新版微分積分演習 解答

1章 微分法 I

2節　導関数

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(1) $f'(-1)=\lim_{h\to 0}\dfrac{\{(-1+h)^2-(-1+h)\}-\{(-1)^2-(-1)\}}{h}=\lim_{h\to 0}\dfrac{-3h+h^2}{h}=\lim_{h\to 0}(-3+h)=-3$

(2) $f'(-1)=\lim_{h\to 0}\dfrac{\{2(-1+h)^2-1\}-\{2(-1)^2-1\}}{h}=\lim_{h\to 0}\dfrac{-4h+2h^2}{h}=\lim_{h\to 0}(-4+2h)=-4$

(3) $f'(-1)=\lim_{h\to 0}\dfrac{\{(-1+h)^3-1\}-\{(-1)^3-1\}}{h}=\lim_{h\to 0}\dfrac{3h-3h^2+h^3}{h}=\lim_{h\to 0}(3-3h+h^2)=3$

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(1) $f'(x)=\lim_{h\to 0}\dfrac{\{2(x+h)+1\}-\{2x+1\}}{h}=\lim_{h\to 0}\dfrac{2h}{h}=\lim_{h\to 0}2=2$

(2) $f'(x)=\lim_{h\to 0}\dfrac{\{(x+h)^2+(x+h)\}-\{x^2+x\}}{h}=\lim_{h\to 0}\dfrac{2xh+h^2+h}{h}=\lim_{h\to 0}(2x+h+1)=2x+1$

(3) $f'(x)=\lim_{h\to 0}\dfrac{\{(x+h)^3-(x+h)\}-\{x^3-x\}}{h}=\lim_{h\to 0}\dfrac{3x^2h+3xh^2+h^3-h}{h}$
$=\lim_{h\to 0}(3x^2+3xh+h^2-1)=3x^2-1$

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(1) $y'=6x-5$

(2) $y'=-3x^2+4x+1$

(3) $y'=(9x^2-6x+1)'=18x-6$

(4) $y'=(x^3-3x^2+2x)'=3x^2-6x+2$

(5) $y'=(x^3-2x^2+x-2)'=3x^2-4x+1$

(6) $y'=(8x^3-12x^2+6x-1)'=24x^2-24x+6$

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(1) $y'=3\cdot(x^2-2x-1)+(3x+2)(2x-2)=9x^2-8x-7$

(2) $y'=1\cdot(2x+1)(3x-1)+(x+1)\cdot 2\cdot(3x-1)+(x+1)(2x+1)\cdot 3=18x^2+14x$

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(1) $y'=-\dfrac{4}{(4x-1)^2}$

(2) $y'=\dfrac{2(x+1)-(2x-1)\cdot 1}{(x+1)^2}=\dfrac{3}{(x+1)^2}$

(3) $y'=\dfrac{1\cdot(x^2+2)-(x+1)\cdot 2x}{(x^2+2)^2}=-\dfrac{x^2+2x-2}{(x^2+2)^2}$

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(1) $y'=(x^{-4})'=-4x^{-5}=-\dfrac{4}{x^5}$

(2) $y'=(2x^{-2})'=-4x^{-3}=-\dfrac{4}{x^3}$

(3) $y'=\left(-\dfrac{1}{2}x^{-6}\right)'=3x^{-7}=\dfrac{3}{x^7}$

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(1) $u=5x+4$ とおくと $y=u^2$ であり，$\dfrac{dy}{du}=2u$，$\dfrac{du}{dx}=5$ だから　$y'=2u\cdot 5=10(5x+4)$

(2) $u=4x-1$ とおくと $y=u^3$ であり，$\dfrac{dy}{du}=3u^2$，$\dfrac{du}{dx}=4$ だから　$y'=3u^2\cdot 4=12(4x-1)^2$

(3) $u=2x^2+1$ とおくと $y=u^4$ であり，$\dfrac{dy}{du}=4u^3$，$\dfrac{du}{dx}=4x$ だから　$y'=4u^3\cdot 4x=16x(2x^2+1)^3$

