７章　微分方程式

２節　１階微分方程式

A

266　c は任意定数

⑴　$y' - \sqrt{y} = 0$

$$\int \frac{1}{\sqrt{y}}\,dy = \int dx$$

$$2\sqrt{y} = x + c$$

$$4y = (x + c)^2$$

$$\therefore \quad y = \frac{1}{4}(x + c)^2$$

⑵　$xy' - y = 0$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\log|y| = \log|x| + c$$

$$\log\left|\frac{y}{x}\right| = c$$

$$\frac{y}{x} = \pm e^c = c$$

$$\therefore \quad y = cx$$

⑶　$yy' = \sqrt{1 - y^2}$

$$\int \frac{y}{\sqrt{1 - y^2}}\,dy = \int dx$$

$$-\sqrt{1 - y^2} = x + c$$

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⑴　$yy' + x = 0$　$(x = 1,\ \ y = 1)$

$$yy' = -x$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$$

$x = 1$　$y = 1$　を代入して　$\frac{1}{2} = -\frac{1}{2} + c$　　$\therefore$　$c = 1$

したがって　$x^2 + y^2 = 2$

(2)　$y' = e^{2x+y}$　$(x=0,\ \ y=0)$

$\int e^{-y}\,dy = \int e^{2x}\,dx$

$-e^{-y} = \dfrac{1}{2}e^{2x} + c$

$x=0,\ \ y=0$　を代入して　$-1 = \dfrac{1}{2} + c$　　$\therefore\ \ c = -\dfrac{3}{2}$

したがって　$e^{2x} + 2e^{-y} = 3$

(3)　$\cos x \cos y \dfrac{dy}{dx} = \sin x \sin y$　$\left(x=0,\ \ y=\dfrac{\pi}{2}\right)$

$\displaystyle\int \frac{\cos y}{\sin y}\,dy = \int \frac{\sin x}{\cos x}\,dx$

$\log|\sin y| = -\log|\cos x| + c$

$\log|\sin y \cos x| = c$

$\sin y \cos x = \pm e^{c} = c$

$\sin y \cos x = c$

$x=0,\ \ y=\dfrac{\pi}{2}$　を代入して　$\sin\dfrac{\pi}{2}\cos 0 = c$

$1 = c$

したがって　$\sin y \cos x = 1$

268　c は任意定数

(1)　$y' = \dfrac{2y}{x-y} = \dfrac{2\dfrac{y}{x}}{1-\dfrac{y}{x}}$

$\dfrac{y}{x} = u$　とおくと　$y' = u'x + u$

$u'x + u = \dfrac{2u}{1-u}$

$u'x = \dfrac{u+u^2}{1-u}$

$\displaystyle\int \frac{1-u}{u+u^2}\,du = \int \frac{1}{x}\,dx$

$\displaystyle\int \left(\frac{1}{u} - \frac{2}{1+u}\right)du = \int \frac{1}{x}\,dx$

$\log|u| - 2\log|1+u| = \log|x| + c$

$\log\left|\dfrac{u}{(1+u)^2 x}\right| = c$

$\dfrac{u}{(1+u)^2 x} = \pm e^c = c$

$\dfrac{\dfrac{y}{x}}{\left(1+\dfrac{y}{x}\right)^2 x} = c$

$\dfrac{y}{(x+y)^2} = c$

$\therefore\ y = c(x+y)^2$

(2)　$xy^2y' = x^3 + y^3$

$y' = \dfrac{x^3+y^3}{xy^2} = \dfrac{1+\left(\dfrac{y}{x}\right)^3}{\left(\dfrac{y}{x}\right)^2}$

