[\frac{1}{s-2}-\frac{1}{s-1}\right](t)=e^{2t}-e^{t}$$

(2) $\dfrac{2s}{s^2-1}=\dfrac{1}{s-1}+\dfrac{1}{s+1}$ だから $\mathscr{L}^{-1}\left[\dfrac{2s}{s^2-1}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s-1}+\dfrac{1}{s+1}\right](t)=e^{t}+e^{-t}$

(3) $\dfrac{1}{s^2-2}=\dfrac{1}{2\sqrt{2}}\left(\dfrac{1}{s-\sqrt{2}}-\dfrac{1}{s+\sqrt{2}}\right)$ だから

$$\mathscr{L}^{-1}\left[\frac{1}{s^2-2}\right](t)=\mathscr{L}^{-1}\left[\frac{1}{2\sqrt{2}}\left(\frac{1}{s-\sqrt{2}}-\frac{1}{s+\sqrt{2}}\right)\right](t)=\frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}t}+e^{-\sqrt{2}t}\right)$$

練習 22

(1) $\dfrac{s}{(s+1)^2}=\dfrac{1}{s+1}-\dfrac{1}{(s+1)^2}$ だから

$$\mathscr{L}^{-1}\left[\frac{s}{(s+1)^2}\right](t)=\mathscr{L}^{-1}\left[\frac{1}{s+1}-\frac{1}{(s+1)^2}\right](t)=e^{-t}-te^{-t}=(1-t)e^{-t}$$

(2) $\dfrac{s^2+s-1}{(s-1)s^2}=\dfrac{1}{s-1}+\dfrac{1}{s^2}$ だから，$\mathscr{L}^{-1}\left[\dfrac{s^2+s-1}{(s-1)s^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s-1}+\dfrac{1}{s^2}\right](t)=e^{t}+t$

(3) $\dfrac{1}{s^2(s^2+1)}=\dfrac{1}{s^2}-\dfrac{1}{s^2+1}$ だから，$\mathscr{L}^{-1}\left[\dfrac{1}{s^2(s^2+1)}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s^2}-\dfrac{1}{s^2+1}\right](t)=t-\sin t$

練習 23

(1) $\mathscr{L}^{-1}\left[\dfrac{1}{s^2-4s+5}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{(s-2)^2+1}\right](t)=e^{2t}\sin t$

(2) $\mathscr{L}^{-1}\left[\dfrac{s+1}{s^2+2s+5}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s+1}{(s+1)^2+4}\right](t)=e^{-t}\cos 2t$

(3) $\mathscr{L}^{-1}\left[\dfrac{s}{s^2+4s+5}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s+2}{(s+2)^2+1}-\dfrac{2}{(s+2)^2+1}\right](t)=e^{-2t}\cos t-2e^{-2t}\sin t$

$=e^{-2t}(\cos t-2\sin t)$

練習 24

(1) $\mathscr{L}^{-1}\left[\dfrac{2s}{(s^2+4)^2}\right](t)=\mathscr{L}^{-1}\left[-\dfrac{d}{ds}\left(\dfrac{1}{2}\cdot\dfrac{2}{s^2+4}\right)\right](t)=\dfrac{1}{2}t\sin 2t$

(2) $\mathscr{L}^{-1}\left[\dfrac{s^2-1}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[-\dfrac{d}{ds}\left(\dfrac{s}{s^2+1}\right)\right](t)=t\cos t$

(3) $\mathscr{L}^{-1}\left[\dfrac{s^2-4}{(s^2+4)^2}\right](t)=\mathscr{L}^{-1}\left[-\dfrac{d}{ds}\left(\dfrac{s}{s^2+4}\right)\right](t)=t\cos 2t$

P.163〜164 節末問題

1

(1) $\mathscr{L}\left[\dfrac{1}{e^t}\right](s)=\mathscr{L}\left[e^{-t}\right](s)=\dfrac{1}{s+1}$　(2) $\mathscr{L}\left[e^{\frac{t}{2}}\right](s)=\dfrac{1}{s-\dfrac{1}{2}}=\dfrac{2}{2s-1}$

(3) $\mathscr{L}\left[\sin\dfrac{t}{2}\right](s)=\dfrac{\dfrac{1}{2}}{s^2+\dfrac{1}{4}}=\dfrac{2}{4s^2+1}$　(4) $\mathscr{L}\left[\cos\dfrac{t}{3}\right](s)=\dfrac{s}{s^2+\dfrac{1}{9}}=\dfrac{9s}{9s^2+1}$

(5) $\mathscr{L}\left[\sin t\cos t\right](s)=\mathscr{L}\left[\dfrac{1}{2}\sin 2t\right](s)=\dfrac{1}{\cancel{2}}\cdot\dfrac{\cancel{2}}{s^2+4}=\dfrac{1}{s^2+4}$

(6) $\mathscr{L}\left[\cos^2 t\right](s)=\mathscr{L}\left[\dfrac{1}{2}(1+\cos 2t)\right](s)=\dfrac{1}{2}\left(\dfrac{1}{s}+\dfrac{s}{s^2+4}\right)$

