

2 章 積分法 I 解答

1 節 不定積分と定積分(1)

A 問題

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$$(1) \int 2x \, dx = 2 \cdot \frac{1}{2} x^2 + C = x^2 + C$$

$$(2) \int dx = x + C$$

$$(3) \int (-6x + 5) \, dx = -6 \cdot \frac{1}{2} x^2 + 5x + C \\ = -3x^2 + 5x + C$$

$$(4) \int (3x^2 - 4x + 1) \, dx = 3 \cdot \frac{1}{3} x^3 - 4 \cdot \frac{1}{2} x^2 + x + C \\ = x^3 - 2x^2 + x + C$$

$$(5) \int x(x + 3) \, dx = \int (x^2 + 3x) \, dx \\ = \frac{1}{3} x^3 + \frac{3}{2} x^2 + C$$

$$(6) \int (x - 2)(2x - 3) \, dx = \int (2x^2 - 7x + 6) \, dx \\ = \frac{2}{3} x^3 - \frac{7}{2} x^2 + 6x + C$$

$$(7) \int (y - 2)^2 \, dy = \int (y^2 - 4y + 4) \, dy \\ = \frac{1}{3} y^3 - 2y^2 + 4y + C$$

$$(8) \int (t + a)(t - a) \, dt = \int (t^2 - a^2) \, dt \\ = \frac{1}{3} t^3 - a^2 t + C$$

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$$(1) \int (2x^2 + 5x + 3) \, dx + \int (x^2 - 5x - 2) \, dx \\ = \int \{ (2x^2 + 5x + 3) + (x^2 - 5x - 2) \} \, dx \\ = \int (3x^2 + 1) \, dx = x^3 + x + C$$

$$(2) \int (x + 1)^2 \, dx - \int (x - 1)^2 \, dx \\ = \int \{ (x + 1)^2 - (x - 1)^2 \} \, dx \\ = \int 4x \, dx = 2x^2 + C$$

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$$(1) \int (3x^5 + \sqrt[3]{x^2}) \, dx = \int \left(3x^5 + x^{\frac{2}{3}} \right) \, dx \\ = \frac{1}{2} x^6 + \frac{3}{5} x^{\frac{5}{3}} + C \\ = \frac{1}{2} x^6 + \frac{3}{5} x \sqrt[3]{x^2} + C$$

$$(2) \int (\sqrt{x} + 1)^2 \, dx = \frac{1}{2} x^2 + \frac{4}{3} x^{\frac{3}{2}} + x + C \\ = \frac{1}{2} x^2 + \frac{4}{3} x \sqrt{x} + x + C$$

$$(3) \int (3x - 1)^4 \, dx = \frac{1}{15} (3x - 1)^5 + C$$

$$(4) \int \sqrt[3]{x} (\sqrt[3]{x} - 1) \, dx = \int \left(x^{\frac{2}{3}} - x^{\frac{1}{3}} \right) \, dx \\ = \frac{3}{5} x^{\frac{5}{3}} - \frac{3}{4} x^{\frac{4}{3}} + C \\ = \frac{3}{5} x \sqrt[3]{x^2} - \frac{3}{4} x \sqrt[3]{x} + C$$

$$(5) \quad \int \frac{(x+2)^2}{x} dx = \int \left(x + 4 + \frac{4}{x} \right) dx \\ = \frac{1}{2} x^2 + 4x + 4 \log |x| + C$$

$$(6) \quad \int \frac{(x-1)^2}{\sqrt{x}} dx = \int \left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} \right) dx \\ = \frac{2}{5} x^{\frac{5}{2}} - \frac{4}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C \\ = \frac{2}{5} x^2 \sqrt{x} - \frac{4}{3} x \sqrt{x} + 2\sqrt{x} + C$$

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$$(1) \quad \int (4 \sin x - 5 \cos x) dx = -4 \cos x - 5 \sin x + C$$

$$(2) \quad \int (3 \cos x + \sin 2x) dx = 3 \sin x - \frac{1}{2} \cos 2x + C$$

$$(3) \quad \int \frac{1 + \cos^3 x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} + \cos x \right) dx = \tan x + \sin x + C$$

$$(4) \quad \int (1 - \tan^2 x) dx = \int \left\{ 1 - \left(\frac{1}{\cos^2 x} - 1 \right) \right\} dx = 2x - \tan x + C$$

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$$(1) \quad \int (2e^x - x^3) dx = 2e^x - \frac{1}{4} x^4 + C$$

$$(2) \quad \int (3^x \log 3 - 1) dx = 3^x - x + C$$

$$(3) \quad \int e^{\frac{x}{2}} dx = 2e^{\frac{x}{2}} + C$$

$$(4) \quad \int 2^{x+1} dx = \frac{2^{x+1}}{\log 2} + C$$

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$$(1) \quad \int \sqrt[3]{2x-5} dx = \int (2x-5)^{\frac{1}{3}} dx \\ = \frac{3}{8} (2x-5)^{\frac{4}{3}} + C \\ = \frac{3}{8} (2x-5) \sqrt[3]{2x-5} + C$$

$$(2) \quad 2x-1=t \quad \text{とおく} \quad x = \frac{t+1}{2} \\ \frac{dx}{dt} = \frac{1}{2} \quad \text{より} \quad dx = \frac{1}{2} dt \\ \int x(2x-1)^3 dx = \int \frac{t+1}{2} \cdot t^3 \cdot \frac{1}{2} dt \\ = \frac{1}{4} \int (t^4 + t^3) dt \\ = \frac{1}{20} (2x-1)^5 + \frac{1}{16} (2x-1)^4 + C$$

$$(3) \quad \int \frac{dx}{(2+3x)^3} = \int (2+3x)^{-3} dx \\ = -\frac{1}{6(2+3x)^2} + C$$

$$(4) \quad x+2=t \quad \text{とおく} \quad x = t-2 \\ \frac{dx}{dt} = 1 \quad \text{より} \quad dx = dt \\ \int \frac{x}{(x+2)^2} dx = \int \frac{t-2}{t^2} dt = \int \left(\frac{1}{t} - \frac{2}{t^2} \right) dt \\ = \log |t| + 2t^{-1} + C \\ = \log |x+2| + \frac{2}{x+2} + C$$

$$(5) \quad \int \sin\left(\frac{1}{3}x + \pi\right) dx = -3 \cos\left(\frac{1}{3}x + \pi\right) + C$$

$$(6) \quad \int \cos \frac{3}{2} \pi x dx = \frac{2}{3\pi} \sin \frac{3}{2} \pi x + C$$

$$(7) \quad \int e^{-2x+1} dx = -\frac{1}{2} e^{-2x+1} + C$$

$$(8) \quad \int 2^{4x-1} dx = \frac{2^{4x-1}}{4 \log 2} + C$$

$$(9) \quad \int 5^{1-x} dx = -\frac{5^{1-x}}{\log 5} + C$$

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$$(1) \quad \int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{(x^2+1)'}{x^2+1} dx \\ = \frac{1}{2} \log(x^2+1) + C$$

$$(2) \quad \int \frac{e^x}{e^x+1} dx = \int \frac{(e^x+1)'}{e^x+1} dx \\ = \log(e^x+1) + C$$

$$(3) \quad \int \frac{\sin x}{1-\cos x} dx = \int \frac{(1-\cos x)'}{1-\cos x} dx \\ = \log|1-\cos x| + C$$

$$(4) \quad \int \tan x dx = -\int \frac{(\cos x)'}{\cos x} dx \\ = -\log|\cos x| + C$$

$$(5) \quad \int \frac{\sin 2x}{\sin^2 x + 1} dx = \int \frac{(\sin^2 x + 1)'}{\sin^2 x + 1} dx \\ = \log(\sin^2 x + 1) + C$$

$$(6) \quad \int \frac{2^x \log 2 - 1}{2^x - x} dx = \int \frac{(2^x - x)'}{2^x - x} dx \\ = \log|2^x - x| + C$$

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$$(1) \quad \int x e^{2x} dx = \int x \left(\frac{1}{2} e^{2x} \right)' dx \\ = x \left(\frac{1}{2} e^{2x} \right) - \frac{1}{2} \int e^{2x} dx \\ = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

$$(2) \quad \int \log 2x dx = \int x' \log 2x dx \\ = x \log 2x - \int x \cdot \frac{1}{2x} \times 2 dx \\ = x \log 2x - x + C$$

$$\text{(別解)} \quad \int \log 2x dx = \int (\log x + \log 2) dx \\ = \int \log x dx + \log 2 \int dx \\ = x \log x - x + x \log 2 + C \\ = x \log 2x - x + C$$

$$\begin{aligned}
 (1) \quad \int \frac{x-3}{x+1} dx &= \int \frac{x+1-4}{x+1} dx \\
 &= \int \left(1 - \frac{4}{x+1}\right) dx \\
 &= x - 4 \log|x+1| + C
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \int \frac{x^2-1}{x^2+x} dx &= \int \frac{(x+1)(x-1)}{x(x+1)} dx \\
 &= \int \left(1 - \frac{1}{x}\right) dx \\
 &= x - \log|x| + C
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int \frac{x^2-x+1}{x+1} dx &= \int \left(x-2 + \frac{3}{x+1}\right) dx \\
 &= \frac{1}{2} x^2 - 2x + 3 \log|x+1| + C
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int \frac{1}{x(x-2)} dx &= \frac{1}{2} \int \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
 &= \frac{1}{2} (\log|x-2| - \log|x|) + C \\
 &= \frac{1}{2} \log \left| \frac{x-2}{x} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \frac{3x-1}{(x+3)(x-1)} &= \frac{a}{x+3} + \frac{b}{x-1} \text{ とおくと} \\
 3x-1 &= a(x-1) + b(x+3)
 \end{aligned}$$

