6章 1節 重積分

243 **(1)** $\displaystyle\int_0^3\int_0^1 (4-y^2)\,dy\,dx$
$\displaystyle=\int_0^3\Big[4y-\frac{y^3}{3}\Big]_0^1 dx$
$\displaystyle=\int_0^3\left(4-\frac{1}{3}\right)\,dx$
$\displaystyle=\frac{11}{3}\int_0^3 dx$
$=\mathbf{11}$

(2) $\displaystyle\int_{-1}^1\int_{-1}^0 (x+y+1)\,dy\,dx$
$\displaystyle=\int_{-1}^1\Big[(x+1)y+\frac{y^2}{2}\Big]_{-1}^0 dx$
$\displaystyle=\int_{-1}^1\left(x+\frac{1}{2}\right)\,dx$
$\displaystyle=2\int_0^1\frac{1}{2}\,dx$
$\displaystyle=\int_0^1 dx$
$=\mathbf{1}$

(3) $\displaystyle\int_{-2}^{-1}\int_0^1 \big(\sin(\pi y)+\cos(\pi x)\big)\,dy\,dx$
$\displaystyle=\int_{-2}^{-1}\Big[-\frac{1}{\pi}\cos(\pi y)$
$\displaystyle\qquad+y\cos(\pi x)\Big]_0^1 dx$
$\displaystyle=\int_{-2}^{-1}\left(\frac{2}{\pi}+\cos(\pi x)\right)\,dx$
$\displaystyle=\Big[\frac{2}{\pi}x+\frac{1}{\pi}\sin(\pi x)\Big]_{-2}^{-1}$
$\displaystyle=\boldsymbol{\frac{2}{\pi}}$

(4) $\displaystyle\int_{-1}^2\int_0^1 xe^{xy}\,dy\,dx$
$\displaystyle=\int_{-1}^2\Big[e^{xy}\Big]_0^1 dx$
$\displaystyle=\int_{-1}^2 (e^x-1)\,dx$
$\displaystyle=\Big[e^x-x\Big]_{-1}^2$
$=\boldsymbol{e^2-e^{-1}-3}$

244 **(1)** $\displaystyle\int_{-2}^0\int_0^{x+2} dy\,dx$
$\displaystyle=\int_{-2}^0\Big[y\Big]_0^{x+2} dx$
$\displaystyle=\int_{-2}^0 (x+2)\,dx$
$\displaystyle=\Big[\frac{(x+2)^2}{2}\Big]_{-2}^0$
$=\mathbf{2}$

(2) $\displaystyle\int_0^1\int_0^{1-x} (x^2+y^2)\,dy\,dx$
$\displaystyle=\int_0^1\Big[x^2y+\frac{y^3}{3}\Big]_0^{1-x} dx$
$\displaystyle=\int_0^1\left(x^2(1-x)+\frac{(1-x)^3}{3}\right)\,dx$
$\displaystyle=\int_0^1\left(x^2-x^3+\frac{(1-x)^3}{3}\right)\,dx$
$\displaystyle=\Big[\frac{x^3}{3}-\frac{x^4}{4}-\frac{(1-x)^4}{12}\Big]_0^1$
$\displaystyle=\left(\frac{1}{3}-\frac{1}{4}\right)-\left(-\frac{1}{12}\right)$
$\displaystyle=\boldsymbol{\frac{1}{6}}$

(3) $\displaystyle\int_1^2\int_x^{2x}\frac{x}{y}\,dy\,dx$
$\displaystyle=\int_1^2\Big[x\log y\Big]_x^{2x} dx$
$\displaystyle=(\log 2)\int_1^2 x\,dx$
$\displaystyle=\log 2\cdot\Big[\frac{x^2}{2}\Big]_1^2$
$\displaystyle=\boldsymbol{\frac{3}{2}\log 2}$

