１章　ベクトル解析

４節　積分公式

p.63～73　練習

練習 1

⑴ $$\int_V y\,dV = \int_{x=0}^{x=1}\int_{y=0}^{y=-x+1} y\int_{z=2}^{z=4-2x-2y} dz$$

$$= \int_{x=0}^{x=1}\left\{\int_{y=0}^{y=-x+1} y\,(2-2x-2y)\,dy\right\}dx = 2\int_{x=0}^{x=1}\left[\frac{1}{2}y^2(1-x)-\frac{1}{3}y^3\right]_{y=0}^{y=-x+1}dx$$

$$= \int_0^1\left\{\frac{1}{2}(-x+1)^3-\frac{1}{3}(-x+1)^3\right\}dx = \int_0^1 \frac{1}{6}(-x+1)^3\,dx = \frac{1}{6}\left[\frac{-1}{4}(-x+1)^4\right]_0^1 = \frac{1}{24}$$

練習 2

$$\int_V 0\,dV = 0,\ \int_V 1\,dV = \int_{x=0}^{x=1}\left\{\int_{y=0}^{y=2}\left(\int_{z=0}^{z=3} 1\,dz\right)dy\right\}dx$$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=2}\Big[\ z\ \Big]_{z=0}^{z=3} dy\ dx = 6\int_0^1 dx = 6,$$

$$\int_V z\ dV = \int_{x=0}^{x=1}\int_{y=0}^{y=2}\left[\frac{1}{2}z^2\right]_{z=0}^{z=3} dy\ dx = \frac{9}{2}\quad \text{よって}\quad \int_V \boldsymbol{f}\ dV = \left(0,\ 6,\ \frac{9}{2}\right)$$

練習 3

$$\int_S \boldsymbol{f}\cdot\boldsymbol{n}\,dS = \int_V \mathrm{div}\,\boldsymbol{f}\ dV \quad \text{（ガウスの発散定理より）}$$

$$= \int_V \left\{(x-z)_x + (xy)_y + (yz)_z\right\} dV$$

$$= \int_V (1+x+y)\,dV = \int_0^1\left\{\int_0^1\left(\int_0^1 1+x+y\,dz\right)dy\right\}dx$$

$$= \int_0^1\left[(1+x)y+\frac{1}{2}y^2\right]_0^1 dx = \int_0^1\left(1+x+\frac{1}{2}\right)dx$$

$$= \left[\frac{1}{2}x^2+\frac{3}{2}x\right]_0^1 = 2$$

練習 4

(1) S の境界を $C : \boldsymbol{r} = \boldsymbol{r}(t) = (\cos t, \sin t, 0)$ $(0 \leqq t \leqq 2\pi)$ とおくと

C 上で $\boldsymbol{f} = (\sin t, 0, \cos t)$ となり $\dfrac{d\boldsymbol{r}}{dt} = (-\sin t, \cos t, 0)$ なので

ストークスの定理を用いて

$$\int_S \mathrm{rot}\ \boldsymbol{f}\ dS = \int_C \boldsymbol{f} \cdot d\boldsymbol{r} = \int_0^{2x} \boldsymbol{f} \cdot \frac{d\boldsymbol{r}}{dt}\, dt$$

$$= \int_0^{2\pi} (\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt = \int_0^{2\pi} (-\sin^2 t + \cos^2 t)\, dt$$

$$= \int_0^{2\pi} \cos 2t\ dt = \left[\frac{1}{2} \sin 2t \right]_0^{2\pi} = 0$$

(2) S の境界 C を $\boldsymbol{r} = \boldsymbol{r}(t) = (\cos t, \sin t, 0)$ $(0 \leqq t \leqq 2\pi)$ とおくと

$\boldsymbol{r}'(t) = (-\sin t, \cos t, 0)$, $\boldsymbol{f}(t) = (0^2, \cos^3 t, \sin^2 t)$

なので，ストークスの定理より

$$与式 = \int_C \boldsymbol{f} \cdot \frac{d\boldsymbol{r}}{dt}\, dt$$

$$= \int_{t=0}^{t=2\pi} (0, \cos^3 t, \sin^2 t) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} \cos^4 t\ dt = 4\int_0^{\frac{\pi}{2}} \cos^4 t\ dt$$

$$= 4 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{4}\pi$$

㊟ 「新版微分積分Ｉ」p.126 例題 15，p.129 問題 9 より

次の公式が成立する。

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \cdots \cdot \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2} & (n\text{が}2\text{以上の偶数}) \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \cdots \cdot \dfrac{4}{5} \cdot \dfrac{2}{3} \cdot 1 & (n\text{が}3\text{以上の奇数}) \end{cases}$$

練習 5

$f_1 = -y$, $f_2 = x$ とおくとグリーンの定理より

$$\iint_D \left\{ (f_2)_x - (f_1)_y \right\} dx\, dy = \iint_D \left\{ 2 - (-1) \right\} dx\, dy = 3\pi$$