これが x についての恒等式だから

$$\begin{aligned}
 a &= \frac{5}{2}, \quad b = \frac{1}{2} \\
 \int \frac{3x-1}{(x+3)(x-1)} dx &= \int \left(\frac{5}{2} \cdot \frac{1}{x+3} + \frac{1}{2} \cdot \frac{1}{x-1} \right) dx \\
 &= \frac{5}{2} \log|x+3| + \frac{1}{2} \log|x-1| + C
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad \frac{1}{x(x-1)^2} &= \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2} \text{ とおくと} \\
 1 &= a(x-1)^2 + bx(x-1) + cx
 \end{aligned}$$

これが x についての恒等式だから

$$\begin{aligned}
 a &= 1, \quad b = -1, \quad c = 1 \\
 \int \frac{1}{x(x-1)^2} dx &= \int \left\{ \frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2} \right\} dx \\
 &= \log|x| - \log|x-1| - \frac{1}{x-1} + C \\
 &= \log \left| \frac{x}{x-1} \right| - \frac{1}{x-1} + C
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad \int 2 \cos^2 x dx &= \int 2 \cdot \frac{1 + \cos 2x}{2} dx \\
 &= x + \frac{1}{2} \sin 2x + C
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \int \sin 5x \cos 3x dx &= \frac{1}{2} \int (\sin 8x + \sin 2x) dx \\
 &= -\frac{1}{16} \cos 8x - \frac{1}{4} \cos 2x + C
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int \cos 4x \cos 2x dx &= \frac{1}{2} \int (\cos 6x + \cos 2x) dx \\
 &= \frac{1}{12} \sin 6x + \frac{1}{4} \sin 2x + C
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int \sin 3x \sin 2x dx &= \frac{1}{2} \int (\cos 5x - \cos x) dx \\
 &= -\frac{1}{10} \sin 5x + \frac{1}{2} \sin x + C
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & \int \frac{1}{\sqrt{x+1} + \sqrt{x-1}} dx \\
 &= \frac{1}{2} \int (\sqrt{x+1} - \sqrt{x-1}) dx \\
 &= \frac{1}{3} \left\{ \sqrt{(x+1)^3} - \sqrt{(x-1)^3} \right\} + C
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad & e^x + 2 = t \quad \text{とおくと} \quad \frac{dt}{dx} = e^x \\
 & \text{より} \quad dx = \frac{1}{e^x} dt = \frac{1}{t-2} dt \\
 & \int \frac{1}{e^x + 2} dx = \int \frac{1}{t(t-2)} dt \\
 &= \frac{1}{2} \int \left(\frac{1}{t-2} - \frac{1}{t} \right) dt \\
 &= \frac{1}{2} \{ \log |t-2| - \log |t| \} + C \\
 &= \frac{1}{2} \log \left| \frac{t-2}{t} \right| + C \\
 &= \frac{1}{2} \log \frac{e^x}{e^x + 2} + C
 \end{aligned}$$

B 問題

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$$\begin{aligned}
 (1) \quad & \int x \cos 3x dx = \int x \left(\frac{1}{3} \sin 3x \right)' dx \\
 &= x \left(\frac{1}{3} \sin 3x \right) - \frac{1}{3} \int \sin 3x dx \\
 &= \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \int \log_3 x dx = \int x' \log_3 x dx \\
 &= x \log_3 x - \int x \cdot \frac{1}{x \log 3} dx \\
 &= x \log_3 x - \frac{x}{\log 3} + C
 \end{aligned}$$

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$$\begin{aligned}
 (1) \quad & \int \frac{x - \sqrt[4]{x}}{\sqrt{x}} dx = \int \left(x^{\frac{1}{2}} - x^{\frac{1}{4}} \right) dx \\
 &= \frac{2}{3} x^{\frac{3}{2}} - \frac{4}{3} x^{\frac{3}{4}} + C \\
 &= \frac{2}{3} x \sqrt{x} - \frac{4}{3} \sqrt[4]{x^3} + C
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \int \frac{(2\sqrt{x} - 1)^3}{x} dx = \int \frac{8x^{\frac{3}{2}} - 12x + 6x^{\frac{1}{2}} - 1}{x} dx \\
 &= \int \left(8x^{\frac{1}{2}} - 12 + 6x^{-\frac{1}{2}} - x^{-1} \right) dx \\
 &= \frac{16}{3} x^{\frac{3}{2}} - 12x + 12x^{\frac{1}{2}} - \log |x| + C \\
 &= \frac{16}{3} x \sqrt{x} - 12x + 12\sqrt{x} - \log |x| + C
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \int \sqrt{x} \left(1 - \frac{1}{\sqrt[3]{x}} \right) dx \\
 &= \int \left(x^{\frac{1}{2}} - x^{\frac{1}{6}} \right) dx = \frac{2}{3} x^{\frac{3}{2}} - \frac{6}{7} x^{\frac{7}{6}} + C \\
 &= \frac{2}{3} x \sqrt{x} - \frac{6}{7} x \sqrt[6]{x} + C
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & \int \frac{x-1}{\sqrt{x}+1} dx = \int \frac{(x-1)(\sqrt{x}-1)}{(\sqrt{x}+1)(\sqrt{x}-1)} dx \\
 &= \int (\sqrt{x}-1) dx \\
 &= \int \left(x^{\frac{1}{2}} - 1 \right) dx = \frac{2}{3} x^{\frac{3}{2}} - x + C \\
 &= \frac{2}{3} x \sqrt{x} - x + C
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad & \int (1 + \tan x) \cos x dx = \int (\cos x + \sin x) dx \\
 &= \sin x - \cos x + C
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad & \int \frac{\sin^2 x}{1 - \cos x} dx = \int \frac{1 - \cos^2 x}{1 - \cos x} dx \\
 &= \int (1 + \cos x) dx = x + \sin x + C
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad \int (\sin x + \cos x)^2 dx &= \int (1 + 2 \sin x \cos x) dx & (2) \quad \int \frac{x + \cos 2x + 1}{x \cos^2 x} dx &= \int \frac{x + 2 \cos^2 x}{x \cos^2 x} dx \\
 &= \int (1 + \sin 2x) dx & &= \int \left(\frac{1}{\cos^2 x} + \frac{2}{x} \right) dx \\
 &= x - \frac{1}{2} \cos 2x + C & &= \tan x + \log x^2 + C
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int \frac{1}{\sin^2 2x} dx &= -\frac{1}{2} \frac{1}{\tan 2x} + C & (4) \quad \int \frac{1}{\cos^2(2-4x)} dx &= -\frac{1}{4} \tan(2-4x) + C
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \int (e^x + e^{-x})^2 dx &= \int (e^{2x} + 2 + e^{-2x}) dx \\
 &= \frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} + C
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad \int (2^x + 1)^2 dx &= \int (2^{2x} + 2^{x+1} + 1) dx & (2) \quad \int \frac{9^x - 1}{3^x - 1} dx &= \int \frac{(3^x + 1)(3^x - 1)}{3^x - 1} dx \\
 &= \frac{2^{2x}}{2 \log 2} + \frac{2^{x+1}}{\log 2} + x + C & &= \int (3^x + 1) dx = \frac{3^x}{\log 3} + x + C
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int \frac{e^{3x} + 1}{e^x + 1} dx &= \int \frac{(e^x + 1)(e^{2x} - e^x + 1)}{e^x + 1} dx \\
 &= \int (e^{2x} - e^x + 1) dx \\
 &= \frac{1}{2} e^{2x} - e^x + x + C
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad x^2 - 1 = t \quad \text{とおくと} \quad \frac{dt}{dx} &= 2x \quad \text{より} & (2) \quad \cos x = t \quad \text{とおくと} \quad \frac{dt}{dx} &= -\sin x \\
 x dx &= \frac{1}{2} dt & \text{より} \quad \sin x dx &= -dt \\
 \int x \sqrt{x^2 - 1} dx &= \frac{1}{2} \int \sqrt{t} dt = \frac{1}{3} t^{\frac{3}{2}} + C & \int \cos^4 x \sin x dx &= -\int t^4 dt \\
 &= \frac{1}{3} (x^2 - 1) \sqrt{x^2 - 1} + C & &= -\frac{1}{5} \cos^5 x + C
 \end{aligned}$$

$$(3) \quad \log x = t \quad \text{とおくと} \quad \frac{dt}{dx} = \frac{1}{x} \quad \text{より}$$

$$\frac{1}{x} dx = dt$$

$$\begin{aligned} \int \frac{1}{x \log x} dx &= \int \frac{1}{t} dt = \log |t| + C \\ &= \log |\log x| + C \end{aligned}$$

$$(4) \quad \cos x = t \quad \text{とおくと} \quad \frac{dt}{dx} = -\sin x$$

$$\text{より} \quad \sin x dx = -dt$$

$$\begin{aligned} \int \sin^3 x dx &= \int (1 - \cos^2 x) \sin x dx \\ &= -\int (1 - t^2) dt = -t + \frac{1}{3} t^3 + C \\ &= -\cos x + \frac{1}{3} \cos^3 x + C \end{aligned}$$