2

(1) $\mathscr{L}\left[H(t-3)\right](s)=\dfrac{e^{-3s}}{s}$

(2) $\mathscr{L}\left[\delta(2t)\right](s)=\mathscr{L}\left[\dfrac{1}{2}\delta(t)\right](s)=\dfrac{1}{2}$ 　　別解： $\mathscr{L}\left[\delta(2t)\right](s)=\dfrac{1}{2}\mathscr{L}\left[\delta(t)\right]\left(\dfrac{s}{2}\right)=\dfrac{1}{2}$

(3) $\mathscr{L}\left[\delta(2t-4)\right](s)=\mathscr{L}\left[\delta(2(t-2))\right](s)=\mathscr{L}\left[\dfrac{1}{2}\delta(t-2)\right](s)=\dfrac{1}{2}e^{-2s}$

別解： $\mathscr{L}\left[\delta(2t-4)\right](s)=\dfrac{1}{2}\mathscr{L}\left[\delta(t-4)\right]\left(\dfrac{s}{2}\right)=\dfrac{1}{2}e^{-4\cdot\frac{s}{2}}=\dfrac{1}{2}e^{-2s}$

3

(1) $\mathscr{L}\left[1+2t-t^2\right](s)=\dfrac{1}{s}+\dfrac{2}{s^2}-\dfrac{2}{s^3}$ 　　(2) $\mathscr{L}\left[e^{2t}-1\right](s)=\dfrac{1}{s-2}-\dfrac{1}{s}$

(3) $\mathscr{L}\left[\left(e^t-e^{-t}\right)^2\right](s)=\mathscr{L}\left[e^{2t}-2+e^{-2t}\right](s)=\dfrac{1}{s-2}-\dfrac{2}{s}+\dfrac{1}{s+2}$

(4) $\sqrt{2}\cos\left(2t+\dfrac{\pi}{4}\right)=\sqrt{2}\left(\cos 2t\cos\dfrac{\pi}{4}-\sin 2t\sin\dfrac{\pi}{4}\right)=\sqrt{2}\left(\cos 2t\dfrac{1}{\sqrt{2}}-\sin 2t\dfrac{1}{\sqrt{2}}\right)$

$=\cos 2t-\sin 2t$ より

$\mathscr{L}\left[\sqrt{2}\cos\left(2t+\dfrac{\pi}{4}\right)\right](s)=\mathscr{L}\left[\cos 2t-\sin 2t\right](s)=\dfrac{s}{s^2+4}-\dfrac{2}{s^2+4}$

(5) $\mathscr{L}\left[e^{2t}\cos\sqrt{3}\,t\right](s)=\dfrac{s-2}{(s-2)^2+3}$ 　　(6) $\mathscr{L}\left[(2t+1)e^{-t}\right](s)=\dfrac{2}{(s+1)^2}+\dfrac{1}{s+1}$

(7) $\mathscr{L}\left[H(t-1)\sin(t-1)\right](s)=\dfrac{e^{-s}}{s^2+1}$

(8) $\mathscr{L}\left[H(t-\pi)\cos t\right](s)=\mathscr{L}\left[-H(t-\pi)\cos(t-\pi)\right](s)=-\dfrac{se^{-\pi s}}{s^2+1}$

4

(1) $\mathscr{L}\left[\displaystyle\int_0^t te^t\,dt\right](s)=\dfrac{1}{s(s-1)^2}$ 　　(2) $\mathscr{L}\left[\displaystyle\int_0^t e^t\sin t\,dt\right](s)=\dfrac{1}{s\left\{(s-1)^2+1\right\}}$

(3) $\mathscr{L}\left[te^t\sin t\right](s)=-\left\{\dfrac{1}{(s-1)^2+1}\right\}'=\dfrac{2(s-1)}{\left\{(s-1)^2+1\right\}^2}$

(4) $\mathscr{L}\left[t\sin^2 t\right](s)=\mathscr{L}\left[t\cdot\dfrac{1-\cos 2t}{2}\right](s)=-\dfrac{1}{2}\left\{\dfrac{1}{s}-\dfrac{s}{s^2+4}\right\}'=-\dfrac{1}{2}\left\{-\dfrac{1}{s^2}-\dfrac{s^2+4-s\cdot 2s}{(s^2+4)^2}\right\}$

$=\dfrac{1}{2}\left\{\dfrac{1}{s^2}-\dfrac{s^2-4}{(s^2+4)^2}\right\}$

5

(1) $\mathscr{L}^{-1}\left[\dfrac{s-2}{s^2+4}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s}{s^2+4}-\dfrac{2}{s^2+4}\right](t)=\cos 2t-\sin 2t$

(2) $\mathscr{L}^{-1}\left[\dfrac{s-3}{(s-2)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s-2}-\dfrac{1}{(s-2)^2}\right](t)=e^{2t}-te^{2t}=(1-t)e^{2t}$

(3) $\mathscr{L}^{-1}\left[\dfrac{4}{s^2-6s+5}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s-5}-\dfrac{1}{s-1}\right](t)=e^{5t}-e^t$