$\dfrac{y}{x} = u$　とおくと　$y' = u + xu'$

$u + xu' = \dfrac{1+u^3}{u^2}$

$xu' = \dfrac{1+u^3}{u^2} - u = \dfrac{1}{u^2}$

$\displaystyle\int u^2\,du = \int \frac{1}{x}\,dx$

$\dfrac{1}{3}u^3 = \log|x| + c$

$\left(\dfrac{y}{x}\right)^3 = 3\log|x| + 3c$

$\therefore\ y^3 = 3x^3(\log|x| + c)$

(3)　$(xy' - y)e^{\frac{y}{x}} = x$

$xy' - y = xe^{-\frac{y}{x}}$

$xy' = xe^{-\frac{y}{x}} + y$

$y' = e^{-\frac{y}{x}} + \dfrac{y}{x}$

$\dfrac{y}{x} = u$　とおくと　$y' = u + xu'$

$u + xu' = e^{-u} + u$

$xu' = e^{-u}$

$\displaystyle\int e^u\,du = \int \frac{1}{x}\,dx$

$e^u = \log|x| + c$

$\therefore\ e^{\frac{y}{x}} = \log|x| + c$

(4)　$y' = \dfrac{y}{x} + \tan\dfrac{y}{x}$

$\dfrac{y}{x} = u$　とおくと　$y' = u + xu'$

$u + xu' = u + \tan u$

$\displaystyle\int \frac{1}{\tan u}\,du = \int \frac{1}{x}\,dx$

$\log|\sin u| = \log|x| + c$

$\dfrac{\sin u}{x} = \pm e^c = c$

$\therefore\ \sin\dfrac{y}{x} = cx$

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(1)　$2xy\,y' = y^2 - x^2$

$$y' = \frac{y^2 - x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\frac{y}{x}}$$