(4) $u=3x^2-x+1$ とおくと $y=u^3$ であり，$\dfrac{dy}{du}=3u^2$，$\dfrac{du}{dx}=6x-1$ だから
$y'=3u^2\cdot(6x-1)=3(6x-1)(3x^2-x+1)^2$

(5) $u=x-1$ とおくと $y=u^{-2}$ であり，$\dfrac{dy}{du}=-2u^{-3}$，$\dfrac{du}{dx}=1$ だから

$y'=-2u^{-3}\cdot 1=-\dfrac{2}{(x-1)^3}$

(6) $u=x^2+3$ とおくと $y=u^{-4}$ であり，$\dfrac{dy}{du}=-4u^{-5}$，$\dfrac{du}{dx}=2x$ だから

$y'=-4u^{-5}\cdot 2x=-\dfrac{8x}{(x^2+3)^5}$

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(1) $y'=\left(x^{\frac{3}{2}}\right)'=\dfrac{3}{2}u^{\frac{1}{2}}=\dfrac{3}{2}\sqrt{x}$

(2) $y'=\left\{\left(x^2+1\right)^{\frac{1}{2}}\right\}'=\dfrac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}}\cdot 2x=\dfrac{x}{\sqrt{x^2+1}}$

(3) $y'=\left\{\left(3x^2+1\right)^{\frac{1}{3}}\right\}'=\dfrac{1}{3}\left(3x^2+1\right)^{-\frac{2}{3}}\cdot 6x=\dfrac{2x}{\sqrt[3]{(3x^2+1)^2}}$

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(1) 両辺を 2 乗して　$y^2=x+1$　　$\therefore\ x=y^2-1$　　$\therefore\ \dfrac{dx}{dy}=2y$　　$\therefore\ y'=\dfrac{1}{2y}=\dfrac{1}{2\sqrt{x+1}}$

(2) 両辺を 3 乗して　$y^3=\dfrac{27}{x}$　　$\therefore\ x=\dfrac{27}{y^3}$　　$\therefore\ \dfrac{dx}{dy}=-\dfrac{81}{y^4}$

$y'=\dfrac{1}{-\dfrac{81}{y^4}}=-\dfrac{1}{81}y^4=-\dfrac{1}{81}\cdot\dfrac{81}{\sqrt[3]{x^4}}=-\dfrac{1}{\sqrt[3]{x^4}}$

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(1) $y'=-\sin 2x\cdot 2=-2\sin 2x$

(2) $y'=\cos(1-x)\cdot(-1)=-\cos(1-x)$

(3) $y'=\dfrac{1}{\cos^2 3x}\cdot 3=\dfrac{3}{\cos^2 3x}$

(4) $y'=2\sin x\cdot\cos x=2\sin x\cos x$

(5) $y'=3\cos^2 x\cdot(-\sin x)=-3\cos^2 x\sin x$

(6) $y'=2\tan x\cdot\dfrac{1}{\cos^2 x}=\dfrac{2\tan x}{\cos^2 x}$

(7) $y'=-\dfrac{\cos x}{\sin^2 x}$

(8) $y'=-\dfrac{-\sin x}{\cos^2 x}=\dfrac{\sin x}{\cos^2 x}$

(9) $y'=\dfrac{-\sin x\cdot x-\cos x\cdot 1}{x^2}=-\dfrac{x\sin x+\cos x}{x^2}$

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(1) $y'=\dfrac{1}{\sqrt{1-(2x)^2}}\cdot 2=\dfrac{2}{\sqrt{1-4x^2}}$

(2) $y'=-\dfrac{1}{\sqrt{1-(3x)^2}}\cdot 3=-\dfrac{3}{\sqrt{1-9x^2}}$

(3) $y'=\dfrac{1}{1+(2x)^2}\cdot 2=\dfrac{2}{1+4x^2}$

(4) $y'=\dfrac{1}{\sqrt{1-\left(\dfrac{x}{3}\right)^2}}\cdot\dfrac{1}{3}=\dfrac{1}{\sqrt{9-x^2}}$