(4) $\mathscr{L}^{-1}\left[\dfrac{s+1}{s^3-s^2+s-1}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s-1}-\dfrac{s}{s^2+1}\right](t)=e^t-\cos t$

(5) $\mathscr{L}^{-1}\left[\dfrac{s^2-2}{s^3-3s^2+2s}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s-1}+\dfrac{1}{s-2}-\dfrac{1}{s}\right](t)=e^t+e^{2t}-1$

(6) $\mathscr{L}^{-1}\left[\dfrac{4}{s^4-1}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s-1}-\dfrac{1}{s+1}-\dfrac{2}{s^2+1}\right](t)=e^t-e^{-t}-2\sin t$

6

(1) $\mathscr{L}^{-1}\left[\dfrac{1}{s^3-s^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s-1}-\dfrac{1}{s}-\dfrac{1}{s^2}\right](t)=e^t-1-t$

(2) $\mathscr{L}^{-1}\left[\dfrac{3s-1}{s^3-s^2-s+1}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{(s-1)^2}+\dfrac{1}{s-1}-\dfrac{1}{s+1}\right](t)=(t+1)e^t-e^{-t}$

(3) $\mathscr{L}^{-1}\left[\dfrac{2s-2}{s^4-2s^3+2s^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{(s-1)^2+1}-\dfrac{1}{s^2}\right](t)=e^t\sin t-t$

(4) $\mathscr{L}^{-1}\left[\dfrac{(s+1)^2}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s^2+1}+\dfrac{2s}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s^2+1}-\left(\dfrac{1}{s^2+1}\right)'\right](t)$

$=(1-t)\sin t$

(5) $\mathscr{L}^{-1}\left[\dfrac{2}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{s^2+1+1-s^2}{(s^2+1)^2}\right](t)=\mathscr{L}^{-1}\left[\dfrac{1}{s^2+1}-\dfrac{s^2-1}{(s^2+1)^2}\right](t)$

$=\sin t-t\cos t$

(6) $\mathscr{L}^{-1}\left[\dfrac{2s+2}{(s^2+2s+2)^2}\right](t)=\mathscr{L}^{-1}\left[-\left(\dfrac{1}{s^2+2s+2}\right)'\right](t)=\mathscr{L}^{-1}\left[-\left(\dfrac{1}{(s+1)^2+1}\right)'\right](t)$

$=te^{-t}\sin t$

7

(1) $\mathscr{L}^{-1}[1](t)=\delta(t)$　　(2) $\mathscr{L}^{-1}[e^{-s}](t)=\delta(t-1)$　　(3) $\mathscr{L}^{-1}\left[\dfrac{e^{-s}}{s}\right](t)=H(t-1)$

8

証明） $u=t-\tau$ とおくと，$du=-d\tau$，$\begin{array}{c|ccc}\tau & 0 & \to & t\\ \hline u & t & \to & 0\end{array}$ だから

$(f*g)(t)=\displaystyle\int_0^t f(\tau)g(t-\tau)\,d\tau=-\int_t^0 f(t-u)g(u)\,du=\int_0^t g(u)f(t-u)\,du=(g*f)(t)$　　//

9

(1) $\displaystyle\int_0^T e^{-t}t^x\,dt=\left[-e^{-t}t^x\right]_0^T-\int_0^T(-e^{-t})xt^{x-1}\,dt=-e^{-T}T^x+x\int_0^T e^{-t}t^{x-1}\,dt$ より

$\displaystyle\Gamma(x+1)=\int_0^\infty e^{-t}t^{(x+1)-1}\,dt=\lim_{T\to\infty}\int_0^T e^{-t}t^x\,dt=\lim_{T\to\infty}\left\{-e^{-T}T^x+x\int_0^T e^{-t}t^{x-1}\,dt\right\}$

$\displaystyle=x\int_0^\infty e^{-t}t^{x-1}\,dt=x\Gamma(x)$

(2) ① $n=1$ のとき

(1) より，$\displaystyle\Gamma(1+1)=1\cdot\Gamma(1)=\int_0^\infty e^{-t}t^{1-1}\,dt=\lim_{T\to\infty}\int_0^T e^{-t}\,dt=\lim_{T\to\infty}\left[-e^{-t}\right]_0^T=1=1!$　だから，成り立つ。

② $n=k$ のとき成り立つと仮定すると

(1) より，$\Gamma\big((k+1)+1\big)=(k+1)\Gamma(k+1)=(k+1)\cdot k!=(k+1)!$　だから，$n=k+1$ のときも成り立つ。

①，②から，数学的帰納法により，任意の自然数 n について，$\Gamma(n+1)=n!$ が成り立つ。

(3) $u=\sqrt{st}$ とおくと，$u^2=st$，$\dfrac{2}{\sqrt{s}}\,du=\dfrac{1}{\sqrt{t}}\,dt$，$\begin{array}{c|ccc}t & 0 & \to & \infty\\ \hline u & 0 & \to & \infty\end{array}$ だから