$\frac{y}{x} = u$　とおくと　$y' = u + xu'$

$$u + xu' = \frac{u^2 - 1}{2u}$$

$$xu' = \frac{u^2 - 1}{2u} - u = \frac{-1 - u^2}{2u}$$

$$\int \frac{2u}{1+u^2}\,du = -\int \frac{1}{x}\,dx$$

$$\log|1 + u^2| = -\log|x| + c$$

$$\log|(1 + u^2)x| = c$$

$$(1 + u^2)x = \pm e^c = c$$

$$\left(1 + \left(\frac{y}{x}\right)^2\right)x = c$$

$$\frac{x^2 + y^2}{x} = c$$

$x = 1,\ \ y = -1$　を代入すると　$c = 2$

$\therefore$　$x^2 + y^2 = 2x$　　$\therefore$　$(x-1)^2 + y^2 = 1$

(2) $(x+y)+(x-y)y'=0$

$$y'=\frac{-x-y}{x-y}=\frac{-1-\frac{y}{x}}{1-\frac{y}{x}}$$

$\frac{y}{x}=u$ とおくと $y'=u+xu'$

$$u+xu'=\frac{-1-u}{1-u}$$

$$xu'=\frac{u^2-2u-1}{1-u}$$

$$\int\frac{1-u}{u^2-2u-1}\,du=\int\frac{1}{x}\,dx$$

$$-\frac{1}{2}\log\left|u^2-2u-1\right|=\log|x|+c$$

$$\log\left|x\sqrt{u^2-2u-1}\right|=-c$$

$$x\sqrt{u^2-2u-1}=\pm e^{-c}=c$$

$$x\sqrt{\left(\frac{y}{x}\right)^2-\frac{2y}{x}-1}=c$$

$$\sqrt{y^2-2xy-x^2}=c$$

$$y^2-2xy-x^2=c^2=c$$

$x=0,\ \ y=0$ を代入して $c=0$

$\therefore\ \ y^2-2xy-x^2=0$

(3) $y'=\frac{y}{x}\left(1+\log\frac{y}{x}\right)$ （$x=1,\ \ y=1$）

$\frac{y}{x}=u$ とおくと $y'=u+xu'$

$$u+xu'=u(1+\log u)$$

$$xu'=u\log u$$

$$\int\frac{1}{u\log u}\,du=\int\frac{1}{x}\,dx$$

$$\log|\log u|=\log|x|+c$$

$$\log\left|\frac{\log u}{x}\right|=c$$

$$\frac{\log u}{x}=\pm e^{c}=c$$

$$\log\frac{y}{x}=cx$$

$$\frac{y}{x}=e^{cx}\quad\therefore\ \ y=xe^{cx}$$

$x=1,\ \ y=1$ を代入して $1=e^{c}$ $\therefore\ \ c=0$

$\therefore\ \ y=x$

270　c は任意定数

(1)　$xy' + y = x^3$

$$y' + \frac{1}{x}y = x^2$$

$$y' + \frac{1}{x}y = 0$$

$$y' = -\frac{1}{x}y$$

$$\int \frac{1}{y}dy = -\int \frac{1}{x}dx$$

$$\log|y| = -\log|x| + c$$

$$xy = c$$

c を x の関数 u とおく　$y = \dfrac{u}{x}$　$y' = \dfrac{u'}{x} - \dfrac{u}{x^2}$

$$\frac{u'}{x} - \frac{u}{x^2} + \frac{u}{x^2} = x^2$$

$$u' = x^3$$

$$u = \frac{1}{4}x^4 + c$$

$$\therefore\ y = \frac{1}{x}\left(\frac{1}{4}x^4 + c\right) = \frac{1}{4}x^3 + \frac{c}{x}$$

(2)　$x^2y' + 2xy = 1$

$$y' + \frac{2}{x}y = \frac{1}{x^2}$$

$$y' + \frac{2}{x}y = 0$$

$$\int \frac{1}{y}dy = -\int \frac{2}{x}dx$$

$$\log|y| = -2\log|x| + c$$

$$yx^2 = c$$

$$y = \frac{c}{x^2}$$

c を x の関数 u とおく

$$y = \frac{u}{x^2}$$

$$y' = \frac{u'}{x^2} - \frac{2u}{x^3}$$

$$\frac{u'}{x^2} - \frac{2u}{x^3} + \frac{2u}{x^3} = \frac{1}{x^2}$$

$$u' = 1$$

$$u = x + c$$

$$\therefore\ y = \frac{1}{x^2}(x + c) = \frac{1}{x} + \frac{c}{x^2}$$

(3) $xy'\log x + y = \log x$

$$y' + \frac{1}{x\log x}y = \frac{1}{x}$$

$$y' + \frac{1}{x\log x}y = 0$$

$$\int \frac{1}{y}dy = -\int \frac{1}{x\log x}dx$$

$$\log|y| = -\log|\log x| + c$$

$$y\log x = c$$

c を x の関数 u とおく

$$y = \frac{u}{\log x} \qquad y' = -\frac{u}{x(\log x)^2} + \frac{u'}{\log x}$$

$$\frac{u'}{\log x} - \frac{c}{x(\log x)^2} + \frac{c}{x(\log x)^2} = \frac{1}{x}$$

$$u' = \frac{1}{x}\log x$$

$$u = \int \frac{1}{x}\log x\,dx$$

$$= (\log x)^2 - \int \frac{\log x}{x}dx \quad \text{より}$$

$$u = \int \frac{1}{x}\log x\,dx = \frac{1}{2}(\log x)^2 + c$$

したがって $y = \dfrac{1}{\log x}\left(\dfrac{1}{2}(\log x)^2 + c\right)$

$$y = \frac{1}{2}\log x + \frac{c}{\log x}$$

(4) $y' - 3y = \sin x$

$$y' - 3y = 0$$

$$\int \frac{1}{y}dy = 3\int dx$$

$$\log|y| = 3x + c$$

$$y = ce^{3x}$$

c を x の関数 u とおく

$$y = ue^{3x}$$

$$y' = u'e^{3x} + 3ue^{3x}$$

$$u'e^{3x} + 3ue^{3x} - 3ue^{3x} = \sin x$$

$$u' = e^{-3x}\sin x$$

$$u = \int e^{-3x}\sin x\,dx$$

$$= -\frac{1}{10}e^{-3x}(3\sin x + \cos x) + c$$

$$\therefore\ y = -\frac{1}{10}(3\sin x + \cos x) + ce^{3x}$$

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(1)　$xy' + 3y + x = 0$　（$x = 1,\ \ y = 1$）