(別解) 3 倍角の公式

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\text{より} \quad \sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

$$\begin{aligned} \int \sin^3 x dx &= \frac{1}{4} \int (3 \sin x - \sin 3x) dx \\ &= \frac{1}{12} \cos 3x - \frac{3}{4} \cos x + C \end{aligned}$$

$$(5) \quad \cos x = t \quad \text{とおくと} \quad \frac{dt}{dx} = -\sin x$$

$$\text{より} \quad \sin x dx = -dt$$

$$\begin{aligned} \int \frac{\tan x}{\cos x} dx &= \int \frac{\sin x}{\cos^2 x} dx = -\int \frac{1}{t^2} dt \\ &= \frac{1}{t} + C = \frac{1}{\cos x} + C \end{aligned}$$

$$(6) \quad e^x = t \quad \text{とおくと} \quad \frac{dt}{dx} = e^x \quad \text{より}$$

$$e^x dx = dt$$

$$\begin{aligned} \int \frac{e^x}{e^x + e^{-x}} dx &= \int \frac{dt}{t + t^{-1}} = \frac{1}{2} \int \frac{2t}{t^2 + 1} dt \\ &= \frac{1}{2} \log(e^{2x} + 1) + C \end{aligned}$$

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$$\begin{aligned} (1) \quad \int \frac{x}{\cos^2 x} dx &= \int x (\tan x)' dx \\ &= x \tan x - \int \tan x dx \\ &= x \tan x + \int \frac{(\cos x)'}{\cos x} dx \\ &= x \tan x + \log |\cos x| + C \end{aligned}$$

$$\begin{aligned} (2) \quad \int x^2 \cos x dx &= \int x^2 (\sin x)' dx \\ &= x^2 \sin x - 2 \int x \sin x dx \\ &= x^2 \sin x - 2 \left\{ x (-\cos x) + \int \cos x dx \right\} \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

$$\begin{aligned} (3) \quad \int x^3 e^x dx &= \int x^3 (e^x)' dx \\ &= x^3 e^x - 3 \int x^2 e^x dx \\ &= x^3 e^x - 3 \left(x^2 e^x - 2 \int x e^x dx \right) \\ &= x^3 e^x - 3x^2 e^x + 6 \left(x e^x - \int e^x dx \right) \\ &= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \\ &= (x^3 - 3x^2 + 6x - 6) e^x + C \end{aligned}$$

$$\begin{aligned} (4) \quad \int x \log(x^2 + 1) dx &= \int \left(\frac{x^2 + 1}{2} \right)' \log(x^2 + 1) dx \\ &= \frac{x^2 + 1}{2} \log(x^2 + 1) - \int \frac{x^2 + 1}{2} \cdot \frac{2x}{x^2 + 1} dx \\ &= \frac{1}{2} (x^2 + 1) \log(x^2 + 1) - \int x dx \\ &= \frac{1}{2} (x^2 + 1) \log(x^2 + 1) - \frac{1}{2} x^2 + C \end{aligned}$$

$$(1) \quad \cos x = t \quad \text{とおくと} \quad \frac{dt}{dx} = -\sin x$$

$$\text{よ} \text{り} \quad \sin x \, dx = -dt$$

$$\begin{aligned} \int \frac{\sin^5 x}{1 + \cos x} dx &= \int \frac{\sin x (1 - \cos^2 x)^2}{1 + \cos x} dx \\ &= -\int \frac{(1 + t^2)^2}{1 + t} dt \\ &= \int (-t^3 + t^2 + t - 1) dt \\ &= -\frac{1}{4} \cos^4 x + \frac{1}{3} \cos^3 x + \frac{1}{2} \cos^2 x - \cos x + C \end{aligned}$$

$$(2) \quad \sin x = t \quad \text{とおくと} \quad \frac{dt}{dx} = \cos x$$

$$\text{よ} \text{り} \quad \cos x \, dx = dt$$

$$\begin{aligned} \int (\sin x + \cos^2 x) \cos x \, dx &= \int (\sin x + 1 - \sin^2 x) \cos x \, dx \\ &= \int (t + 1 - t^2) dt \\ &= \frac{t^2}{2} + t - \frac{t^3}{3} + C \\ &= -\frac{1}{3} \sin^3 x + \frac{1}{2} \sin^2 x + \sin x + C \end{aligned}$$

$$(3) \quad e^x + 1 = t \quad \text{とおくと} \quad \frac{dt}{dx} = e^x$$

$$\text{よ} \text{り} \quad e^x dx = dt$$

$$\begin{aligned} \int \frac{e^x \cdot e^x}{\sqrt{e^x + 1}} dx &= \int \frac{t-1}{\sqrt{t}} dt = \int \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) dt \\ &= \frac{2}{3} t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + C \\ &= \frac{2}{3} (t-3) t^{\frac{1}{2}} + C \\ &= \frac{2}{3} (e^x - 2) \sqrt{e^x + 1} + C \end{aligned}$$

$$(4) \quad \log x + 1 = t \quad \text{とおくと} \quad \frac{dt}{dx} = \frac{1}{x}$$

$$\text{よ} \text{り} \quad \frac{1}{x} dx = dt$$

$$\begin{aligned} \int \frac{\log x}{\log x + 1} \cdot \frac{1}{x} dx &= \int \frac{t-1}{t} dt = \int \left(1 - \frac{1}{t} \right) dt \\ &= t - \log |t| + C \\ &= \log x + 1 - \log |\log x + 1| + C \\ &= \log x - \log |\log x + 1| + C \end{aligned}$$

$$\begin{aligned} (5) \quad \int \frac{2x^2 + x - 1}{x^3 + 1} dx &= \int \frac{(x+1)(2x-1)}{(x+1)(x^2 - x + 1)} dx \\ &= \int \frac{(x^2 - x + 1)'}{x^2 - x + 1} dx \\ &= \log(x^2 - x + 1) + C \end{aligned}$$

$$(1) \quad \sin x = t \quad \text{とおくと} \quad \frac{dt}{dx} = \cos x \quad \text{より}$$

$$\cos x \, dx = dt$$

$$\begin{aligned} \int \frac{\cos x}{4 - \sin^2 x} dx &= \int \frac{dt}{4 - t^2} = \frac{1}{4} \int \left(\frac{1}{t+2} - \frac{1}{t-2} \right) dt \\ &= \frac{1}{4} \left\{ \log |t+2| - \log |t-2| \right\} + C \\ &= \frac{1}{4} \log \left| \frac{t+2}{t-2} \right| + C \\ &= \frac{1}{4} \log \left| \frac{\sin x + 2}{\sin x - 2} \right| + C \\ &= \frac{1}{4} \log \left(\frac{2 + \sin x}{2 - \sin x} \right) + C \end{aligned}$$

$$(2) \quad \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1 - \sin^2 x} dx$$

$$\sin x = t \quad \text{とおけば} \quad \frac{dt}{dx} = \cos x \quad \text{より}$$

$$\cos x \, dx = dt$$

$$\begin{aligned} \int \frac{1}{\cos x} dx &= \int \frac{1}{1-t^2} dt = \frac{1}{2} \int \left(\frac{1}{t+1} - \frac{1}{t-1} \right) dt \\ &= \frac{1}{2} \left(\log |t+1| - \log |t-1| \right) + C \\ &= \log \sqrt{\left| \frac{\sin x + 1}{\sin x - 1} \right|} + C \\ &= \log \sqrt{\frac{1 + \sin x}{1 - \sin x}} + C \end{aligned}$$

$$(3) \quad e^x + 1 = t \quad \text{とおくと} \quad \frac{dt}{dx} = e^x \quad \text{より}$$

$$dx = \frac{1}{e^x} dt = \frac{1}{t-1} dt$$

$$\begin{aligned} \int \frac{e^{3x}}{(e^x + 1)^2} dx &= \int \frac{(t-1)^3}{t^2(t-1)} dt \\ &= \int \frac{(t^2 - 2t + 1)(t-1)}{t^2(t-1)} dt = \int \left(1 - \frac{2}{t} + \frac{1}{t^2} \right) dt \\ &= t - 2 \log |t| - \frac{1}{t} + C \\ &= e^x - 2 \log(e^x + 1) - \frac{1}{e^x + 1} + C \end{aligned}$$

$$\begin{aligned} (i) \quad \int \cos^4 x \, dx &= \int (\cos^2 x)^2 dx = \int \left(\frac{1 + \cos 2x}{2} \right)^2 dx \\ &= \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4} \int \left(1 + 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) dx \\ &= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C \end{aligned}$$

$$\begin{aligned}
(\text{ii}) \quad \int \cos^4 x \, dx &= \int \cos^3 x \cos x \, dx = \int \cos^3 x (\sin x)' \, dx \\
&= \cos^3 x \sin x - \int (\cos^3 x)' \sin x \, dx \\
&= \cos^3 x \sin x - \int (-3 \cos^2 x \sin x) \sin x \, dx \\
&= \cos^3 x \sin x + 3 \int (\sin x \cos x)^2 \, dx \\
&= \cos^3 x \sin x + \frac{3}{4} \int \sin^2 2x \, dx \\
&= \cos^3 x \sin x + \frac{3}{4} \int \frac{1 - \cos 4x}{2} \, dx \\
&= \cos^3 x \sin x + \frac{3}{8} x - \frac{3}{32} \sin 4x + C
\end{aligned}$$

(参考) (ii) の答えは次のように変形することによって

(i) の答えと同じであることがわかる。

$$\begin{aligned}
&\frac{3}{8} x + (\sin x \cos x) \cos^2 x - \frac{3}{32} \sin 4x + C \\
&= \frac{3}{8} x + \frac{1}{2} \sin 2x \cdot \frac{1 + \cos 2x}{2} - \frac{3}{32} \sin 4x + C \\
&= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{4} \sin 2x \cos 2x - \frac{3}{32} \sin 4x + C \\
&= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{8} \sin 4x - \frac{3}{32} \sin 4x + C \\
&= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C
\end{aligned}$$

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$$\frac{x^2 + 4x - 1}{(x^2 + 1)(x + 2)} = \frac{ax + b}{x^2 + 1} + \frac{c}{x + 2} \quad \text{の両辺に}$$

