

3 章 微分法Ⅱ

3 節 テイラーの定理とその応用

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$$(1) \quad f(x) = \sqrt[4]{x} \quad \text{より}$$

$$\sqrt[4]{x} \doteq \sqrt[4]{a} + \frac{1}{4 \times \sqrt[4]{a^3}} (x - a)$$

$$(2) \quad f(x) = \sin x \quad \text{より}$$

$$\sin x \doteq \sin a + (x - a) \cos a$$

$$(3) \quad f(x) = \cos x \quad \text{より}$$

$$\cos x \doteq \cos a - (x - a) \sin a$$

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$$(1) \quad f(x) = \sqrt{x} \quad \text{とすると} \quad f(x) \doteq \sqrt{a} + \frac{1}{2\sqrt{a}} (x - a) \quad \text{より}$$

$$\sqrt{4.1} \doteq \sqrt{4} + \frac{1}{2\sqrt{4}} (4.1 - 4)$$

$$= 2 + \frac{1}{4} \times 0.1$$

$$= 2.025 \quad (\sqrt{4.1} = 2.024845673 \dots)$$

$$(2) \quad f(x) = \sqrt[4]{x} \quad \text{とすると} \quad f(x) \doteq \sqrt[4]{a} + \frac{1}{4 \times \sqrt[4]{a^3}} (x - a) \quad \text{より}$$

$$\sqrt[4]{16.1} \doteq \sqrt[4]{16} + \frac{1}{4 \times \sqrt[4]{16^3}} (16.1 - 16)$$

$$= 2 + \frac{1}{32} \times 0.1$$

$$= 2.003125 \quad (\sqrt[4]{16.1} = 2.00311770 \dots)$$

$$(3) \quad f(x) = \log x \quad \text{とすると} \quad f(x) \doteq \log a + (x - a) \times \frac{1}{a} \quad \text{より}$$

$$\log(1.1) \doteq \log 1 + (1.1 - 1) \times \frac{1}{1}$$

$$= 0 + 0.1$$

$$= 0.1 \quad (\log(1.1) = 0.09531 \dots)$$

$$(1) \quad f(x) = \frac{1}{1+x} \text{ について}$$

$$f(x) \doteq \frac{1}{1+a} - \frac{1 \times (x-a)}{(1+a)^2} + \frac{1}{2} \times 2 \cdot \frac{1}{(1+a)^3} (x-a)^2 \quad \text{より}$$

$$f(x) \doteq 1 - 1 \times x + x^2$$

$$f(x) = 1 - x + x^2$$

$$(2) \quad \frac{1}{1.05} \doteq \frac{1}{1+0} - \frac{1}{(1+0)^2} (0.05-0) + \frac{1}{(1+0)^3} (0.05-0)^2$$

$$= \frac{1}{1} - \frac{1}{1^2} \times 0.05 + \frac{1}{1^3} \times 0.05^2$$

$$= 0.9525$$

$$(1) \quad x=1 \text{ のまわりなので } f(x) \doteq \log a + \frac{1}{a} (x-a) - \frac{1}{2!a^2} (x-a)^2 \quad \text{より}$$

$$f(x) \doteq 0 + \frac{1}{1} (x-1) - \frac{1}{2} (x-1)^2$$

$$\therefore f(x) \doteq x-1 - \frac{1}{2} (x-1)^2$$

$$(2) \quad f(x) = (1+x)^r \quad (r>0) \text{ とすると}$$

$$f(x) \doteq (1+a)^r + r(1+a)^{r-1} \times (x-a) + \frac{1}{2!} \times r(r-1)(1+a)^{r-2} \times (x-a)^2 \quad \text{より}$$

$$f(x) \doteq (1+1)^r + r(1+1)^{r-1} (x-1) + \frac{r(r-1)(1+1)^{r-2}}{2!} \times (x-1)^2$$

$$= 2^r + 2^{r-1} r (x-1) + \frac{2^{r-2} r(r-1)}{2!} (x-1)^2$$

$$\therefore f(x) \doteq 2^r + 2^{r-1} r (x-1) + 2^{r-3} r(r-1) (x-1)^2$$

$$(3) \quad f(x) = \sqrt[3]{1+x}$$

$$f(x) \doteq \sqrt[3]{1+a} + \frac{1}{3 \times \sqrt[3]{(1+a)^2}} (x-a) - \frac{1}{2!} \times \frac{2}{9 \times \sqrt[3]{(1+a)^5}} (x-a)^2 \quad \text{より}$$

$$f(x) \doteq \sqrt[3]{1+1} + \frac{1}{3 \times \sqrt[3]{(1+1)^2}} (x-1) - \frac{2}{2! \times 9 \times \sqrt[3]{(1+1)^5}} (x-1)^2$$

$$= \sqrt[3]{2} + \frac{1}{3\sqrt[3]{4}} (x-1) - \frac{1}{2 \times 9 \times \sqrt[3]{4}} (x-1)^2$$

$$\therefore f(x) \doteq \sqrt[3]{2} + \frac{1}{3 \times \sqrt[3]{4}} (x-1) - \frac{1}{18 \times \sqrt[3]{4}} (x-1)^2$$