(5) $y'=\dfrac{1}{1+\left(\dfrac{x}{2}\right)^2}\cdot\dfrac{1}{2}=\dfrac{2}{4+x^2}$

(6) $y'=\dfrac{1}{1+x}\cdot\dfrac{1}{2\sqrt{x}}=\dfrac{1}{2(x+1)\sqrt{x}}$

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(1) $y'=\dfrac{1}{2x}\cdot 2=\dfrac{1}{x}$

(2) $y'=\dfrac{1}{3x+1}\cdot 3=\dfrac{3}{3x+1}$

(3) $y'=\dfrac{1}{x\log 3}$

(4) $y' = 3x^2 \log x + x^3 \cdot \dfrac{1}{x} = x^2(3\log x + 1)$

(5) $y' = 1 \cdot \log_2 x + x \cdot \dfrac{1}{x \log 2} = \dfrac{\log x}{\log 2} + \dfrac{1}{\log 2} = \dfrac{1}{\log 2}(\log x + 1)$

(6) $y' = \dfrac{\dfrac{1}{x} \cdot x - \log x \cdot 1}{x^2} = \dfrac{1 - \log x}{x^2}$

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(1) $y' = \dfrac{1}{2x-1} \cdot 2 = \dfrac{2}{2x-1}$

(2) $y' = \dfrac{1}{x^2 - x} \cdot (2x-1) = \dfrac{2x-1}{x^2-x}$

(3) $y' = \dfrac{1}{\cos x} \cdot (-\sin x) = -\tan x$

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両辺の対数をとると　$\log y = \log \dfrac{(x-2)^3}{(x-1)^2} = \log(x-2)^3 - \log(x-1)^2 = 3\log(x-2) - 2\log(x-1)$

この両辺を x で微分すると　$\dfrac{y'}{y} = \dfrac{3}{x-2} - \dfrac{2}{x-1} = \dfrac{x+1}{(x-2)(x-1)}$

$\therefore \quad y' = \dfrac{x+1}{(x-2)(x-1)} \cdot \dfrac{(x-2)^3}{(x-1)^2} = \dfrac{(x+1)(x-2)^2}{(x-1)^3}$

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(1) $y' = e^{3x+1} \cdot 3 = 3e^{3x+1}$

(2) $y' = 1 \cdot e^x + xe^x = (x+1)e^x$

(3) $y' = 2^{1-x} \log 2 \cdot (-1) = -2^{1-x} \log 2$

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(1) $y' = e^x \cos x + e^x(-\sin x) = e^x(\cos x - \sin x)$

(2) $y' = \dfrac{e^x(x+1) - e^x \cdot 1}{(x+1)^2} = \dfrac{xe^x}{(x+1)^2}$

(3) $y' = e^{-x^2} \cdot (-2x) = -2xe^{-x^2}$

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(1) $y'' = \left(6x^2 - 6x + 4\right)' = 12x - 6$

(2) $y'' = \left(\dfrac{1}{1+x^2}\right)' = -\dfrac{2x}{(1+x^2)^2}$

(3) $y'' = (\sin x + x\cos x)' = \cos x + \cos x + x(-\sin x) = 2\cos x - x\sin x$

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(1) $y''' = (5x^4 + 8x^3 - 9x^2)'' = (20x^3 + 24x^2 - 18x)' = 60x^2 + 48x - 18$

(2) $y''' = (2\cos 2x)'' = (-4\sin 2x)' = -8\cos 2x$

(3) $y''' = \left(\dfrac{3}{2}\sqrt{x}\right)'' = \left(\dfrac{3}{4\sqrt{x}}\right)' = -\dfrac{3}{8\sqrt{x^3}}$

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(1) $y' = -e^{-x}, \quad y'' = e^{-x}, \quad y''' = -e^{-x}, \quad y^{(4)} = e^{-x}, \quad \cdots \therefore \quad y^{(n)} = (-1)^n e^{-x}$