$(x^2 + 1)(x + 2)$ を掛けて

$$\begin{aligned}
x^2 + 4x - 1 &= (ax + b)(x + 2) + c(x^2 + 1) \\
&= (a + c)x^2 + (2a + b)x + 2b + c
\end{aligned}$$

これが x についての恒等式だから

$$a + c = 1 \cdots \textcircled{1} \quad 2a + b = 4 \cdots \textcircled{2} \quad 2b + c = -1 \cdots \textcircled{3}$$

①, ③より $a - 2b = 2$ ②と連立して

$$a = 2, \quad b = 0 \quad \textcircled{1} \text{ から } c = -1$$

$$\begin{aligned}
\int \frac{x^2 + 4x - 1}{(x^2 + 1)(x + 2)} \, dx &= \int \left(\frac{2x}{x^2 + 1} - \frac{1}{x + 2} \right) \, dx \\
&= \log(x^2 + 1) - \log|x + 2| + C \\
&= \log \left| \frac{x^2 + 1}{x + 2} \right| + C
\end{aligned}$$

(1) $I = \int e^{-x} \sin x \, dx$ とおくと

$$I = \int \left(-e^{-x} \right)' \sin x \, dx = -e^{-x} \sin x + \int e^{-x} \cos x \, dx$$

$$\begin{aligned} \text{ここで } \int e^{-x} \cos x \, dx &= \int \left(-e^{-x} \right)' \cos x \, dx \\ &= -e^{-x} \cos x - \int e^{-x} \sin x \, dx \end{aligned}$$

$$I = -e^{-x} \sin x + \left\{ -e^{-x} \cos x - \int e^{-x} \sin x \, dx \right\}$$

ゆえに $I = -e^{-x} \sin x - e^{-x} \cos x - I$

右辺の I を移項して両辺を 2 で割ると

$$I = -\frac{1}{2} e^{-x} (\sin x + \cos x) + C$$

(2) $\int \log(x + \sqrt{x^2 + 1}) \, dx = \int (x)' \log(x + \sqrt{x^2 + 1}) \, dx$

$$= x \log(x + \sqrt{x^2 + 1}) - \int x \left\{ \log(x + \sqrt{x^2 + 1}) \right\}' dx$$

$$\begin{aligned} \text{ここで } \left\{ \log(x + \sqrt{x^2 + 1}) \right\}' &= \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} \\ &= \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

$$\begin{aligned} \text{よって } \int \log(x + \sqrt{x^2 + 1}) \, dx &= x \log(x + \sqrt{x^2 + 1}) - \int \frac{x}{\sqrt{x^2 + 1}} \, dx \\ &= x \log(x + \sqrt{x^2 + 1}) - \sqrt{x^2 + 1} + C \end{aligned}$$

発展問題

- 90 点 (x, y) における接線の傾きは $f'(x)$ であるから

$$f'(x) = 6x^2 - 2x + 3$$

$$\text{よって } f(x) = \int (6x^2 - 2x + 3) dx$$

$$= 2x^3 - x^2 + 3x + C$$

ここで、点 $(-1, 3)$ を通るから $f(-1) = 3$ より

$$-2 - 1 - 3 + C = 3 \quad \therefore C = 9$$

$$\text{したがって } f(x) = 2x^3 - x^2 + 3x + 9$$

$$\therefore y = 2x^3 - x^2 + 3x + 9$$

- 91 $I_n = \int x^n (e^x)' dx = x^n e^x - \int nx^{n-1} e^x dx$

$$= x^n e^x - n \int x^{n-1} e^x dx$$

$$\text{ゆえに } I_n = x^n e^x - n I_{n-1} \quad (\text{証明終})$$

1 節 不定積分と定積分(2)

A 問題

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$$\begin{aligned}(1) \quad \int_0^1 (3x^2 - 1) dx &= \left[x^3 - x \right]_0^1 \\ &= (1 - 1) - 0 = 0\end{aligned}$$

$$\begin{aligned}(2) \quad \int_1^4 (x - 2)(2x + 1) dx &= \int_1^4 (2x^2 - 3x - 2) dx \\ &= \left[\frac{2}{3} x^3 - \frac{3}{2} x^2 - 2x \right]_1^4 \\ &= \frac{2}{3} (64 - 1) - \frac{3}{2} (16 - 1) - 2(4 - 1) \\ &= 42 - \frac{45}{2} - 6 = \frac{27}{2}\end{aligned}$$

$$\begin{aligned}(3) \quad \int_{-3}^2 (x - 1)^2 dx &= \int_{-3}^2 (x^2 - 2x + 1) dx \\ &= \left[\frac{1}{3} x^3 - x^2 + x \right]_{-3}^2 \\ &= \left(\frac{8}{3} - 4 + 2 \right) - (-9 - 9 - 3) \\ &= \frac{65}{3}\end{aligned}$$

$$\begin{aligned}(4) \quad \int_2^1 (6x^2 + 2x - 1) dx &= \left[2x^3 + x^2 - x \right]_2^1 \quad (\text{別解}) \quad \int_2^1 (6x^2 + 2x - 1) dx = - \int_1^2 (6x^2 + 2x - 1) dx \\ &= (2 + 1 - 1) - (16 + 4 - 2) \\ &= -16 \\ &= - \left[2x^3 + x^2 - x \right]_1^2 \\ &= - \{ (16 + 4 - 2) - (2 + 1 - 1) \} \\ &= -16\end{aligned}$$

$$\begin{aligned}(5) \quad \int_1^3 (4x^3 + 2x) dx &= \left[x^4 + x^2 \right]_1^3 \\ &= (81 + 9) - (1 + 1) \\ &= 88\end{aligned}$$

$$\begin{aligned}(6) \quad \int_{-1}^2 (x^4 + 3x^2 + 2) dx &= \left[\frac{1}{5} x^5 + x^3 + 2x \right]_{-1}^2 \\ &= \left(\frac{32}{5} + 8 + 4 \right) - \left(-\frac{1}{5} - 1 - 2 \right) \\ &= \frac{108}{5}\end{aligned}$$

$$(1) \int_1^1 (2x^2 - 5x + 3) dx = 0$$

$$(2) \int_{-2}^{-2} (x^3 + x^2 + x + 1) dx = 0$$

$$\begin{aligned} (3) \int_{-2}^2 (3x^2 + 2x + 4) dx &= 2 \int_0^2 (3x^2 + 4) dx \\ &= 2 \left[x^3 + 4x \right]_0^2 \\ &= 2 \{ (8 + 8) - 0 \} \\ &= 32 \end{aligned}$$

$$\begin{aligned} (4) \int_{-1}^1 (x^2 - 3)(2x + 5) dx &= \int_{-1}^1 (2x^3 + 5x^2 - 6x - 15) dx \\ &= 2 \int_0^1 (5x^2 - 15) dx \\ &= 2 \left[\frac{5}{3} x^3 - 15x \right]_0^1 \\ &= 2 \left\{ \left(\frac{5}{3} - 15 \right) - 0 \right\} = -\frac{80}{3} \end{aligned}$$

$$\begin{aligned} (1) \int_0^2 (3x^2 + 2x - 1) dx - \int_0^2 (3x^2 - x) dx &= \int_0^2 \{ (3x^2 + 2x - 1) - (3x^2 - x) \} dx \\ &= \int_0^2 (3x - 1) dx \\ &= \left[\frac{3}{2} x^2 - x \right]_0^2 \\ &= (6 - 2) - 0 = 4 \end{aligned}$$

$$\begin{aligned} (2) \int_1^2 (x^3 - 2x^2) dx + \int_2^1 (x^3 - x^2 + 5) dx &= \int_1^2 (x^3 - 2x^2) dx - \int_1^2 (x^3 - x^2 + 5) dx \\ &= \int_1^2 \{ (x^3 - 2x^2) - (x^3 - x^2 + 5) \} dx \\ &= \int_1^2 (-x^2 - 5) dx \\ &= \left[-\frac{1}{3} x^3 - 5x \right]_1^2 \\ &= -\frac{1}{3} (8 - 1) - 5(2 - 1) = -\frac{22}{3} \end{aligned}$$

$$\begin{aligned} (3) \int_1^2 (x^2 - 1) dx + \int_2^3 (x^2 - 1) dx &= \int_1^3 (x^2 - 1) dx \\ &= \left[\frac{1}{3} x^3 - x \right]_1^3 \\ &= (9 - 3) - \left(\frac{1}{3} - 1 \right) = \frac{20}{3} \end{aligned}$$

$$\begin{aligned} (4) \int_{-2}^4 (x^2 + x) dx - \int_1^4 (x^2 + x) dx &= \int_{-2}^4 (x^2 + x) dx + \int_4^1 (x^2 + x) dx \\ &= \int_{-2}^1 (x^2 + x) dx \\ &= \left[\frac{1}{3} x^3 + \frac{1}{2} x^2 \right]_{-2}^1 \\ &= \frac{1}{3} (1 + 8) + \frac{1}{2} (1 - 4) = \frac{3}{2} \end{aligned}$$

$$(1) \int_0^4 x\sqrt{x} dx = \int_0^4 x^{\frac{3}{2}} dx = \left[\frac{2}{5} x^{\frac{5}{2}} \right]_0^4 = \frac{64}{5}$$

$$(2) \int_1^e \frac{1}{x} dx = [\log x]_1^e = 1$$

$$\begin{aligned}
 (3) \quad \int_{-1}^0 \frac{x+3}{x+2} dx &= \int_{-1}^0 \left(\frac{1}{x+2} + 1 \right) dx \\
 &= \left[\log |x+2| + x \right]_{-1}^0 \\
 &= \log 2 + 1
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_4^5 \frac{1}{(x-3)(x-2)} dx &= \int_4^5 \left(\frac{1}{x-3} - \frac{1}{x-2} \right) dx \\
 &= \left[\log \left| \frac{x-3}{x-2} \right| \right]_4^5 \\
 &= \log \frac{2}{3} - \log \frac{1}{2} \\
 &= \log \frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad \int_0^8 \frac{x-5}{\sqrt[3]{x}} dx &= \int_0^8 \left(x^{\frac{2}{3}} - 5x^{-\frac{1}{3}} \right) dx \\
 &= \left[\frac{3}{5} x^{\frac{5}{3}} - \frac{15}{2} x^{\frac{2}{3}} \right]_0^8 \\
 &= -\frac{54}{5}
 \end{aligned}$$

$$\begin{aligned}
 (6) \quad \int_0^1 \frac{dx}{\sqrt{x+1} - \sqrt{x}} &= \int_0^1 \left(\sqrt{x+1} + \sqrt{x} \right) dx \\
 &= \left[\frac{2}{3} (x+1)^{\frac{3}{2}} + \frac{2}{3} x^{\frac{3}{2}} \right]_0^1 \\
 &= \frac{4\sqrt{2}}{3}
 \end{aligned}$$