$$y' + \frac{3}{x}y = -1$$

$$y' + \frac{3}{x}y = 0$$

$$\int \frac{1}{y}dy = -\int \frac{3}{x}dx$$

$$\log|y| = -3\log|x| + c$$

$$yx^3 = c$$

$$y = \frac{c}{x^3}$$

cをxの関数uとおく　$y = \dfrac{u}{x^3}$

$$y' = \frac{u'}{x^3} - \frac{3u}{x^4}$$

$$\frac{u'}{x^3} - \frac{3u}{x^4} + \frac{3u}{x^4} = -1$$

$$u' = -x^3$$

$$u = -\frac{1}{4}x^4 + c$$

$$\therefore\quad y = \frac{1}{x^3}\left(-\frac{1}{4}x^4 + c\right)$$

$$= -\frac{1}{4}x + \frac{c}{x^3}$$

$x = 1,\ \ y = 1$　を代入すると　$1 = -\dfrac{1}{4} + c$

$$\therefore\ \ c = \frac{5}{4}$$

$$\therefore\ \ y = -\frac{1}{4}x + \frac{5}{4x^3}$$

(2)　$y' + y = e^x$

$$y' + y = 0$$

$$\int \frac{1}{y}dy = -\int dx$$

$$\log|y| = -x + c$$

$$y = ce^{-x}$$

cをxの関数uとおく　$y = ue^{-x}$，$y' = u'e^{-x} - ue^{-x}$

$$u'e^{-x} - ue^{-x} + ue^{-x} = e^x$$

$$u' = e^{2x}$$

$$u = \frac{1}{2}e^{2x} + c$$

$$\therefore\ \ y = \frac{1}{2}e^x + ce^{-x}$$

$x = 0,\ \ y = 0$ を代入すると

$$0 = \frac{1}{2} + c \qquad \therefore\ \ c = -\frac{1}{2}$$

$$\therefore\ \ y = \frac{1}{2}(e^x - e^{-x})$$

(3) $y'\cos x + y\sin x = 1$

$$y' + \frac{\sin x}{\cos x}y = \frac{1}{\cos x}$$

$$y' + \frac{\sin x}{\cos x}y = 0$$

$$\int \frac{1}{y}dy = -\int \frac{\sin x}{\cos x}dx$$

$$\log|y| = \log|\cos x| + c$$

$$\frac{y}{\cos x} = c$$

$$y = c\cos x$$

c を x の関数 u とおく

$$y = u\cos x$$

$$y' = u'\cos x - u\sin x$$

$$u'\cos x - u\sin x + u\sin x = \frac{1}{\cos x}$$

$$u' = \frac{1}{\cos^2 x}$$

$$u = \tan x + c$$

$\therefore \quad y = \sin x + c\cos x$

$x = 0,\ \ y = 1$ を代入して　$1 = c$

$\therefore \quad y = \sin x + \cos x$

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点 P における接線　　$Y - y = y'(X - x)$

点 R の x 座標は　$0 - y = y'(X - x)$ より　$X = x + \dfrac{y}{y'}$

$$\therefore \quad \mathrm{QR} = \left| x - \left(x + \frac{y}{y'} \right) \right| = \left| \frac{y}{y'} \right| = 2$$

$$\therefore \quad \frac{y}{y'} = \pm 2$$

$$\int \frac{1}{y}dy = \pm\frac{1}{2}\int dx$$

$$\log|y| = \pm\frac{1}{2}x + c$$

$$y = ce^{\pm\frac{1}{2}x}$$

$x = 0,\ \ y = 2$ を代入して　$2 = c$

$$\therefore y = 2e^{\pm\frac{1}{2}x}$$

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(1) $$\frac{dy}{dx} = y^2\left(xy - \frac{1}{xy}\right)$$