(2) $y' = e^{2x} + x \cdot 2e^{2x} = (2x+1)e^{2x}, \quad y'' = 2e^{2x} + (2x+1) \cdot 2e^{2x} = 2(2x+2)e^{2x},$

$y''' = 2^2 e^{2x} + 2(2x+2) \cdot 2e^{2x} = 2^2(2x+3)e^{2x}, \quad \cdots \quad y^{(4)} = \quad \therefore \quad y^{(n)} = 2^{n-1}(2x+n)e^{2x}$

(3) $y' = -\dfrac{1}{(x-1)^2}, \quad y'' = \dfrac{2 \cdot 1}{(x-1)^3}, \quad y''' = -\dfrac{3 \cdot 2 \cdot 1}{(x-1)^4}, \quad y^{(4)} = \dfrac{4 \cdot 3 \cdot 2 \cdot 1}{(x-1)^5} \quad \cdots$

$\therefore \quad y^{(n)} = \dfrac{(-1)^n n!}{(x-1)^{n+1}}$

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$y' = e^{-x} + x \cdot \left(-e^{-x}\right) = (1-x)e^{-x}, \quad y'' = -1 \cdot e^{-x} + (1-x) \cdot \left(-e^{-x}\right) = (x-2)e^{-x}$

$\therefore \quad y'' + 2y' + y = (x-2)e^{-x} + 2 \cdot (1-x)e^{-x} + xe^{-x} = 0$

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(1) $y' = \lim_{h\to 0} \dfrac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} = \lim_{h\to 0} \dfrac{\left(\sqrt[3]{x+h} - \sqrt[3]{x}\right)\left(\sqrt[3]{(x+h)^2} + \sqrt[3]{x+h}\sqrt[3]{x} + \sqrt[3]{x^2}\right)}{h\left(\sqrt[3]{(x+h)^2} + \sqrt[3]{x+h}\sqrt[3]{x} + \sqrt[3]{x^2}\right)}$

$= \lim_{h\to 0} \dfrac{(\cancel{x} + \cancel{h}) - \cancel{x}}{\cancel{h}\left(\sqrt[3]{(x+h)^2} + \sqrt[3]{(x+h)x} + \sqrt[3]{x^2}\right)} = \dfrac{1}{\sqrt[3]{x^2} + \sqrt[3]{x^2} + \sqrt[3]{x^2}} = \dfrac{1}{3\sqrt[3]{x^2}}$

(2) $y' = \lim_{h\to 0} \dfrac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim_{h\to 0} \dfrac{\cancel{x} - (\cancel{x} + \cancel{h})}{\cancel{h}\cdot(x+h)x} = \lim_{h\to 0} \dfrac{-1}{(x+h)x} = -\dfrac{1}{x^2}$

(3) $y' = \lim_{h\to 0} \dfrac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \dfrac{\cos x\cos h - \sin x\sin h - \cos x}{h}$

$= \lim_{h\to 0} \left(\dfrac{-\cos x(1-\cos h)}{h} - \dfrac{\sin x\sin h}{h}\right) = \lim_{h\to 0} \left(-2\cos x\cdot\dfrac{1-\cos h}{2}\cdot\dfrac{1}{h} - \sin x\cdot\dfrac{\sin h}{h}\right)$

$= \lim_{h\to 0} \left(-2\cos x\cdot\sin^2\dfrac{h}{2}\cdot\dfrac{h}{h^2} - \sin x\cdot\dfrac{\sin h}{h}\right) = \lim_{h\to 0} \left(-2\cos x\cdot\dfrac{\sin^2\frac{h}{2}}{\left(\frac{h}{2}\right)^2}\cdot\dfrac{h}{4} - \sin x\cdot\dfrac{\sin h}{h}\right)$

$= -\sin x$

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(1) $y' = 3x^2\left(x^2+1\right)^3 + x^3\cdot 3\left(x^2+1\right)^2\cdot 2x = 3x^2(x^2+1)^2(3x^2+1)$