$$(1) \quad f(x) = x^2 + 4x - 5$$

$$f'(x) = 2x + 4$$

$$f'(x) = 0 \text{ のとき } x = -2$$

$$f''(x) = 2$$

$$\text{よって, } x = -2 \text{ のとき } f''(-2) = 2 > 0, \quad f(-2) = -9$$

$x = -2$ のとき極小値 -9 をとる

$$(2) \quad f(x) = x^3 - 3x + 7$$

$$f'(x) = 3x^2 - 3$$

$$f''(x) = 6x$$

$$f'(x) = 0 \text{ のとき } x = \pm 1$$

$$x = 1 \text{ のとき } f''(1) = 6 > 0, \quad f(1) = 5$$

$$x = -1 \text{ のとき } f''(-1) = -6 < 0, \quad f(-1) = 9$$

よって, $x = 1$ のとき極小値 5

$x = -1$ のとき極大値 9 をとる

$$(3) \quad f(x) = e^{x^2}$$

$$f'(x) = 2xe^{x^2}$$

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2}$$

$$f'(x) = 0 \text{ のとき } x = 0$$

$$x = 0 \text{ のとき, } f''(0) = 2 > 0, \quad f(0) = 1 \text{ なので}$$

$x = 0$ のとき極小値 1 をとる

$$(4) \quad f(x) = \frac{\log x}{x}$$

$$f'(x) = \frac{1}{x^2} - \frac{\log x}{x^2}$$

$$f''(x) = \frac{-\frac{1}{x} \times x^2 - (1 - \log x) \times 2x}{x^4} = \frac{2 \log x - 3}{x^3}$$

$$f'(x) = 0 \text{ となるのは } x = e$$

$$x = e \text{ のとき, } f''(e) = \frac{-1}{e^3} < 0, \quad f(e) = \frac{1}{e} \text{ なので}$$

$x = e$ のとき極大値 $\frac{1}{e}$ をとる

$$(5) \quad f(x) = \frac{1}{x^3 + 1}$$

$$f'(x) = -\frac{1}{(x^3 + 1)^2} \times 3x^2$$

$$f'(x) = 0 \text{ となるのは } x = 0$$

$$f''(x) = \frac{6x(2x^3 - 1)}{(x^3 + 1)^3}$$

$$x = 0 \text{ のとき, } f''(0) = 0, \quad f'''(0) \neq 0 \quad \text{より極値をもたない}$$

$$(6) \quad f(x) = x + 2 \cos x$$

$$f'(x) = 1 - 2 \sin x$$

$$f''(x) = -2 \cos x$$

$$f'(x) = 0 \text{ となるのは } x = \frac{\pi}{6}, \frac{5}{6}\pi \quad (0 \leq x < 2\pi)$$

$$x = \frac{\pi}{6} \text{ のとき } f''\left(\frac{\pi}{6}\right) = -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} < 0, \quad f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$$

$$x = \frac{5}{6}\pi \text{ のとき } f''\left(\frac{5}{6}\pi\right) = -2 \times \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3} > 0, \quad f\left(\frac{5}{6}\pi\right) = \frac{5}{6}\pi - \sqrt{3},$$

$$\text{よって, } x = \frac{\pi}{6} \text{ のとき極大値 } \frac{\pi}{6} + \sqrt{3}$$

$$x = \frac{5}{6}\pi \text{ のとき極小値 } \frac{5}{6}\pi - \sqrt{3} \text{ をとる}$$

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$$(1) \quad f(x) = e^{-2x}$$

$$f'(x) = -2e^{-2x}$$

$$f''(x) = 4e^{-2x}$$

$$f''(1) = 4e^{-2} > 0$$

よって $f(x)$ は $x = 1$ において下に凸

$$(2) \quad f(x) = x\sqrt{x+3}$$

$$f'(x) = \sqrt{x+3} + \frac{x}{2\sqrt{x+3}}$$

$$f''(x) = \frac{1}{2\sqrt{x+3}} + \frac{x+6}{4(x+3)\sqrt{x+3}}$$

$$f''(1) = \frac{15}{32} > 0$$

よって $f(x)$ は $x = 1$ において下に凸

$$(1) \quad f(x) = 2x^3 + 9x^2 - 5$$

$$f'(x) = 6x^2 + 18x$$

$$f''(x) = 12x + 18$$

$$f''(x) < 0 \quad \text{すなわち} \quad x < -\frac{3}{2} \quad \text{のとき} \quad f(x) \text{ は上に凸}$$

$$f''(x) > 0 \quad \text{すなわち} \quad x > -\frac{3}{2} \quad \text{のとき} \quad f(x) \text{ は下に凸}$$

$$(2) \quad f(x) = x^4 - 2x^2 + 5$$

$$f'(x) = 4x^3 - 4x$$

$$f''(x) = 12x^2 - 4$$

$$f''(x) < 0 \quad \text{すなわち} \quad -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \quad \text{のとき} \quad f(x) \text{ は上に凸}$$