$\displaystyle\mathscr{L}\left[\frac{1}{\sqrt{t}}\right](s)=\int_0^\infty e^{-st}\frac{1}{\sqrt{t}}\,dt=\frac{2}{\sqrt{s}}\int_0^\infty e^{-u^2}\,du=\frac{2}{\sqrt{s}}\cdot\frac{\sqrt{\pi}}{2}=\sqrt{\frac{\pi}{s}}$

(4) (3) より，$\displaystyle\Gamma\left(\frac{1}{2}\right)=\int_0^\infty e^{-t}t^{\frac{1}{2}-1}\,dt=\int_0^\infty e^{-1\cdot t}\frac{1}{\sqrt{t}}\,dt=\mathscr{L}\left[\frac{1}{\sqrt{t}}\right](1)=\sqrt{\pi}$

(5) $u=st$ とおくと，$s>0$ より，$\dfrac{1}{s}\,du=dt$，$\begin{array}{c|ccc}t & 0 & \to & \infty\\ \hline u & 0 & \to & \infty\end{array}$ だから，$r>-1$ ならば

$\displaystyle\mathscr{L}[t^r](s)=\int_0^\infty e^{-st}t^r\,dt=\int_0^\infty e^{-u}\cdot\frac{u^r}{s^r}\cdot\frac{1}{s}\,du=\frac{1}{s^{r+1}}\int_0^\infty e^{-u}u^{(r+1)-1}\,du=\frac{\Gamma(r+1)}{s^{r+1}}$

10

(1) $\mathscr{L}\left[\dfrac{\sin t}{t}\right](s)=\mathrm{Tan}^{-1}\dfrac{1}{s}$ より

$\displaystyle\int_0^\infty\frac{1}{t}e^{-t}\sin t\,dt=\int_0^\infty e^{-1\cdot t}\frac{\sin t}{t}\,dt=\mathscr{L}\left[\frac{\sin t}{t}\right](1)=\mathrm{Tan}^{-1}1=\frac{\pi}{4}$

(2) $\left(\dfrac{1}{s^2+1}\right)'=-\dfrac{2s}{(s^2+1)^2}$

$\left(\dfrac{1}{s^2+1}\right)''=-\dfrac{2(s^2+1)^2-2s\cdot2(s^2+1)\cdot2s}{(s^2+1)^4}=-\dfrac{2(s^2+1)-8s^2}{(s^2+1)^3}=\dfrac{6s^2-2}{(s^2+1)^3}$

$$\left(\frac{1}{s^2+1}\right)''' = \frac{12s(s^2+1)^3-(6s^2-2)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6} = \frac{12s(s^2+1)-6s(6s^2-2)}{(s^2+1)^4}$$
$$= -\frac{24s(s^2-1)}{(s^2+1)^4}$$

だから，$\mathscr{L}\big[t^n f(t)\big](s) = (-1)^n \dfrac{d^n}{ds^n}\Big\{\mathscr{L}\big[f(t)\big](s)\Big\}$ より

$$\int_0^\infty t^3 e^{-t}\sin t\,dt = \int_0^\infty e^{-1\cdot t} t^3 \sin t\,dt = \mathscr{L}\big[t^3\sin t\big](1) = (-1)^3\Big\{\mathscr{L}\big[\sin t\big]\Big\}'''(1)$$
$$= -\left(\frac{1}{s^2+1}\right)'''(1) = 0$$

P.166〜179 練習　　以下，特に断らなくても $X = \mathscr{L}\big[x(t)\big](s)$ とする。

練習 1

(1)　微分方程式をラプラス変換すると，$sX - 0 - X = \dfrac{1}{s-1}$　　$\therefore X = \dfrac{1}{(s-1)^2}$

両辺の逆ラプラス変換を考えると，$x = te^t$

(2)　微分方程式をラプラス変換すると，$sX - 1 + X = \dfrac{2}{s}$　　$\therefore X = \dfrac{s+2}{s(s+1)} = \dfrac{2}{s} - \dfrac{1}{s+1}$

両辺の逆ラプラス変換を考えると，$x = 2 - e^{-t}$

練習 2

(1)　$C = x(0)$ とおき，微分方程式をラプラス変換すると，$sX - C - X = \dfrac{1}{s-1}$　　$\therefore X = \dfrac{1}{(s-1)^2} + \dfrac{C}{s-1}$

両辺の逆ラプラス変換を考えると，$x = (t+C)e^t$

(2)　$C' = x(0)$ とおき，微分方程式をラプラス変換すると，$sX - C' + X = \dfrac{2}{s}$　　$\therefore X = \dfrac{C's+2}{s(s+1)}$

右辺を部分分数分解すると，$X = \dfrac{2}{s} + \dfrac{C'-2}{s+1}$

$C = C' - 2$ とし，両辺の逆ラプラス変換を考えると，$x = 2 + Ce^{-t}$

練習 3

(1)　微分方程式をラプラス変換すると，$s^2X - s\cdot 0 - 0 + X = \dfrac{2}{s+1}$　　$\therefore X = \dfrac{2}{(s+1)(s^2+1)}$

右辺を部分分数分解すると，$X = \dfrac{1}{s+1} + \dfrac{1}{s^2+1} - \dfrac{s}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x = e^{-t} + \sin t - \cos t$