$u = xy$ とおくと $u' = y + xy'$ より $y' = \dfrac{u' - y}{x} = \dfrac{u' - \dfrac{u}{x}}{x} = \dfrac{u'x - u}{x^2}$

$$\therefore \quad \frac{u'}{x} - \frac{u}{x^2} = \left(\frac{u}{x}\right)^2\left(u - \frac{1}{u}\right)$$

$$\frac{u'}{x} = \frac{u^3}{x^2} \qquad \therefore \ u' = \frac{u^3}{x}$$

(2) $$\int \frac{1}{u^3}\,du = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}u^{-2} = \log|x| + c$$

$$\frac{1}{u^2} = -2\log|x| + c$$

$$\frac{1}{x^2y^2} = -2\log|x| + c$$

$$\frac{1}{y^2} = x^2(-\log x^2 + c)$$

$\therefore \ y^2 = \dfrac{1}{x^2(-\log x^2 + c)}$ （c は任意定数）

B

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(1) $y' = xe^{x+y}$

$$\int e^{-y}\,dy = \int xe^{x}\,dx$$

$$-e^{-y} = xe^{x} - e^{x} + c$$

$$\therefore\ e^{-y} = e^{x}(1-x) + c$$ （cは任意定数）

(2) $x^2y' = y^2 - xy$

$$y' = \left(\frac{y}{x}\right)^2 - \frac{y}{x}$$

$\dfrac{y}{x} = u$ とおくと $y' = xu' + u$

$$xu' + u = u^2 - u$$

$$xu' = u^2 - 2u$$

$$\int \frac{1}{u^2 - 2u}\,du = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\left(\int\left(\frac{-1}{u} + \frac{1}{u-2}\right)du\right) = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}\left(-\log|u| + \log|u-2|\right) = \log|x| + c$$

$$\log\left|\frac{u-2}{u}\right| = 2\log|x| + 2c$$

$$\log\left|\frac{u-2}{ux^2}\right| = c$$

$$\frac{u-2}{ux^2} = \pm e^{c} = c$$

$$\frac{\frac{y}{x} - 2}{\frac{y}{x}\cdot x^2} = c$$

より $y = \dfrac{2x}{1-cx^2}$ （cは任意定数）

(3) $(1+x^2)\dfrac{dy}{dx} = xy + 1$

$$\frac{dy}{dx} = \frac{x}{1+x^2}y + \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{x}{1+x^2}y$$

$$\int \frac{1}{y}\,dy = \int \frac{x}{1+x^2}\,dx$$

$$\log|y| = \frac{1}{2}\log|1+x^2| + c$$

$$\log\left|\frac{y}{\sqrt{1+x^2}}\right| = c$$

$$y = c\sqrt{1+x^2}$$

cをxの関数uとおく

$$y = u\sqrt{1+x^2}$$

$$y' = u'\sqrt{1+x^2} + \frac{xu}{\sqrt{1+x^2}}$$

$$u'\sqrt{1+x^2} + \frac{xu}{\sqrt{1+x^2}} = \frac{x}{1+x^2}u\sqrt{1+x^2} + \frac{1}{1+x^2}$$

$$u' = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$u = \frac{x}{\sqrt{1+x^2}} + c$$

$$\therefore\quad y = \left(\frac{x}{\sqrt{1+x^2}} + c\right)\sqrt{1+x^2}$$

$$y = x + c\sqrt{1+x^2}$$ （cは任意定数）

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⑴　$x(y^2-1)+y(x^2-1)\dfrac{dy}{dx}=0 \quad (x=2,\ \ y=2)$