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$$\begin{aligned}
 (1) \quad \int_0^{\frac{\pi}{4}} \sin^2 x dx &= \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 - \cos 2x) dx \\
 &= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \int_0^{\frac{\pi}{3}} \tan^2 x dx &= \int_0^{\frac{\pi}{3}} \left(\frac{1}{\cos^2 x} - 1 \right) dx \\
 &= \left[\tan x - x \right]_0^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int_{-2}^2 (e^x + e^{-x})^2 dx &= \int_{-2}^2 (e^{2x} + 2 + e^{-2x}) dx \\
 &= \left[\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} \right]_{-2}^2 \\
 &= e^4 - \frac{1}{e^4} + 8
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_0^1 (5^x + e^x) dx &= \left[\frac{5^x}{\log 5} + e^x \right]_0^1 \\
 &= \frac{4}{\log 5} + e - 1
 \end{aligned}$$

$$(5) \quad \int_0^{\frac{\pi}{3}} \sin 2x dx = \left[-\frac{1}{2} \cos 2x \right]_0^{\frac{\pi}{3}} = \frac{3}{4}$$

$$\begin{aligned}
 (6) \quad \int_0^{\pi} \sin^2 \frac{x}{2} dx &= \int_0^{\pi} \frac{1 - \cos x}{2} dx \\
 &= \frac{1}{2} \left[x - \sin x \right]_0^{\pi} = \frac{\pi}{2}
 \end{aligned}$$

$$(7) \quad \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - \sin x \right) dx = \left[\tan x + \cos x \right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}$$

$$\begin{aligned}
 (8) \quad \int_0^{\frac{\pi}{3}} \frac{\sin 2x}{\cos x} dx &= \int_0^{\frac{\pi}{3}} \frac{2 \sin x \cos x}{\cos x} dx \\
 &= \left[-2 \cos x \right]_0^{\frac{\pi}{3}} = 1
 \end{aligned}$$

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$$(1) \quad \int_0^1 (3x+1)^2 dx = \left[\frac{1}{9} (3x+1)^3 \right]_0^1 = 7$$

$$\begin{aligned}
 (2) \quad \int_1^3 \frac{1}{(x+1)^2} dx &= \int_1^3 (x+1)^{-2} dx \\
 &= \left[-\frac{1}{x+1} \right]_1^3 = \frac{1}{4}
 \end{aligned}$$

(3) $\sqrt{1-x} = t$ とおくと

$$x = 1 - t^2 \quad \text{より} \quad dx = (-2t) dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow 1 \\ \hline t & 1 \rightarrow 0 \\ \hline \end{array}$$

$$\begin{aligned} \int_0^1 (x+1)\sqrt{1-x} dx &= \int_1^0 (2-t^2)t(-2t) dt \\ &= \int_0^1 (4t^2 - 2t^4) dt \\ &= \left[\frac{4}{3}t^3 - \frac{2}{5}t^5 \right]_0^1 \\ &= \frac{4}{3} - \frac{2}{5} = \frac{14}{15} \end{aligned}$$

(4) $x^2 - 1 = t$ とおくと $2x \frac{dx}{dt} = 1$ より

$$x dx = \frac{1}{2} dt \quad \begin{array}{|c|c|} \hline x & 1 \rightarrow 3 \\ \hline t & 0 \rightarrow 8 \\ \hline \end{array}$$

$$\begin{aligned} \int_1^3 x\sqrt{x^2-1} dx &= \frac{1}{2} \int_0^8 \sqrt{t} dt \\ &= \frac{1}{3} \left[t^{\frac{3}{2}} \right]_0^8 = \frac{16\sqrt{2}}{3} \end{aligned}$$

(5) $x^2 + 1 = t$ とおくと $2x \frac{dx}{dt} = 1$ より

$$x dx = \frac{1}{2} dt \quad \begin{array}{|c|c|} \hline x & -1 \rightarrow 0 \\ \hline t & 2 \rightarrow 1 \\ \hline \end{array}$$

$$\begin{aligned} \int_{-1}^0 x(x^2+1)^3 dx &= \frac{1}{2} \int_2^1 t^3 dt \\ &= \frac{1}{8} [t^4]_2^1 = -\frac{15}{8} \end{aligned}$$

(6) $x^3 - 3x^2 + 1 = t$ とおくと

$$3(x^2 - 2x) \frac{dx}{dt} = 1 \quad \text{より}$$

$$(x^2 - 2x) dx = \frac{1}{3} dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow 2 \\ \hline t & 1 \rightarrow -3 \\ \hline \end{array}$$

$$\begin{aligned} \int_0^2 \frac{x^2 - 2x}{x^3 - 3x^2 + 1} dx &= \frac{1}{3} \int_1^{-3} \frac{1}{t} dt \\ &= \frac{1}{3} \left[\log |t| \right]_1^{-3} = \frac{1}{3} \log 3 \end{aligned}$$

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$$\begin{aligned} (1) \quad \int_0^{\frac{\pi}{2}} \sin 3x \cos x dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin 4x + \sin 2x) dx \\ &= \frac{1}{2} \left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} \right]_0^{\frac{\pi}{2}} \\ &= \frac{1}{2} \end{aligned}$$

(2) $t = \cos x$ とおくと $\frac{dt}{dx} = -\sin x$ より

$$\sin x dx = -dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow \frac{\pi}{3} \\ \hline t & 1 \rightarrow \frac{1}{2} \\ \hline \end{array}$$

$$\begin{aligned} \int_0^{\frac{\pi}{3}} \sin x \cos^2 x dx &= -\int_1^{\frac{1}{2}} t^2 dt \\ &= \int_{\frac{1}{2}}^1 t^2 dt = \left[\frac{1}{3} t^3 \right]_{\frac{1}{2}}^1 = \frac{7}{24} \end{aligned}$$

(3) $e^x = t$ とおくと $e^x \frac{dx}{dt} = 1$ より

$$e^x dx = dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow 1 \\ \hline t & 1 \rightarrow e \\ \hline \end{array}$$

$$\begin{aligned} \int_0^1 \frac{e^{2x}}{e^x + 1} dx &= \int_1^e \frac{t}{t+1} dt \\ &= \int_1^e \left(1 - \frac{1}{t+1} \right) dt \\ &= \left[t - \log |t+1| \right]_1^e \\ &= e - 1 + \log \frac{2}{e+1} \end{aligned}$$

(4) $e^x = t$ とおくと $e^x \frac{dx}{dt} = 1$ より

$$e^x dx = dt \quad \begin{array}{|c|c|} \hline x & 1 \rightarrow 2 \\ \hline t & e \rightarrow e^2 \\ \hline \end{array}$$

$$\begin{aligned} \int_1^2 \frac{1}{e^x - 1} dx &= \int_e^{e^2} \frac{e^x}{e^x(e^x - 1)} dx \\ &= \int_e^{e^2} \frac{1}{t(t-1)} dt = \int_e^{e^2} \left(\frac{1}{t-1} - \frac{1}{t} \right) dt \\ &= \left[\log \left| \frac{t-1}{t} \right| \right]_e^{e^2} = \log \frac{e^2 - 1}{e^2} - \log \frac{e-1}{e} \\ &= \log \frac{e+1}{e} \end{aligned}$$

$$(5) \quad \log x = t \quad \text{とおくと} \quad \frac{1}{x} \frac{dx}{dt} = 1 \quad \text{より}$$

$$\frac{1}{x} dx = dt \quad \begin{array}{|c|c|} \hline x & e^2 \rightarrow e^3 \\ \hline t & 2 \rightarrow 3 \\ \hline \end{array}$$

$$\begin{aligned} \int_{e^2}^{e^3} \frac{1}{x \log x} dx &= \int_2^3 \frac{1}{t} dt = \left[\log |t| \right]_2^3 \\ &= \log 3 - \log 2 = \log \frac{3}{2} \end{aligned}$$

$$(6) \quad x^2 = t \quad \text{とおくと} \quad 2x \frac{dx}{dt} = 1 \quad \text{より}$$

$$x dx = \frac{1}{2} dt \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow 1 \\ \hline t & 0 \rightarrow 1 \\ \hline \end{array}$$

$$\begin{aligned} \int_0^1 x e^{x^2} dx &= \frac{1}{2} \int_0^1 e^t dt \\ &= \frac{1}{2} \left[e^t \right]_0^1 = \frac{1}{2} (e - 1) \end{aligned}$$