(2)　微分方程式をラプラス変換すると，$s^2X - s\cdot 1 - 0 + X = \dfrac{2}{s+1}$

$$\therefore X = \frac{s}{s^2+1} + \frac{2}{(s+1)(s^2+1)} = \cancel{\frac{s}{s^2+1}} + \frac{1}{s+1} + \frac{1}{s^2+1} - \cancel{\frac{s}{s^2+1}} = \frac{1}{s+1} + \frac{1}{s^2+1}$$

両辺の逆ラプラス変換を考えると，$x = e^{-t} + \sin t$

練習 4

(1)　微分方程式をラプラス変換すると，$s^2X + sX + \dfrac{1}{s} = 0$　　$\therefore X = -\dfrac{1}{s^2(s+1)} = \dfrac{1}{s} - \dfrac{1}{s^2} - \dfrac{1}{s+1}$

両辺の逆ラプラス変換を考えると，$x = 1 - t - e^{-t}$

(2)　微分方程式をラプラス変換すると，$s^2X - s\cdot 1 - 3 - 3(sX - 1) + 2X = \dfrac{1}{s-1}$

$$\therefore (s^2-3s+2)X = s + \frac{1}{s-1} = \frac{s^2-s+1}{s-1} \qquad \therefore X = \frac{s^2-s+1}{(s-2)(s-1)^2} = \frac{3}{s-2} - \frac{2}{s-1} - \frac{1}{(s-1)^2}$$

両辺の逆ラプラス変換を考えると，$x = 3e^{2t} - 2e^t - te^t$

(3)　微分方程式をラプラス変換すると，$s^2X + X = \dfrac{2}{(s-1)^2} + \dfrac{2}{s-1}$　　$\therefore X = \dfrac{2s}{(s-1)^2(s^2+1)}$

右辺を部分分数分解すると，$X = \dfrac{1}{(s-1)^2} - \dfrac{1}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x = te^t - \sin t$

練習 5

(1) $\alpha = x'(0)$ とおき，微分方程式をラプラス変換すると，$s^2X - \alpha + X = \dfrac{1}{s}$ $\quad \therefore X = \dfrac{\alpha s + 1}{s(s^2+1)}$

右辺を部分分数分解すると，$X = \dfrac{1}{s} - \dfrac{s}{s^2+1} + \dfrac{\alpha}{s^2+1}$

両辺の逆ラプラス変換を考えると，$x = 1 - \cos t + \alpha \sin t$

境界条件 $x\left(\dfrac{\pi}{2}\right) = 1$ より，$1 = x\left(\dfrac{\pi}{2}\right) = 1 - \cos\dfrac{\pi}{2} + \alpha\sin\dfrac{\pi}{2} = 1 + \alpha$ $\quad \therefore \alpha = 0$

よって，$x = 1 - \cos t$

(2) 途中までは (1) と同じである。境界条件 $x\left(\dfrac{\pi}{2}\right) = 2$ より，$2 = x\left(\dfrac{\pi}{2}\right) = 1 + \alpha$ $\quad \therefore \alpha = 1$

よって，$x = 1 - \cos t + \sin t$

練習 6

$\alpha = x'(0)$ とおき，微分方程式をラプラス変換すると，$s^2X - \alpha - X = \dfrac{1}{s}$ $\quad \therefore X = \dfrac{\alpha s + 1}{(s-1)(s+1)s}$

右辺を部分分数分解すると，$X = \dfrac{\alpha+1}{2} \cdot \dfrac{1}{s-1} + \dfrac{1-\alpha}{2} \cdot \dfrac{1}{s+1} - \dfrac{1}{s}$

両辺の逆ラプラス変換を考えると，$x = \dfrac{\alpha+1}{2}e^t + \dfrac{1-\alpha}{2}e^{-t} - 1 = \dfrac{\alpha}{2}\left(e^t - e^{-t}\right) + \dfrac{1}{2}\left(e^t + e^{-t} - 2\right)$

境界条件 $x(1) = e - 1$ より，$x(1) = \dfrac{\alpha}{2}\left(e - e^{-1}\right) + \dfrac{1}{2}\left(e + e^{-1} - 2\right) = e - 1$

$\therefore \alpha\left(e - e^{-1}\right) = 2(e-1) - \left(e + e^{-1} - 2\right) = e - e^{-1}$ $\quad \therefore \alpha = 1$

よって，$x = e^t - 1$

練習 7

この回路において，コンデンサの電荷 $q(t)$ が満たす微分方程式は，$q' + \dfrac{1}{RC}q = \dfrac{E}{R}$

$Q(s) = \mathscr{L}[q(t)](s)$ とし，ラプラス変換すると，$sQ + \dfrac{1}{RC}Q = \dfrac{E}{Rs}$ $\quad \therefore Q = \dfrac{CE}{s(RCs+1)}$

右辺を部分分数分解すると，$Q = \dfrac{CE}{s} - \dfrac{RC^2E}{RCs+1} = \dfrac{CE}{s} - \dfrac{CE}{s + \dfrac{1}{RC}}$