$$\int \frac{y}{y^2-1}\,dy=-\int \frac{x}{x^2-1}\,dx$$

$$\frac{1}{2}\log\left|y^2-1\right|=-\frac{1}{2}\log\left|x^2-1\right|+c$$

$$\log\left|\sqrt{y^2-1}\sqrt{x^2-1}\right|=c$$

$$\sqrt{(x^2-1)(y^2-1)}=\pm e^c=c$$

$$(x^2-1)(y^2-1)=c$$

$x=2,\ \ y=2$ を代入して　$c=9$

$\therefore\ \ (x^2-1)(y^2-1)=9$

⑵　$xy'\cos\dfrac{y}{x}=y\cos\dfrac{y}{x}+x \quad \left(x=1,\ \ y=\dfrac{\pi}{2}\right)$

$$y'\cos\frac{y}{x}=\frac{y}{x}\cos\frac{y}{x}+1$$

$\dfrac{y}{x}=u$　とおくと　$y'=u+xu'$

$$(u+xu')\cos u=u\cos u+1$$

$$xu'\cos u=1$$

$$\int\cos u\,du=\int\frac{1}{x}\,dx$$

$$\sin u=\log|x|+c$$

$$\sin\frac{y}{x}=\log|x|+c$$

$x=1,\ \ y=\dfrac{\pi}{2}$ を代入して

$$\sin\frac{\pi}{2}=\log 1+c \qquad 1=c$$

$\therefore\ \ \sin\dfrac{y}{x}=\log|x|+1$

⑶ $y' + \dfrac{\cos x}{\sin x} y = \dfrac{1}{\cos^2 x} \quad \left(x = \dfrac{\pi}{4},\ \ y = 2 \right)$

$$y' = \frac{\cos x}{\sin x} y = 0$$

$$\int \frac{1}{y} dy = -\int \frac{\cos x}{\sin x} dx$$

$$\log |y| = -\log |\sin x| + c$$

$$\log |y \sin x| = c$$

$$y \sin x = \pm e^c = c$$

$$y = \frac{c}{\sin x}$$

cをx の関数uとおく

$$y = \frac{u}{\sin x}, \quad y' = \frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x}$$

$$\frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x} + \frac{\cos x}{\sin x} \frac{u}{\sin x} = \frac{1}{\cos^2 x}$$

$$u' = \frac{\sin x}{\cos^2 x}$$

$$u = \frac{1}{\cos x} + c$$

$$\therefore \ y = \frac{1}{\sin x}\left(\frac{1}{\cos x} + c \right) = \frac{1}{\sin x \cos x} + \frac{c}{\sin x}$$

$x = \dfrac{\pi}{4},\ \ y = 2$ を代入すると

$$2 = \frac{1}{\frac{1}{2}} + \sqrt{2}\, c \qquad c = 0$$

$$\therefore \ y = \frac{1}{\sin x \cos x}$$

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(1) $y' = ay - 4x$

$y' = ay$

$\int \frac{1}{y}\,dy = \int a\,dx$

$\log|y| = ax + c$

$y = ce^{ax}$

cをxの関数 uとして

$y = ue^{ax}, \quad y' = u'e^{ax} + aue^{ax}$

$u'e^{ax} + aue^{ax} = aue^{ax} - 4x$

$u' = -4e^{ax}x$

$u = \frac{4}{a}e^{-ax}x - \frac{4}{a}\int e^{-ax}\,dx$

$= \frac{4}{a}e^{-ax}\left(x + \frac{1}{a}\right) + c$

$\therefore \ y = \frac{4}{a^2}(ax+1) + ce^{ax}$ （c は任意定数）

(2) $\begin{cases} y = \frac{4}{a^2}(ax+1) + ce^{ax} \\ y' = \frac{4}{a} + cae^{ax} \end{cases}$