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$$\begin{aligned} (1) \quad \int_0^\pi x \sin x dx &= \int_0^\pi x (-\cos x)' dx \\ &= \left[x(-\cos x) \right]_0^\pi + \int_0^\pi \cos x dx \\ &= \pi + \left[\sin x \right]_0^\pi = \pi \end{aligned}$$

$$\begin{aligned} (2) \quad \int_{-1}^0 x e^{-x} dx &= \int_{-1}^0 x (-e^{-x})' dx \\ &= \left[x(-e^{-x}) \right]_{-1}^0 + \int_{-1}^0 e^{-x} dx \\ &= -e - \left[e^{-x} \right]_{-1}^0 = -1 \end{aligned}$$

$$\begin{aligned} (3) \quad \int_0^2 \log(x+1) dx &= \int_0^2 (x+1)' \cdot \log(x+1) dx \\ &= \left[(x+1) \log(x+1) \right]_0^2 - \int_0^2 \frac{x+1}{x+1} dx \\ &= 3 \log 3 - 2 \end{aligned}$$

$$\begin{aligned} (4) \quad \int_1^e x \log x dx &= \left[\frac{x^2}{2} \log x \right]_1^e - \int_1^e \frac{1}{2} x dx \\ &= \frac{1}{2} e^2 - \frac{1}{4} \left[x^2 \right]_1^e = \frac{1}{4} (e^2 + 1) \end{aligned}$$

$$\begin{aligned} (5) \quad \int_0^{\frac{\pi}{2}} (x+1) \cos x dx &= \int_0^{\frac{\pi}{2}} (x+1) (\sin x)' dx \\ &= \left[(x+1) \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx \\ &= \frac{\pi}{2} + 1 + \left[\cos x \right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} (6) \quad \int_{\frac{1}{e}}^1 x^2 \log x dx &= \int_{\frac{1}{e}}^1 \left(\frac{1}{3} x^3 \right)' \log x dx \\ &= \left[\frac{1}{3} x^3 \log x \right]_{\frac{1}{e}}^1 - \frac{1}{3} \int_{\frac{1}{e}}^1 x^2 dx \\ &= \frac{1}{3e^3} - \frac{1}{9} \left[x^3 \right]_{\frac{1}{e}}^1 = \frac{1}{9} \left(\frac{4}{e^3} - 1 \right) \end{aligned}$$

100

$$\begin{aligned} (1) \quad \int_1^3 x(x-3)^4 dx &= \int_1^3 x \left\{ \frac{1}{5} (x-3)^5 \right\}' dx \\ &= \left[x \cdot \frac{1}{5} (x-3)^5 \right]_1^3 - \frac{1}{5} \int_1^3 (x-3)^5 dx \\ &= \frac{32}{5} - \frac{1}{30} \left[(x-3)^6 \right]_1^3 = \frac{128}{15} \end{aligned}$$

$$\begin{aligned} (\text{参考}) \quad \int_1^3 x(x-3)^4 dx &= \int_1^3 \{(x-3)+3\} (x-3)^4 dx \\ &= \int_1^3 (x-3)^5 dx + 3 \int_1^3 (x-3)^4 dx \\ &= \frac{1}{6} \left[(x-3)^6 \right]_1^3 + \frac{3}{5} \left[(x-3)^5 \right]_1^3 = \frac{128}{15} \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \int_{\alpha}^{\beta} (x-\alpha)(x-\beta)^3 dx \\
 &= \int_{\alpha}^{\beta} (x-\alpha) \left\{ \frac{1}{4} (x-\beta)^4 \right\}' dx \\
 &= \left[(x-\alpha) \cdot \frac{1}{4} (x-\beta)^4 \right]_{\alpha}^{\beta} - \frac{1}{4} \int_{\alpha}^{\beta} (x-\beta)^4 dx \\
 &= 0 - \frac{1}{20} \left[(x-\beta)^5 \right]_{\alpha}^{\beta} = \frac{1}{20} (\alpha-\beta)^5
 \end{aligned}$$

$$\begin{aligned}
 (\text{参考}) \quad & \int_{\alpha}^{\beta} (x-\alpha)(x-\beta)^3 dx \\
 &= \int_{\alpha}^{\beta} \{ (x-\beta) - (\alpha-\beta) \} (x-\beta)^3 dx \\
 &= \int_{\alpha}^{\beta} (x-\beta)^4 dx - (\alpha-\beta) \int_{\alpha}^{\beta} (x-\beta)^3 dx \\
 &= \frac{1}{5} \left[(x-\beta)^5 \right]_{\alpha}^{\beta} - \frac{\alpha-\beta}{4} \left[(x-\beta)^4 \right]_{\alpha}^{\beta} \\
 &= \frac{1}{20} (\alpha-\beta)^5
 \end{aligned}$$

101

$$(1) \quad F'(x) = x \sin x$$

$$(2) \quad F'(x) = (2x \sin 2x) \times (2x)' = 4x \sin 2x$$

$$\begin{aligned}
 (3) \quad F'(x) &= (3x \sin 3x) \times (3x)' - (x \sin x) \times (x)' \\
 &= 9x \sin 3x - x \sin x
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad F'(x) &= \int_0^x \sin t \, dt + x \sin x \\
 &= \left[-\cos t \right]_0^x + x \sin x \\
 &= 1 - \cos x + x \sin x
 \end{aligned}$$

$$102 \quad \int_0^x t f(t) \, dt = x^3 \sin x - 2x^2$$

$$\text{両辺を } x \text{ で微分して} \quad x f'(x) = 3x^2 \sin x + x^3 \cos x - 4x$$

$$\text{両辺を } x \text{ で割って} \quad f'(x) = 3x \sin x + x^2 \cos x - 4$$

B 問題

103

$$\begin{aligned}
 (1) \quad & \int_1^4 \sqrt{x^2 - 4x + 4} \, dx \\
 &= \int_1^4 \sqrt{(x-2)^2} \, dx \\
 &= \int_1^4 |x-2| \, dx \\
 &= \left[-\frac{1}{2} (x-2)^2 \right]_1^2 + \left[\frac{1}{2} (x-2)^2 \right]_2^4 = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & \int_0^3 \sqrt{|x-2|} \, dx \\
 &= \int_0^2 \sqrt{2-x} \, dx + \int_2^3 \sqrt{x-2} \, dx \\
 &= \left[-\frac{2}{3} (3-x)^{\frac{3}{2}} \right]_0^2 + \left[\frac{2}{3} (x-2)^{\frac{3}{2}} \right]_2^3 \\
 &= \frac{4\sqrt{2}}{3} + \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & \int_0^{\pi} |\cos 2x| \, dx \\
 &= \int_0^{\frac{\pi}{4}} \cos 2x \, dx - \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \cos 2x \, dx + \int_{\frac{3}{4}\pi}^{\pi} \cos 2x \, dx \\
 &= \left[\frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{4}} - \left[\frac{1}{2} \sin 2x \right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi} + \left[\frac{1}{2} \sin 2x \right]_{\frac{3}{4}\pi}^{\pi} = 2
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad & \int_0^{\pi} |\sin x - \cos x| \, dx \\
 &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) \, dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x) \, dx \\
 &= \left[\sin x + \cos x \right]_0^{\frac{\pi}{4}} + \left[-\cos x - \sin x \right]_{\frac{\pi}{4}}^{\pi} \\
 &= 2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 (1) \quad \int_0^3 \frac{x}{\sqrt{x+1}+1} dx &= \int_0^3 \frac{x(\sqrt{x+1}-1)}{(\sqrt{x+1}+1)(\sqrt{x+1}-1)} dx \\
 &= \int_0^3 \frac{x(\sqrt{x+1}-1)}{x} dx \\
 &= \int_0^3 (\sqrt{x+1}-1) dx = \left[\frac{2}{3}(x+1)^{\frac{3}{2}} - x \right]_0^3 = \frac{5}{3}
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad \int_0^1 \frac{x}{x^2+3} dx &= \int_0^1 \frac{1}{2} \cdot \frac{(x^2+3)'}{x^2+3} dx \\
 &= \left[\frac{1}{2} \log(x^2+3) \right]_0^1 = \frac{1}{2} \log \frac{4}{3}
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \int_0^{\frac{\pi}{2}} \sin \frac{3}{2} x \cos \frac{x}{2} dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} (\sin 2x + \sin x) dx \\
 &= \frac{1}{2} \left[-\frac{1}{2} \cos 2x - \cos x \right]_0^{\frac{\pi}{2}} = 1
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_1^{\log^2} e^{2x} dx &= \left[\frac{1}{2} e^{2x} \right]_1^{\log^2} = \frac{1}{2} (e^{\log^4} - e^2) \\
 &= \frac{1}{2} (4 - e^2) = 2 - \frac{1}{2} e^2
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad &\int_0^\pi \sin kx \cos lx dx \\
 &= \frac{1}{2} \int_0^\pi (\sin(k+l)x + \sin(k-l)x) dx \\
 (i) \quad k \neq l \text{ のとき } &\int_0^\pi \sin kx \cos lx dx = -\frac{1}{2} \left[\frac{\cos(k+l)x}{k+l} + \frac{\cos(k-l)x}{k-l} \right]_0^\pi
 \end{aligned}$$

次に、 $k+l$ が偶数ならば $k-l$ も偶数なので

$$\int_0^\pi \sin kx \cos lx dx = -\frac{1}{2} \left\{ \left(\frac{1}{k+l} + \frac{1}{k-l} \right) - \left(\frac{1}{k+l} + \frac{1}{k-l} \right) \right\} = 0$$