(4) $\displaystyle\int_0^\pi\int_0^{\sin x} 2y\,dy\,dx$
$\displaystyle=\int_0^\pi\Big[y^2\Big]_0^{\sin x} dx$
$\displaystyle=\int_0^\pi\sin^2 x\,dx$
$\displaystyle=\frac{1}{2}\int_0^\pi(1-\cos 2x)\,dx$
$\displaystyle=\frac{1}{2}\Big[x-\frac{1}{2}\sin 2x\Big]_0^\pi$
$\displaystyle=\boldsymbol{\frac{\pi}{2}}$

245 **(1)** $\displaystyle\int_0^1\int_{y^2}^y dx\,dy$
$\displaystyle=\int_0^1\Big[x\Big]_{y^2}^y dy$
$\displaystyle=\int_0^1 (y-y^2)\,dy$
$\displaystyle=\Big[\frac{y^2}{2}-\frac{y^3}{3}\Big]_0^1$
$\displaystyle=\boldsymbol{\frac{1}{6}}$

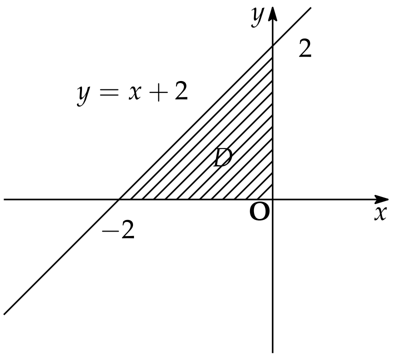
(2) $\displaystyle\int_\pi^{2\pi}\int_0^\pi (\sin x+\cos y)\,dx\,dy$
$\displaystyle=\int_\pi^{2\pi}\Big[-\cos x+x\cos y\Big]_0^\pi dy$
$\displaystyle=\int_\pi^{2\pi}(\pi\cos y+2)\,dy$

$$= \Big[\pi \sin y + 2y\Big]_{\pi}^{2\pi}$$
$$= \mathbf{2\pi}$$

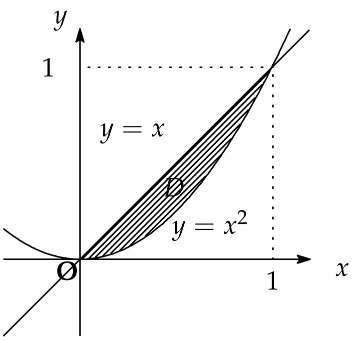
(3) $\displaystyle\int_0^1 \int_0^{\pi} y\cos(xy)\,dx\,dy$
$$= \int_0^1 \Big[\sin(xy)\Big]_0^{\pi} dy$$
$$= \int_0^1 \sin(\pi y)\,dy$$
$$= \Big[-\frac{1}{\pi}\cos(\pi y)\Big]_0^1$$
$$= \boldsymbol{\frac{2}{\pi}}$$

(4) $\displaystyle\int_0^1 \int_0^1 e^{x+y}\,dx\,dy$
$$= \int_0^1 e^x\,dx \cdot \int_0^1 e^y\,dy$$
$$= \Big[e^x\Big]_0^1 \cdot \Big[e^y\Big]_0^1$$
$$= \boldsymbol{(e-1)^2}$$

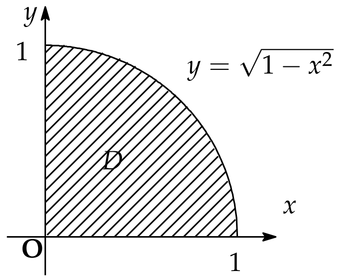
246 **(1)** $D = \{(x,\, y)\,|\, -2 \leqq x \leqq 0,\ 0 \leqq y \leqq x+2\}$



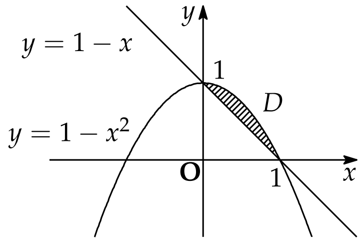
(2) $D = \{(x,\, y)\,|\, 0 \leqq x \leqq 1,\ x^2 \leqq y \leqq x\}$