に $x = 0,\ \ y = 2,\ \ y' = 1$ を代入

$\begin{cases} 2 = \frac{4}{a^2} + c \\ 1 = \frac{4}{a} + ca \end{cases}$ より $a = \frac{1}{2}$， $c = -14$

$\therefore \ y = 16\left(\frac{1}{2}x + 1\right) - 14e^{\frac{x}{2}}$

$= 8x - 14e^{\frac{x}{2}} + 16$

$a = \frac{1}{2}$

$y = 8x - 14e^{\frac{x}{2}} + 16$

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(1) $2xyy' - y^2 + x = 0$

$y' - \frac{1}{2x}y = -\frac{1}{2}y^{-1}$

$z = y^2$ とおくと，例題12より $z' - \frac{1}{x}z = -1$

$z' = \frac{1}{x}z$

$\int \frac{1}{z}\,dz = \int \frac{1}{x}\,dx$

$\log|z| = \log|x| + c$

$\frac{z}{x} = c$

$z = cx$

cをxの関数 uとして

$z = ux,\ \ z' = u'x + u$

$u'x + u - u = -1$

$u' = -\frac{1}{x}$

$u = -\log|x| + c$

$\therefore \ z = -x\log|x| + cx$

$\therefore \ y^2 = -x\log|x| + cx$ （c は任意定数）

(2) $y' + y = y^2(\cos x + \sin x)$

$z = y^{-1}$ とおくと，例題12より $z' - z = -(\cos x + \sin x)$

$z' - z = 0$ の一般解は $z = ce^x$

c をx の関数 uとして

$$z = ue^x,\quad z' = u'e^x + ue^x$$

$$u'e^x + ue^x - ue^x = -(\cos x + \sin x)$$

$$u' = -e^{-x}(\cos x + \sin x)$$

$$u = e^{-x}(\cos x + \sin x) - \int e^{-x}(-\sin x + \cos x)\,dx$$

$$= e^{-x}(\cos x + \sin x)$$
$$- \left(- e^{-x}(-\sin x + \cos x) + \int e^{-x}(-\cos x - \sin x)\,dx \right)$$

$$\therefore\ 2u = 2e^{-x}\cos x$$

$$\therefore\ u = e^{-x}\cos x + c$$

$$z = \cos x + ce^x$$

したがって $y = \dfrac{1}{\cos x + ce^x}$ （c は任意定数）

⑴ $x^2y' + 2xy = 1$

$$y' + \frac{2}{x}y = \frac{1}{x^2}$$

$$y' + \frac{2}{x}y = 0$$

$$\int \frac{1}{y}dy = -\int \frac{2}{x}dx$$

$$\log|y| = -2\log|x| + c$$

$$yx^2 = c$$

$$y = \frac{c}{x^2}$$

cをxの関数uとおいて　$y = \dfrac{u}{x^2}$，$y' = \dfrac{u'}{x^2} - \dfrac{2u}{x^3}$

$$\frac{u'}{x^2} - \frac{2u}{x^3} + \frac{2u}{x^3} = \frac{1}{x^2}$$

$$u' = 1$$

$$\therefore\ u = x + c$$

$$y = \frac{1}{x^2}(x + c) = \frac{1}{x} + \frac{c}{x^2}$$ （cは任意定数）

問題の式は線形方程式なので特異解は存在しない。

したがって　$\displaystyle\lim_{x\to\infty}\left(\frac{1}{x} + \frac{c}{x^2}\right) = 0$

⑵ $y(1) = y(2)$ より

$$1 + c = \frac{1}{2} + \frac{c}{4}$$

$$\therefore\ c = -\frac{2}{3}$$

$$\therefore\ y = \frac{1}{x} - \frac{2}{3x^2}$$

発展問題

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⑴ $y' = \dfrac{Ax + By + C}{ax + by + c}$