また、 $k+l$ が奇数ならば $k-l$ も奇数なので

$$\begin{aligned}
 \int_0^\pi \sin kx \cos lx dx &= -\frac{1}{2} \left\{ \left(\frac{-1}{k+l} + \frac{-1}{k-l} \right) - \left(\frac{1}{k+l} + \frac{1}{k-l} \right) \right\} \\
 &= \frac{1}{k+l} + \frac{1}{k-l} = \frac{2k}{(k+l)(k-l)}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad k=l \text{ のとき } \int_0^\pi \sin kx \cos lx \, dx &= \frac{1}{2} \int_0^\pi \sin 2kx \, dx \\
 &= \frac{1}{2} \left[-\frac{1}{2k} \cos 2kx \right]_0^\pi = 0
 \end{aligned}$$

よって、 $k \neq l$ で $k+l$ が偶数のとき、0

$$k \neq l \text{ で } k+l \text{ が奇数のとき, } \frac{2k}{(k+l)(k-l)}$$

$$k=l \text{ のとき, } 0$$

105

$$(1) \quad x = 3 \sin \theta \text{ とおくと } \frac{dx}{d\theta} = 3 \cos \theta \text{ より}$$

$$dx = 3 \cos \theta \, d\theta$$

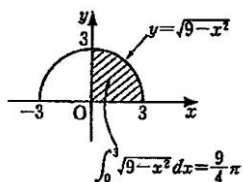
x	$0 \rightarrow 3$
θ	$0 \rightarrow \frac{\pi}{2}$

$$\int_0^3 \sqrt{9-x^2} \, dx = \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2 \theta} \cdot 3 \cos \theta \, d\theta$$

$$= 9 \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 \theta} \cdot \cos \theta \, d\theta$$

$$= 9 \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta = \frac{9}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) \, d\theta$$

$$= \frac{9}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}} = \frac{9}{4} \pi$$



$$(2) \quad x = 3 \sin \theta \text{ とおくと } \frac{dx}{d\theta} = 3 \cos \theta \text{ より}$$

$$dx = 3 \cos \theta \, d\theta$$

$$x = 0 \text{ のとき } \sin \theta = 0 \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow \frac{3}{2} \\ \hline \theta & 0 \rightarrow \frac{\pi}{6} \\ \hline \end{array}$$

$$\text{よって } \theta = 0$$

$$x = \frac{3}{2} \text{ のとき } \sin \theta = \frac{1}{2}$$

$$\text{よって } \theta = \frac{\pi}{6}$$

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}} = \int_0^{\frac{\pi}{6}} \frac{3 \cos \theta}{\sqrt{9-9\sin^2 \theta}} \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{3 \cos \theta}{3\sqrt{1-\sin^2 \theta}} \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}$$

$$(3) \quad x = \sqrt{2} \tan \theta \text{ とおくと } \frac{dx}{d\theta} = \frac{\sqrt{2}}{\cos^2 \theta} \text{ より}$$

$$dx = \frac{\sqrt{2}}{\cos^2 \theta} \, d\theta$$

$$x = 0 \text{ のとき } \tan \theta = 0 \quad \begin{array}{|c|c|} \hline x & 0 \rightarrow \sqrt{6} \\ \hline \theta & 0 \rightarrow \frac{\pi}{3} \\ \hline \end{array}$$

$$\text{よって } \theta = 0$$

$$x = \sqrt{6} \text{ のとき } \tan \theta = \sqrt{3}$$

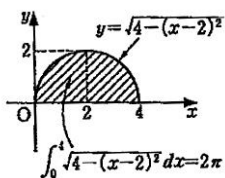
$$\text{よって } \theta = \frac{\pi}{3}$$

$$\int_0^{\sqrt{6}} \frac{1}{x^2+2} \, dx = \int_0^{\frac{\pi}{3}} \frac{1}{2 \tan^2 \theta + 2} \cdot \frac{\sqrt{2}}{\cos^2 \theta} \, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{2(\tan^2 \theta + 1)} \cdot \frac{\sqrt{2}}{\cos^2 \theta} \, d\theta$$

$$= \frac{\sqrt{2}}{2} \int_0^{\frac{\pi}{3}} \cos^2 \theta \cdot \frac{1}{\cos^2 \theta} \, d\theta$$

$$= \frac{\sqrt{2}}{2} \int_0^{\frac{\pi}{3}} d\theta = \frac{\sqrt{2}}{6} \pi$$



$$(4) \quad \int_1^2 \frac{dx}{x^2-2x+2} = \int_1^2 \frac{1}{(x-1)^2+1} \, dx \text{ より}$$

$$x-1 = \tan \theta \text{ とおくと}$$

$$\frac{dx}{d\theta} = \frac{1}{\cos^2 \theta} \text{ より } dx = \frac{1}{\cos^2 \theta} \, d\theta$$

$$\begin{array}{|c|c|} \hline x & 1 \rightarrow 2 \\ \hline \theta & 0 \rightarrow \frac{\pi}{4} \\ \hline \end{array} \quad \text{だから}$$

$$\int_1^2 \frac{1}{(x-1)^2+1} \, dx = \int_0^{\frac{\pi}{4}} \frac{1}{\tan^2 \theta + 1} \cdot \frac{1}{\cos^2 \theta} \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 \theta \cdot \frac{1}{\cos^2 \theta} \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{4}$$

$$(5) \quad \int_0^4 \sqrt{4x - x^2} dx = \int_0^4 \sqrt{4 - (x-2)^2} dx \quad \text{より} \quad x-2 = 2 \sin \theta \quad \text{とおくと}$$

$$\frac{dx}{d\theta} = 2 \cos \theta \quad \text{より} \quad dx = 2 \cos \theta d\theta$$

x	$0 \rightarrow 4$
θ	$-\frac{\pi}{2} \rightarrow \frac{\pi}{2}$

だから

$$\begin{aligned} \int_0^4 \sqrt{4 - (x-2)^2} dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4 - \sin^2 \theta} \cdot 2 \cos \theta d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{1 - \sin^2 \theta} \cdot 2 \cos \theta d\theta = 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta \\ &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta = 2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\pi \end{aligned}$$

$$(6) \quad x = \tan \theta \quad \text{とおくと}$$

$$\frac{dx}{d\theta} = \frac{1}{\cos^2 \theta} \quad \text{より} \quad dx = \frac{1}{\cos^2 \theta} d\theta$$

x	$0 \rightarrow 1$
θ	$0 \rightarrow \frac{\pi}{4}$

だから

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1+x^2}} &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 \theta}} \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 \theta}} \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \cos \theta \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\cos \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos \theta}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta \end{aligned}$$

$$\text{ここで } t = \sin \theta \quad \text{とおくと}$$

$$\frac{dt}{d\theta} = \cos \theta \quad \text{より} \quad \cos \theta d\theta = dt$$

θ	$0 \rightarrow \frac{\pi}{4}$
t	$0 \rightarrow \frac{1}{\sqrt{2}}$

だから

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{\cos \theta}{1 - \sin^2 \theta} d\theta &= \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{1-t^2} dt \\ &= \int_0^{\frac{1}{\sqrt{2}}} \frac{-1}{(t+1)(t-1)} dt = \frac{1}{2} \int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{t+1} - \frac{1}{t-1} \right) dt \\ &= \frac{1}{2} \left[\log \left| \frac{t+1}{t-1} \right| \right]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{2} \log \frac{\sqrt{2}+1}{\sqrt{2}-1} \\ &= \frac{1}{2} \log (\sqrt{2}+1)^2 = \log (\sqrt{2}+1) \end{aligned}$$

(別解) $\sqrt{1+x^2} = t - x$ とおけば

$0 \leq x \leq 1$ のとき $t > 1$ で

$$x = \frac{t^2 - 1}{2t} \quad \sqrt{1+x^2} = \frac{t^2 + 1}{2t}$$

$$\frac{dx}{dt} = \frac{t^2 + 1}{2t^2} \quad \text{より} \quad dx = \frac{t^2 + 1}{2t^2} dt$$

$t = x + \sqrt{x^2 + 1}$ であるから

x	$0 \rightarrow 1$
t	$1 \rightarrow 1 + \sqrt{2}$

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{1+x^2}} &= \int_1^{1+\sqrt{2}} \frac{2t}{t^2+1} \cdot \frac{t^2+1}{2t^2} dt \\ &= \int_1^{1+\sqrt{2}} \frac{1}{t} dt = \left[\log |t| \right]_1^{1+\sqrt{2}} = \log(1 + \sqrt{2}) \end{aligned}$$

106

$$(1) \quad \sin^2 x \cos^3 x = \sin^2 x (1 - \sin^2 x) \cos x$$

より $\sin x = t$ とおくと

$$\frac{dt}{dx} = \cos x \quad \text{より} \quad \cos x dx = dt$$

x	$0 \rightarrow \frac{\pi}{2}$
t	$0 \rightarrow 1$

だから

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x dx &= \int_0^{\frac{\pi}{2}} \sin^2 x (1 - \sin^2 x) \cos x dx \\ &= \int_0^1 t^2 (1 - t^2) dt = \int_0^1 (t^2 - t^4) dt \\ &= \left[\frac{1}{3} t^3 - \frac{1}{5} t^5 \right]_0^1 = \frac{2}{15} \end{aligned}$$

$$(2) \quad \int_0^{\frac{\pi}{4}} \frac{dx}{1 + \sin x} = \int_0^{\frac{\pi}{4}} \frac{1 - \sin x}{1 - \sin^2 x} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1 - \sin x}{\cos^2 x} dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \left[\tan x - \frac{1}{\cos x} \right]_0^{\frac{\pi}{4}} = 2 - \sqrt{2}$$