(2) $y' = 2\left(x^4+2x^2+3\right)\cdot\left(4x^3+4x\right) = 8x(x^4+2x^2+3)(x^2+1)$

(3) $y' = \dfrac{2(1-x) - (2x+1)(-1)}{(1-x)^2} = \dfrac{3}{(1-x)^2}$

(4) $y' = \dfrac{1\cdot\sqrt{x+1} - (x-1)\cdot\dfrac{1}{2\sqrt{x+1}}}{x+1} = \dfrac{x+3}{2(x+1)\sqrt{x+1}}$

(5) $y' = \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} = -\dfrac{x}{\sqrt{1-x^2}}$

(6) $y' = \dfrac{3}{\cancel{4}}(2x^2+1)^{-\frac{1}{4}}\cdot\cancel{4}x = \dfrac{3x}{\sqrt[4]{2x^2+1}}$

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(1) 両辺の対数をとると　$\log y = \log\sqrt[3]{\dfrac{x-1}{x+1}} = \dfrac{1}{3}\Big(\log(x-1) - \log(x+1)\Big)$

この両辺を x で微分すると　$\dfrac{y'}{y} = \dfrac{1}{3}\left(\dfrac{1}{x-1} - \dfrac{1}{x+1}\right) = \dfrac{2}{3(x-1)(x+1)}$

$\therefore\quad y' = \dfrac{2}{3(x-1)(x+1)}\sqrt[3]{\dfrac{x-1}{x+1}}$

(2) 両辺の対数をとると　$\log y = \log x^{\frac{1}{x}} = \dfrac{1}{x}\log x$

この両辺を x で微分すると　$\dfrac{y'}{y} = -\dfrac{1}{x^2}\log x + \dfrac{1}{x}\cdot\dfrac{1}{x} = \dfrac{1}{x^2}(1-\log x)$

$\therefore\quad y' = x^{\frac{1}{x}}\cdot\dfrac{1}{x^2}(1-\log x) = x^{\frac{1}{x}-2}(1-\log x)$

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(1) $y' = \cos x\cdot\cos^2 x + \sin x\cdot 2\cos x(-\sin x) = \cos^3 x - 2\sin^2 x\cos x = 3\cos^3 x - 2$

(2) $y' = \left\{\log\dfrac{x^2+1}{x}\right\}' = \Big\{\log(x^2+1) - \log x\Big\}' = \dfrac{2x}{x^2+1} - \dfrac{1}{x} = \dfrac{2x^2 - (x^2+1)}{x(x^2+1)} = \dfrac{x^2-1}{x(x^2+1)}$

(3) $y' = \dfrac{(\cos x + \sin x)^2 + (\sin x - \cos x)^2}{(\sin x + \cos x)^2} = \dfrac{\cos^2 x + \cancel{2\cos x\sin x} + \sin^2 x + \sin^2 x - \cancel{2\sin x\cos x} + \cos^2 x}{(\sin x + \cos x)^2}$

$= \dfrac{2}{(\sin x + \cos x)^2}$

(4) $y' = \cos x e^{\sin x}$

(5) $y' = 2e^{2x}\sin^2 x + e^{2x}2\sin x\cos x = 2e^x\sin x(\sin x + \cos x)$

(6) $y' = \Big(\log|1-\cos x| - \log|1+\cos x|\Big)' = \dfrac{\sin x}{1-\cos x} - \dfrac{-\sin x}{1+\cos x}$

$= \dfrac{\sin x + \cancel{\sin x\cos x} + \sin x - \cancel{\sin x\cos x}}{1-\cos^2 x} = \dfrac{2\sin x}{\sin^2 x} = \dfrac{2}{\sin x}$

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(1) $y' = \dfrac{1}{2}\left(\sqrt{x^2+1} + \dfrac{x^2}{\sqrt{x^2+1}} + \dfrac{1+\dfrac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}\right)$