$x = X + x_0,\ y = Y + y_0$ を行うと

$$\frac{dY}{dX} = \frac{AX + BY + Ax_0 + By_0 + C}{aX + +bY + ax_0 + by_0 + c}$$

$\dfrac{A}{a} \fallingdotseq \dfrac{B}{b}$ のとき $\begin{cases} Ax_0 + By_0 + C = 0 \\ ax_0 + by_0 + c = 0 \end{cases}$

を満たす $x_0,\ y_0$ が存在するので

$\dfrac{dY}{dX} = \dfrac{AX + BY}{aX + bY}$ と同次形になる。

⑵ $\dfrac{A}{a} = \dfrac{B}{b} \fallingdotseq \dfrac{C}{c}$ のとき

$\dfrac{A}{a} = \dfrac{B}{b} = k$ とおくと

$$\frac{dy}{dx} = \frac{k(ax + by) + C}{ax + by + c}$$

$u = ax + by$ を行うと

$y' = \dfrac{u' - a}{b}$ より

$\dfrac{1}{b}\left(\dfrac{du}{dx} - a\right) = \dfrac{ku + C}{u + c}$ と変数分離形になる。

⑶ $\begin{cases} 2x + y - 3 = 0 \\ x + 2y - 5 = 0 \end{cases}$ より $x = \dfrac{1}{3},\ y = \dfrac{7}{3}$

$x = X + \dfrac{1}{3},\ y = Y + \dfrac{7}{3}$ とすると

$$\frac{dY}{dX} = \frac{2X + Y}{X + 2Y} = \frac{2 + \frac{Y}{X}}{1 + 2\frac{Y}{X}}$$

$\dfrac{Y}{X} = u$ とおくと $Y = Xu$ $Y' = u + Xu'$

$$u + Xu' = \frac{2 + u}{1 + 2u}$$

$$Xu' = \frac{2 + u}{1 + 2u} - u = \frac{2 - 2u^2}{1 + 2u}$$

$$\int \frac{1 + 2u}{1 - u^2}\,du = \int \frac{2}{X}\,dx$$

$$\int\left(\frac{-\frac{1}{2}}{(1 + u)} + \frac{\frac{3}{2}}{(1 - u)}\right)du = \int \frac{2}{X}\,dX$$

$$-\frac{1}{2}\log|1 + u| - \frac{3}{2}\log|1 - u| = 2\log|X| + c$$

$$\log\left|(1 + u)(1 - u)^3 X^4\right| = c$$

$$\therefore\ (1 + u)(1 - u)^3 X^4 = \pm e^c = c$$

$$\left(1 + \frac{Y}{X}\right)\left(1 - \frac{Y}{X}\right)^3 X^4 = c$$

$$(X + Y)(X - Y)^3 = c$$

$X = x - \dfrac{1}{3},\ Y = y - \dfrac{7}{3}$ より

$$\left(x + y - \frac{8}{3}\right)(x - y + 2)^3 = c$$

（cは任意定数）

$y = u + 1$ とおくと例題 13 より

$$u' = \{2(2x-1) - (4x-1)\}u = (2x-1)u^2$$

$$u' + u = (2x-1)u^2$$

$z = u^{-1}$ とおくと例題 12 より

$$z' - z = -(2x-1)$$

$z' - z = 0$ の一般解は $z = ce^x$

c を x の関数 v とおくと　$z = ve^x$

$$z' = v'e^x + ve^x$$

$$v'e^x + ve^x - ve^x = -2x + 1$$

$$v' = e^{-x}(-2x+1)$$

$$v = -e^{-x}(-2x+1) + \int e^{-x}(-2)\,dx$$

$$= -e^{-x}(-2x+1) + 2e^{-x} + c$$

$$= ze^{-x}x + e^{-x} + c$$

したがって　$z = 2x + 1 + ce^x$

ゆえに　$u = \dfrac{1}{2x + 1 + ce^x}$

したがって　$y = \dfrac{1}{2x + 1 + ce^x} + 1$　（c は任意定数）