107

$$(1) \quad \int_0^{\frac{\pi}{2}} x \sin^2 x dx = \int_0^{\frac{\pi}{2}} x \cdot \frac{1 - \cos 2x}{2} dx$$

$$= \int_0^{\frac{\pi}{2}} x \cdot \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right)' dx$$

$$= \left[x \cdot \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) \right]_0^{\frac{\pi}{2}} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(x - \frac{1}{2} \sin 2x \right) dx$$

$$= \frac{\pi^2}{8} - \frac{1}{2} \left[\frac{1}{2} x^2 + \frac{1}{4} \cos 2x \right]_0^{\frac{\pi}{2}} = \frac{1}{16} \pi^2 + \frac{1}{4}$$

$$\begin{aligned}
(2) \quad \int_0^1 (1-x^2)e^x dx &= \int_0^1 (1-x^2)(e^x)' dx \\
&= \left[(1-x^2)e^x \right]_0^1 + 2 \int_0^1 xe^x dx \\
&= -1 + 2 \int_0^1 xe^x dx \\
\text{また } \int_0^1 xe^x dx &= \int_0^1 x(e^x)' dx \\
&= \left[xe^x \right]_0^1 - \int_0^1 e^x dx = e - \left[e^x \right]_0^1 = 1 \\
\text{よって } \int_0^1 (1-x^2)e^x dx &= -1 + 2 \cdot 1 = 1
\end{aligned}$$

$$\begin{aligned}
(3) \quad \int_1^e x(\log x)^2 dx &= \int_1^e \left(\frac{x^2}{2} \right)' (\log x)^2 dx \\
&= \left[\frac{x^2}{2} (\log x)^2 \right]_1^e - \int_1^e \frac{x^2}{2} \cdot 2(\log x) \frac{1}{x} dx \\
&= \frac{1}{2} e^2 - \int_1^e x \log x dx \\
&= \frac{1}{2} e^2 - \int_1^e \left(\frac{x^2}{2} \right)' \log x dx \\
&= \frac{1}{2} e^2 - \left[\frac{x^2}{2} \cdot \log x \right]_1^e + \int_1^e \frac{x^2}{2} \cdot \frac{1}{x} dx \\
&= \frac{1}{2} e^2 - \frac{1}{2} e^2 + \left[\frac{1}{4} x^2 \right]_1^e = \frac{1}{4} (e^2 - 1)
\end{aligned}$$

$$\begin{aligned}
(4) \quad \int_0^1 x \log(x^2 + 1) dx &= \left[\frac{x^2}{2} \log(x^2 + 1) \right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{2x}{x^2 + 1} dx \\
&= \frac{1}{2} \log 2 - \int_0^1 \left(x - \frac{x}{x^2 + 1} \right) dx \\
&= \frac{1}{2} \log 2 - \left[\frac{x^2}{2} - \frac{1}{2} \log(x^2 + 1) \right]_0^1 = \log 2 - \frac{1}{2}
\end{aligned}$$

(別解) 置換積分で求めることもできる。

$x^2 + 1 = t$ とおくと

x	$0 \rightarrow 1$
t	$1 \rightarrow 2$

$$\frac{dt}{dx} = 2x \quad \text{よ} \quad x dx = \frac{1}{2} dt$$

$$\int_0^1 x \log(x^2 + 1) dx = \int_1^2 \log t \cdot \frac{1}{2} dt$$

$$= \frac{1}{2} \int_1^2 \log t dt = \frac{1}{2} [t \log t - t]_1^2 = \log 2 - \frac{1}{2}$$

108 〔部分積分〕

$$\begin{aligned}
 I &= \int_0^1 \left\{ \frac{1}{2} \log(x^2 + 1) \right\}' \log(x^2 + 1) dx \\
 &= \left[\frac{1}{2} \log(x^2 + 1) \cdot \log(x^2 + 1) \right]_0^1 \\
 &\quad - \int_0^1 \frac{1}{2} \log(x^2 + 1) \cdot \frac{2x}{x^2 + 1} dx \\
 &= \frac{1}{2} (\log 2)^2 - \int_0^1 \frac{x}{x^2 + 1} \log(x^2 + 1) dx
 \end{aligned}$$

したがって

$$I = \frac{1}{2} (\log 2)^2 - I$$

$$2I = \frac{1}{2} (\log 2)^2 \quad \text{よって} \quad I = \frac{1}{4} (\log 2)^2$$

〔置換積分〕

$$\log(x^2 + 1) = t \quad \text{とおくと} \quad \frac{2x}{x^2 + 1} \cdot \frac{dx}{dt} = 1 \quad \text{より}$$

$$\frac{x}{x^2 + 1} dx = \frac{1}{2} dt$$

x	$0 \rightarrow 1$
t	$0 \rightarrow \log 2$

$$\begin{aligned}
 I &= \int_0^1 \log(x^2 + 1) \cdot \frac{x}{x^2 + 1} dx \\
 &= \int_0^{\log 2} t \left(\frac{1}{2} dt \right) = \frac{1}{2} \int_0^{\log 2} t dt \\
 &= \frac{1}{2} \left[\frac{1}{2} t^2 \right]_0^{\log 2} = \frac{1}{4} (\log 2)^2
 \end{aligned}$$

109

$$f'(x) = \left(x - \frac{\pi}{2} \right) \sin x$$

x	0	...	$\frac{\pi}{2}$...	π	...	2π
$f'(x)$	0	-	0	+	0	-	0
$f(x)$	0	↘	極小	↗	極大	↘	-2π

$$\begin{aligned}
 f(x) &= \left[\left(t - \frac{\pi}{2} \right) (-\cos t) \right]_0^x + \int_0^x \cos t dt \\
 &= -\left(x - \frac{\pi}{2} \right) \cos x - \frac{\pi}{2} + \left[\sin t \right]_0^x \\
 &= -\left(x - \frac{\pi}{2} \right) \cos x + \sin x - \frac{\pi}{2}
 \end{aligned}$$

$$\text{極大値 } f(\pi) = 0, \text{ 極小値 } f\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{2}$$

110

$$\begin{aligned}
 (1) \quad F(x) &= \int_0^x (x-t) \cos t dt \\
 &= x \int_0^x \cos t dt - \int_0^x t \cos t dt \quad \text{より} \\
 F'(x) &= \int_0^x \cos t dt + x \cos x - x \cos x \\
 &= \left[\sin t \right]_0^x = \sin x
 \end{aligned}$$

$$(2) \quad F(x) = \int_0^x t \sin(x-t) dt \quad \text{において}$$

$$x-t=u \quad \text{とおくと} \quad t=x-u$$

$$\frac{du}{dt} = -1 \quad \text{より} \quad dt = (-1) du$$

t	$0 \rightarrow x$
u	$x \rightarrow 0$

$$\begin{aligned}
 F(x) &= \int_x^0 (x-u) \sin u \cdot (-1) du \\
 &= \int_0^x (x-u) \sin u du \\
 &= x \int_0^x \sin u du - \int_0^x u \sin u du
 \end{aligned}$$

$$\text{よって} \quad F'(x) = \int_0^x \sin u du + x \sin x - x \sin x$$

$$= \left[-\cos u \right]_0^x = -\cos x + 1$$

$$\begin{aligned}
 (3) \quad F(x) &= \int_0^x e^t \log \frac{x+1}{t+1} dt \\
 &= \int_0^x e^t \{ \log(x+1) - \log(t+1) \} dt \\
 &= \log(x+1) \int_0^x e^t dt - \int_0^x e^t \log(t+1) dt \\
 \text{よって } F'(x) &= \frac{1}{x+1} \int_0^x e^t dt + e^x \log(x+1) \\
 &\quad - e^x \log(x+1) \\
 &= \frac{1}{x+1} [e^t]_0^x = \frac{e^x - 1}{x+1}
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \int_1^2 te^{x-t} dt &= e^x \int_1^2 te^{-t} dt \\
 \text{ここで } \int_1^2 te^{-t} dt &= \left[t(-e^{-t}) \right]_1^2 + \int_1^2 e^{-t} dt \\
 &= -2e^{-2} + e^{-1} + \left[-e^{-t} \right]_1^2 = \frac{2}{e} - \frac{3}{e^2} \\
 \text{よって } F(x) &= \cos x + \left(\frac{2}{e} - \frac{3}{e^2} \right) e^x \\
 \text{ゆえに } F'(x) &= -\sin x + \left(\frac{2}{e} - \frac{3}{e^2} \right) e^x
 \end{aligned}$$

$$\begin{aligned}
 111 \quad f(x) &= x \int_0^x \sin^2 t dt - \int_0^x t \sin^2 t dt \quad \text{より} \\
 f'(x) &= \int_0^x \sin^2 t dt + x \sin^2 x - x \sin^2 x \\
 &= \int_0^x \sin^2 t dt \\
 \text{ゆえに } f''(x) &= \sin^2 x
 \end{aligned}$$

$$\begin{aligned}
 112 \quad \int_0^x (x-t)f(t)dt &= a \cos x - 2 \cdots \textcircled{1} \\
 x \int_0^x f(t)dt - \int_0^x tf(t)dt &= a \cos x - 2 \\
 \text{両辺を } x \text{ で微分すると} \\
 \int_0^x f(t)dt + xf(x) - xf(x) &= -a \sin x \\
 \int_0^x f(t)dt &= -a \sin x \\
 \text{さらに両辺を } x \text{ で微分すると} \\
 f(x) &= -a \cos x \cdots \textcircled{2} \\
 \text{また, } \textcircled{1} \text{において } x=0 \text{ とおくと} \\
 0 &= a \cos 0 - 2 \quad \text{より } a=2 \\
 a=2 \text{ を } \textcircled{2} \text{ に代入して } f(x) &= -2 \cos x
 \end{aligned}$$