両辺のラプラス変換を考えると，$q = CE\left(1 - e^{-\frac{t}{RC}}\right)$

練習 8

質点 P の運動方程式は，$2x'' + 2x' + x = 0$

ラプラス変換すると，$2\left(s^2X - s + \dfrac{1}{2}\right) + 2(sX - 1) + X = 0$

$$\therefore X = \frac{2s+1}{2s^2+2s+1} = \frac{s + \dfrac{1}{2}}{\left(s + \dfrac{1}{2}\right)^2 + \dfrac{1}{4}}$$

逆ラプラス変換を考えると，$x = e^{-\frac{t}{2}}\cos\dfrac{t}{2}$

練習 9

$c_1 = x'(0)$，$c_3 = x'''(0)$ とおく。弾性曲線方程式 $\dfrac{d^4x}{dt^4} = -\dfrac{w}{EI}$ をラプラス変換すると

$s^4X - c_1s^2 - c_3 = -\dfrac{w}{EIs}$ $\quad \therefore X = -\dfrac{w}{EIs^5} + \dfrac{c_3}{s^4} + \dfrac{c_1}{s^2}$

両辺の逆ラプラス変換を考えると，$x = -\dfrac{w}{24EI}t^4 + \dfrac{c_3}{6}t^3 + c_1t$ $\quad \therefore x' = -\dfrac{w}{6EI}t^3 + \dfrac{c_3}{2}t^2 + c_1$

境界条件 $x(\ell) = x'(\ell) = 0$ より

$x(\ell) = -\dfrac{w}{24EI}\ell^4 + \dfrac{c_3}{6}\ell^3 + c_1\ell = 0,\quad x'(\ell) = -\dfrac{w}{6EI}\ell^3 + \dfrac{c_3}{2}\ell^2 + c_1 = 0$

$\therefore c_1 = -\dfrac{w\ell^3}{48EI}$，$c_3 = \dfrac{3w\ell}{8EI}$ $\quad \therefore x = -\dfrac{w}{24EI}t^4 + \dfrac{w\ell}{16EI}t^3 - \dfrac{w\ell^3}{48EI}t$

$\therefore x = -\dfrac{w}{48EI}t(t-\ell)^2(2t+\ell)$

練習 10

(1) 積分方程式を $f(t)*\sin t = t^2$ と書き直す。$F(s)=\mathscr{L}[f(t)](s)$ として，両辺のラプラス変換を考えると

$F(s)\cdot\dfrac{1}{s^2+1}=\dfrac{2}{s^3}\qquad \therefore\ F(s)=\dfrac{2}{s}+\dfrac{2}{s^3}$

両辺の逆ラプラス変換を考えると，$f(t)=2+t^2$

(2) 積分方程式を $f(t)*e^t=t$ と書き直す。$F(s)=\mathscr{L}[f(t)](s)$ として，両辺のラプラス変換を考えると

$F(s)\cdot\dfrac{1}{s-1}=\dfrac{1}{s^2}\qquad \therefore\ F(s)=\dfrac{1}{s}-\dfrac{1}{s^2}$

両辺の逆ラプラス変換を考えると，$f(t)=1-t$

P.179〜180 節末問題

1

(1) $sX+X=\dfrac{2}{s^2+1}\qquad \therefore\ X=\dfrac{2}{(s+1)(s^2+1)}=\dfrac{1}{s+1}+\dfrac{1}{s^2+1}-\dfrac{s}{s^2+1}$

$\therefore\ x=e^{-t}+\sin t-\cos t$

(2) $s^2X-1+2sX+X=\dfrac{1}{s}\qquad \therefore\ (s+1)^2X=1+\dfrac{1}{s}=\dfrac{s+1}{s}\qquad \therefore\ X=\dfrac{1}{s(s+1)}=\dfrac{1}{s}-\dfrac{1}{s+1}$

$\therefore\ x=1-e^{-t}$

(3) $s^2X+X=\dfrac{2s}{s^2+1}\qquad \therefore\ (s^2+1)X=\dfrac{2s}{s^2+1}\qquad \therefore\ X=\dfrac{2s}{(s^2+1)^2}=-\left(\dfrac{1}{s^2+1}\right)'$

$\therefore\ x=t\sin t$

(4) $s^2X-2-2sX+X=\dfrac{2}{s^2+1}\qquad \therefore\ (s^2-2s+1)X=2+\dfrac{2}{s^2+1}=\dfrac{2s^2+4}{s^2+1}$

$\therefore\ X=\dfrac{2s^2+4}{(s^2+1)(s-1)^2}=\dfrac{s}{s^2+1}+\dfrac{3}{(s-1)^2}-\dfrac{1}{s-1}\qquad \therefore\ x=\cos t+(3t-1)e^t$

2

(1) $\alpha=x'(0)$ とおくと，$s^2X-\alpha+4X=\dfrac{4}{s^2}\qquad \therefore\ (s^2+4)X=\alpha+\dfrac{4}{s^2}=\dfrac{\alpha s^2+4}{s^2}$