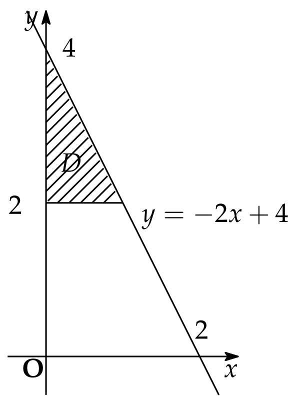
(3) $D = \{(x,\, y)\,|\, 0 \leqq x \leqq 1,\ 0 \leqq y \leqq \sqrt{1-x^2}\}$



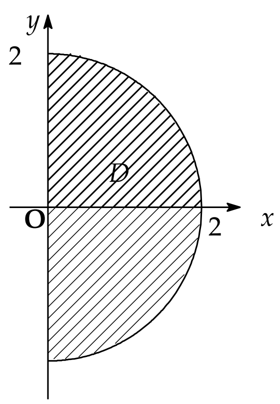
(4) $D = \{(x,\, y)\,|\, 0 \leqq y \leqq 1,\ 1-y \leqq x \leqq \sqrt{1-y}\}$



(5) $D = \{(x,\, y)\,|\, 2 \leqq y \leqq 4,\ x \geqq 0,\ y \leqq 4-2x\}$



(6) $D = \{(x,\, y)\,|\, -2 \leqq y \leqq 2,\ 0 \leqq x \leqq \sqrt{4-y^2}\}$



247 **(1)** $\displaystyle\int_0^{\pi}\int_x^{\pi}\frac{\cos y}{y}\,dy\,dx$

$\displaystyle=\int_0^{\pi}\frac{\cos y}{y}\left(\int_0^{y}dx\right)dy$

$\displaystyle=\int_0^{\pi}\frac{\cos y}{y}\cdot y\,dy$

$\displaystyle=\int_0^{\pi}\cos y\,dy$

$\displaystyle=\Big[\sin y\Big]_0^{\pi}$

$=\sin\pi-\sin 0$

$=\mathbf{0}$

(2) $\displaystyle\int_0^1\int_x^1 y^2\,\sin(\pi xy)\,dy\,dx$

$\displaystyle=\int_0^1\left(\int_0^{y}y^2\,\sin(\pi xy)\,dx\right)dy$

$\displaystyle=\int_0^1\left[-\frac{y}{\pi}\cos(\pi xy)\right]_0^{y}dy$

$\displaystyle=\frac{1}{\pi}\int_0^{\pi}\Big(y-y\cos(\pi y^2)\Big)\,dy$

$\displaystyle=\frac{1}{\pi}\left[\frac{y^2}{2}-\frac{1}{2\pi}\sin(\pi y^2)\right]_0^1$

$\displaystyle=\frac{1}{\pi}\left(\frac{1}{2}-\frac{1}{2\pi}\sin(\pi)\right)$

$\displaystyle=\boldsymbol{\frac{1}{2\pi}}$

(3) $\displaystyle\int_0^1\int_y^1 x^2e^{xy}\,dx\,dy$

$\displaystyle=\int_0^1\left(\int_0^{x}x^2e^{xy}\,dy\right)dx$

$\displaystyle=\int_0^1\Big[xe^{xy}\Big]_0^{x}dx$

$\displaystyle=\int_0^1(xe^{x^2}-x)\,dx$

$\displaystyle=\left[\frac{e^{x^2}}{2}-\frac{x^2}{2}\right]_0^1$

$\displaystyle=\frac{e}{2}-\frac{1}{2}-\frac{1}{2}$

$\displaystyle=\boldsymbol{\frac{e}{2}-1}$

(4) $\displaystyle\int_0^2\int_{\frac{y}{2}}^1 e^{x^2}\,dx\,dy$

$\displaystyle=\int_0^1 e^{x^2}\left(\int_0^{2x}dy\right)dx$

$\displaystyle=\int_0^1 2xe^{x^2}\,dx$

$\displaystyle=\Big[e^{x^2}\Big]_0^1$

$=\boldsymbol{e-1}$

248 **(1)** $\displaystyle\iint_D\sqrt{x^2+y^2}\,dx\,dy$

$\displaystyle=\int_0^1\int_0^{2\pi}r\cdot r\,dr\,d\theta$

$\displaystyle=\int_0^{2\pi}d\theta\cdot\int_0^1 r^2\,dr$

$\displaystyle=2\pi\left[\frac{r^3}{3}\right]_0^1$

$\displaystyle=\boldsymbol{\frac{2}{3}\pi}$

(2) $\displaystyle\iint_D x\,dx\,dy$

$\displaystyle=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r\cos\theta\cdot r\,dr\,d\theta$