$= \dfrac{1}{2}\left(\sqrt{x^2+1} + \dfrac{x^2}{\sqrt{x^2+1}} + \dfrac{\cancel{\sqrt{x^2+1}+x}}{\cancel{\left(x+\sqrt{x^2+1}\right)}\sqrt{x^2+1}}\right) = \dfrac{1}{2}\left(\sqrt{x^2+1} + \dfrac{x^2+1}{\sqrt{x^2+1}}\right)$

$= \dfrac{1}{2}\left(\sqrt{x^2+1} + \sqrt{x^2+1}\right) = \sqrt{x^2+1}$

(2) $y' = \dfrac{1}{2}\left(\sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}} + \dfrac{1}{\sqrt{1-x^2}}\right) = \dfrac{1}{2}\left(\sqrt{1-x^2} + \dfrac{1-x^2}{\sqrt{1-x^2}}\right)$

$= \dfrac{1}{2}\left(\sqrt{1-x^2} + \sqrt{1-x^2}\right) = \sqrt{1-x^2}$

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(1) ① $n=1$ のとき

$y' = -\sin x = \cos\left(x+\dfrac{\pi}{2}\right)$ だから，$n=1$ のとき成り立つ

② $n=k$ のとき成り立つとすると

$y^{(k+1)} = \{y^{(k)}\}' = \left\{\cos\left(x+\dfrac{k\pi}{2}\right)\right\}' = -\sin\left(x+\dfrac{k\pi}{2}\right) = \cos\left\{\left(x+\dfrac{k\pi}{2}\right)+\dfrac{\pi}{2}\right\}$

$= \cos\left(x+\dfrac{(k+1)\pi}{2}\right)$ だから，$n=k+1$ のときも成り立つ

以上より，数学的帰納法によって，任意の自然数 n について $y^{(n)} = \cos\left(x+\dfrac{n\pi}{2}\right)$ が成り立つ

(2) ① $n=1$ のとき

$y' = 2\cos(2x+1) = 2\sin\left(2x+1+\dfrac{\pi}{2}\right)$ だから，$n=1$ のとき成り立つ

② $n=k$ のとき成り立つとすると

$y^{(k+1)} = \{y^{(k)}\}' = \left\{2^k\sin\left(2x+1+\dfrac{k\pi}{2}\right)\right\}' = 2^k\cdot 2\cos\left(2x+1+\dfrac{k\pi}{2}\right)$

$= 2^{k+1}\sin\left\{\left(2x+1+\dfrac{k\pi}{2}\right)+\dfrac{\pi}{2}\right\} = 2^{k+1}\sin\left(2x+1+\dfrac{(k+1)\pi}{2}\right)$

だから，$n=k+1$ のときも成り立つ

以上より，数学的帰納法によって，任意の自然数 n について $y^{(n)} = \sin\left(2x+1+\dfrac{n\pi}{2}\right)$ が成り立つ

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(1) [証明]

左辺 $= (\sinh x)' = \left(\dfrac{e^x-e^{-x}}{2}\right)' = \dfrac{e^x+e^{-x}}{2} = \cosh x =$ 右辺 [証明終]

(2) [証明]

左辺 $= (\cosh x)' = \left(\dfrac{e^x+e^{-x}}{2}\right)' = \dfrac{e^x-e^{-x}}{2} = \sinh x =$ 右辺 [証明終]

(3) [証明]

左辺 $= \cosh^2 x - \sinh^2 x = \left(\dfrac{e^x+e^{-x}}{2}\right)^2 - \left(\dfrac{e^x-e^{-x}}{2}\right)^2 = \dfrac{(\cancel{e^{2x}}+2+\cancel{e^{-x}}) - (\cancel{e^{2x}}-2+\cancel{e^{-x}})}{4}$

$= 1 =$ 右辺 [証明終]

(4) [証明]

左辺 $= (\tanh x)' = \left(\dfrac{\sinh x}{\cosh x}\right)' = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} =$ 右辺 [証明終]