$\therefore\ X=\dfrac{\alpha s^2+4}{s^2(s^2+4)}=\dfrac{1}{s^2}+\dfrac{\alpha-1}{s^2+4}=\dfrac{1}{s^2}+\dfrac{\alpha-1}{2}\cdot\dfrac{2}{s^2+4}\qquad \therefore\ x=t+\dfrac{\alpha-1}{2}\sin 2t$

境界条件 $x\left(\dfrac{\pi}{4}\right)=0$ より，$x\left(\dfrac{\pi}{4}\right)=\dfrac{\pi}{4}+\dfrac{\alpha-1}{2}\sin\dfrac{\pi}{2}=0\qquad \therefore\ \dfrac{\alpha-1}{2}=-\dfrac{\pi}{4}$

$\therefore\ x=t-\dfrac{\pi}{4}\sin 2t$

(2) $\alpha=x'(0)$ とおくと，$s^2X-\alpha+sX=\dfrac{1}{s}\qquad \therefore\ (s^2+s)X=\alpha+\dfrac{1}{s}=\dfrac{\alpha s+1}{s}$

$\therefore\ X=\dfrac{\alpha s+1}{s^2(s+1)}=\dfrac{\alpha-1}{s}+\dfrac{1}{s^2}+\dfrac{1-\alpha}{s+1}$

$\therefore\ x=\alpha-1+t+(1-\alpha)e^{-t}\qquad \therefore\ x'=1-(1-\alpha)e^{-t}$

境界条件 $x(1)+x'(1)=0$ より，$x(1)+x'(1)=\left\{\alpha+(1-\alpha)e^{-1}\right\}+\left\{1-(1-\alpha)e^{-1}\right\}=\alpha+1=0$

$\therefore\ \alpha=-1\qquad \therefore\ x=-2+t+2e^{-t}$

(3) $\alpha=x'(0)$ とおくと，$s^2X-\alpha+sX-2X=\dfrac{1}{s}-\dfrac{2}{s^2}\qquad \therefore\ (s^2+s-2)X=\dfrac{\alpha s^2+s-2}{s^2}$

$\therefore\ X=\dfrac{\alpha x^2+s-2}{s^2(s-1)(s+2)}=\dfrac{1}{s^2}+\dfrac{\alpha-1}{3}\cdot\dfrac{1}{s-1}-\dfrac{\alpha-1}{3}\cdot\dfrac{1}{s+2}$

$\therefore\ x=t+\dfrac{\alpha-1}{3}e^t-\dfrac{\alpha-1}{3}e^{-2t}\qquad \therefore\ x'=1+\dfrac{\alpha-1}{3}e^t+\dfrac{2(\alpha-1)}{3}e^{-2t}$

境界条件 $x'(1)-x(1)=3e^{-2}$ より

$x'(1)-x(1)=\left\{1+\cancel{\dfrac{\alpha-1}{3}e}+\dfrac{2(\alpha-1)}{3}e^{-2}\right\}-\left\{\cancel{1+\dfrac{\alpha-1}{3}e}-\dfrac{\alpha-1}{3}e^{-2}\right\}=(\alpha-1)e^{-2}=3e^{-2}$

$\therefore\ \alpha-1=3\qquad \therefore\ x=t+e^t-e^{-2t}$

(4) $\alpha=x'(0)$ とおくと，$s^2X-\alpha-3sX+2X=\dfrac{6}{s+1}\qquad \therefore\ (s^2-3s+2)X=\dfrac{\alpha s+\alpha+6}{s+1}$

$\therefore\ X=\dfrac{\alpha s+\alpha+6}{(s-2)(s-1)(s+1)}=\dfrac{\alpha+2}{s-2}-\dfrac{\alpha+3}{s-1}+\dfrac{1}{s+1}$

$\therefore x = (\alpha+2)e^{2t} - (\alpha+3)e^t + e^{-t} \qquad \therefore x' = 2(\alpha+2)e^{2t} - (\alpha+3)e^t - e^{-t}$

境界条件 $x'(1) + x(1) = 6e^2 - 6e$ より

$x'(1) + x(1) = \left\{2(\alpha+2)e^2 - (\alpha+3)e - \cancel{e^{-1}}\right\} + \left\{(\alpha+2)e^2 - (\alpha+3)e + \cancel{e^{-1}}\right\} = 3(\alpha+2)e^2 - 2(\alpha+3)e$

$= 6e^2 - 6e$

$\therefore \alpha e(3e-2) = 0 \qquad \therefore \alpha = 0 \qquad \therefore x = 2e^{2t} - 3e^t - e^{-t}$

3

$Q(s) = \mathscr{L}[q(t)](s)$ とする。電荷が満たす微分方程式は，$Rq' + \dfrac{1}{C}q = e^{-t} \qquad \therefore RCq' + q = Ce^{-t}$

$$\therefore RCsQ + Q = \frac{C}{s+1}$$

$$\therefore Q = \frac{C}{(s+1)(RCs+1)} = \frac{C}{RC-1}\left(\frac{RC}{RCs+1} - \frac{1}{s+1}\right) = \frac{C}{RC-1}\left(\frac{1}{s+\dfrac{1}{RC}} - \frac{1}{s+1}\right)$$

$$\therefore q = \frac{C}{RC-1}\left(e^{-\frac{t}{RC}} - e^{-t}\right)$$

4

(1) $c_1 = x'(0)$ とおくと，$s^2X - c_1 + 2asX + (a^2+4)X = 0 \qquad \therefore (s^2 + 2as + a^2 + 4)X = c_1$

$$\therefore X = \frac{c_1}{(s+a)^2+4} = \frac{c_1}{2}\cdot\frac{2}{(s+a)^2+4} \qquad \therefore x = \frac{1}{2}c_1e^{-at}\sin 2t$$