$$f''(x) > 0 \quad \text{すなわち} \quad x < -\frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}} < x \quad \text{のとき} \quad f(x) \text{ は下に凸}$$

$$(1) \quad f(x) = e^x$$

$$f'(x) = e^x, \quad f''(x) = e^x, \quad f'''(x) = e^x, \quad f^{(4)}(x) = e^x$$

$$e^x = e^1 + (x-1)e^1 + \frac{(x-1)^2}{2!}e^1 + \cdots + \frac{(x-1)^n}{n!}e^1$$

$$= e + (x-1)e + \frac{(x-1)^2}{2!}e + \cdots + \frac{(x-1)^n}{n!}e$$

$$(2) \quad f(x) = x^{-1}$$

$$f'(x) = -x^{-2}$$

$$f''(x) = 2x^{-3}$$

$$f'''(x) = -6x^{-4}$$

$$f^{(4)}(x) = 24x^{-5}$$

$$f(x) = \frac{1}{1} + \left(-\frac{1}{1^2}\right)(x-1) + \left(\frac{2}{1^3}\right)\frac{(x-1)^2}{2!} + \left(-\frac{6}{1^4}\right)\frac{(x-1)^3}{3!}$$

$$+ \frac{24}{1^5}\frac{(x-1)^4}{4!} + \cdots + \frac{n!}{1^{n+1}} \times \frac{(x-1)^n}{n!}$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^{n-1}(x-1)^n$$

$$(3) \quad f(x) = \sqrt[3]{x}$$

$$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}, \quad f''(x) = -\frac{2}{9} x^{-\frac{5}{3}}, \quad f'''(x) = \frac{10}{27} x^{-\frac{8}{3}}$$

$$f^{(4)}(x) = -\frac{80}{81} x^{-\frac{11}{3}}$$

$$\sqrt[3]{x} = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{5}{81}(x-1)^3 - \frac{10}{243}(x-1)^4 + \dots$$

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$$(1) \quad f(x) = \log(1+x)$$

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}$$

$$\begin{aligned} \text{よつて} \quad \log(1+x) &= \log 1 + \frac{1}{1} \times x + \frac{-1}{1^2} \times \frac{x^2}{2!} + \frac{2}{1^3} \times \frac{x^3}{3!} + \dots \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots \end{aligned}$$

$$(2) \quad f(x) = \tan x$$

$$f'(x) = \frac{1}{\cos^2 x}, \quad f''(x) = \frac{2 \sin x}{\cos^3 x}, \quad f'''(x) = \frac{2(2 \sin^2 x + 1)}{\cos^4 x}$$

$$\begin{aligned} \text{よつて} \quad \tan x &= \tan 0 + \frac{1}{\cos^2 0} x + \frac{2 \sin 0}{\cos^3 0} \times \frac{x^2}{2!} + \frac{2(2 \sin^2 0 + 1)}{\cos^4 0} \times \frac{x^3}{3!} + \dots \\ &= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots \end{aligned}$$

$$(3) \quad f(x) = e^{x^2}$$

$$f'(x) = 2xe^{x^2}, \quad f''(x) = 2e^{x^2} + 4x^2e^{x^2}, \quad f'''(x) = 12xe^{x^2} + 8x^3e^{x^2},$$

$$f^{(4)}(x) = 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2}$$

$$\begin{aligned} e^{x^2} &= e^{0^2} + 2 \times 0 \times e^{0^2} \times x + \left(2e^{0^2} + 4 \times 0^2 \times e^{0^2} \right) \times \frac{x^2}{2!} \\ &\quad + \left(12 \times 0 \times e^{0^2} + 8 \times 0^3 \times e^{0^2} \right) \times \frac{x^3}{3!} \\ &\quad + \left(12e^{0^2} + 48 \times 0^2 \times e^{0^2} + 16 \times 0^4 \times e^{0^2} \right) \times \frac{x^4}{4!} + \dots \\ &= 1 + x^2 + \frac{1}{2}x^4 + \dots \end{aligned}$$

$$(4) \quad y = \sin^{-1} x$$

$$y' = \frac{1}{\sqrt{1-x^2}}, \quad y'' = \frac{-x}{(x^2-1)\sqrt{1-x^2}}, \quad y''' = \frac{2x^2+1}{(x^2-1)^2\sqrt{1-x^2}}$$

$$\begin{aligned} \sin^{-1} x &= \sin^{-1} 0 + \frac{1}{\sqrt{1-0^2}} x + \frac{-0}{(0^2-1)\sqrt{1-0^2}} \times \frac{x^2}{2!} + \frac{2 \times 0^2 + 1}{(0^2-1)^2\sqrt{1-0^2}} \times \frac{x^3}{3!} \\ &\quad + \frac{-3 \times 0 \times (2 \times 0^2 + 3)}{(0^2-1)^3\sqrt{1-0^2}} \times \frac{x^4}{4!} + \frac{24 \times 0^4 + 72 \times 0^2 + 9}{(0^2-1)^4\sqrt{1-0^2}} \times \frac{x^5}{5!} + \dots \\ &= x + \frac{x^3}{3!} + \frac{9}{5!} x^5 + \dots \end{aligned}$$