境界条件 $x\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}$ より，$x\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}c_1e^{-\frac{\pi a}{4}}\sin\dfrac{\pi}{2} = \dfrac{1}{2}c_1e^{-\frac{\pi a}{4}} = \dfrac{1}{2} \qquad \therefore c_1 = e^{\frac{\pi a}{4}}$

$$\therefore x = \frac{1}{2}e^{-a\left(t-\frac{\pi}{4}\right)}\sin 2t$$

(2) $c = x(0)$ とおくと，$s^2X - cs + 2a(sX - c) + (a^2+4)X = 0 \qquad \therefore (s^2 + 2as + a^2 + 4)X = cs + 2ac$

$$\therefore X = \frac{cs+2ac}{(s+a)^2+4} = \frac{ac}{(s+a)^2+4} + \frac{c(s+a)}{(s+a)^2+4} = \frac{ac}{2}\cdot\frac{2}{(s+a)^2+4} + c\cdot\frac{s+a}{(s+a)^2+4}$$

$$\therefore x = \frac{ac}{2}e^{-at}\sin 2t + ce^{-at}\cos 2t = e^{-at}\left(\frac{ac}{2}\sin 2t + c\cos 2t\right)$$

境界条件 $x(\pi) = 2e^{-\pi a}$ より，$x(\pi) = ce^{-a\pi}\cos 2\pi = ce^{-\pi a} = 2e^{-\pi a} \qquad \therefore c = 2$

$\therefore x = e^{-at}(a\sin 2t + 2\cos 2t)$

5

$c_2 = x''(0)$，$c_3 = x'''(0)$ とおいて，弾性曲線方程式 $\dfrac{d^4x}{dt^4} = -\dfrac{w}{EI}$ をラプラス変換すると

$$s^4X - c_2s - c_3 = -\frac{w}{EIs} \qquad \therefore X = \frac{c_2}{s^3} + \frac{c_3}{s^4} - \frac{w}{EIs^5}$$

$$\therefore x = \frac{c_2}{2}t^2 + \frac{c_3}{6}t^3 - \frac{w}{24EI}t^4 \qquad \therefore x' = c_2t + \frac{c_3}{2}t^2 - \frac{w}{6EI}t^3 \qquad \therefore x'' = c_2 + c_3t - \frac{w}{2EI}t^2$$

境界条件 $x(\ell) = 0$，$x''(\ell) = 0$ より

$$x(\ell) = \frac{c_2}{2}\ell^2 + \frac{c_3}{6}\ell^3 - \frac{w}{24EI}\ell^4 = 0 \qquad \therefore \frac{c_2}{2} + \frac{c_3}{6}\ell - \frac{w}{24EI}\ell^2 = 0 \quad \cdots ①$$

$$x''(\ell) = c_2 + c_3\ell - \frac{w}{2EI}\ell^2 = 0 \quad \cdots ②$$

① $\times 6 -$ ② より，$c_2 = -\dfrac{w}{8EI}\ell^2$　　これを②に代入して，$c_3 = \dfrac{5w}{8EI}\ell$

$$\therefore x = -\frac{w}{16EI}\ell^2t^2 + \frac{5w}{48EI}\ell t^3 - \frac{w}{24EI}t^4 = -\frac{w}{48EI}t^2(2t^2 - 5\ell t + 3\ell^2)$$

$$\therefore x = -\frac{w}{48EI}t^2(t-\ell)(2t-3\ell)$$

6

(1) $F(s) = \mathscr{L}[f(t)](s)$ とする。積分方程式を $f(t) * \dfrac{1}{\sqrt{t}} = \sqrt{t}$ と書き直し，3 章 1 節の節末問題 9 を用いて，ラプラス変換すると

$$F(s)\cdot\sqrt{\frac{\pi}{s}} = \frac{\Gamma\left(\dfrac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} = \frac{\dfrac{1}{2}\Gamma\left(\dfrac{1}{2}\right)}{s\sqrt{s}} = \frac{\sqrt{\pi}}{2s\sqrt{s}} \qquad \therefore F(s) = \frac{1}{2s} \qquad \therefore f(t) = \frac{1}{2}$$

(2) $F(s) = \mathscr{L}[f(t)](s)$ とする。積分方程式を $f(t) * e^t = \sin t$ と書き直し，ラプラス変換すると

$$F(s)\cdot\frac{1}{s-1} = \frac{1}{s^2+1} \qquad \therefore F(s) = \frac{s-1}{s^2+1} = \frac{s}{s^2+1} - \frac{1}{s^2+1} \qquad \therefore f(t) = \cos t - \sin t$$