$\displaystyle=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\,d\theta\cdot\int_0^1 r^2\,dr$

$\displaystyle=\Big[\sin\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\left[\frac{r^3}{3}\right]_0^1$

$\displaystyle=\boldsymbol{\frac{2}{3}}$

249 **(1)** $\displaystyle\iint_D 6xy^2\,dx\,dy$

$\displaystyle=\int_2^4\int_1^2 6xy^2\,dy\,dx$

$\displaystyle=\int_2^4 2x\,dx\cdot\int_1^2 3y^2\,dy$

$\displaystyle=\Big[x^2\Big]_2^4\cdot\Big[y^3\Big]_1^2$

$=(16-4)\cdot(8-1)$

$=\mathbf{84}$

(2) $\displaystyle\iint_D\frac{1}{(2x+3y)^2}\,dx\,dy$

$\displaystyle=\int_0^1\int_1^2(2x+3y)^{-2}\,dy\,dx$

$\displaystyle=\int_0^1\left[-\frac{1}{3}(2x+3y)^{-1}\right]_1^2dx$

$\displaystyle=\frac{1}{3}\int_0^1\left(\frac{1}{2x+3}-\frac{1}{2x+6}\right)dx$

$\displaystyle=\frac{1}{6}\Big[\log(2x+3)-\log(2x+6)\Big]_0^1$

$\displaystyle=\frac{1}{6}\left(\log 5-\log 8-\log 3+\log 6\right)$

$\displaystyle=\boldsymbol{\frac{1}{6}\log\left(\frac{5}{4}\right)}$

(3) $\displaystyle\iint_D x\cos^2 y\,dx\,dy$

$\displaystyle=\int_{-2}^3 x\,dx\cdot\int_0^{\frac{\pi}{2}}\cos^2 y\,dy$

$\displaystyle=\int_{-2}^3 x\,dx\cdot\int_0^{\frac{\pi}{2}}\frac{1+\cos 2y}{2}\,dy$

$\displaystyle=\left[\frac{x^2}{2}\right]_{-2}^3\cdot\frac{1}{2}\left[y+\frac{1}{2}\sin 2y\right]_0^{\frac{\pi}{2}}$

$$= \frac{9-4}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$
$$= \boldsymbol{\frac{5}{8}\pi}$$

(4) $\displaystyle\iint_D e^{\frac{x}{y}}\,dx\,dy$
$$= \int_1^2 \int_y^{y^3} e^{\frac{x}{y}}\,dx\,dy$$
$$= \int_1^2 \left[ye^{\frac{x}{y}}\right]_y^{y^3} dy$$
$$= \int_1^2 (ye^{y^2} - ey)\,dy$$
$$= \left[\frac{1}{2}e^{y^2} - \frac{e}{2}y^2\right]_1^2$$
$$= \boldsymbol{\frac{e^4}{2} - 2e}$$

(5) $\displaystyle\iint_D 4xy\,dx\,dy$
$$= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy\,dy\,dx$$
$$= \int_0^1 \left[2xy^2\right]_{x^3}^{\sqrt{x}} dx$$
$$= \int_0^1 (2x^2 - 2x^7)\,dx$$
$$= \left[\frac{2}{3}x^2 - \frac{2}{8}x^8\right]_0^1$$
$$= \frac{2}{3} - \frac{2}{8}$$
$$= \boldsymbol{\frac{5}{12}}$$

(6) $\displaystyle\iint_D dx\,dy$
$$= \int_1^3 \int_{-\frac{y}{2}+\frac{3}{2}}^{2y-1} dx\,dy$$
$$= \int_1^3 \left(2y - 1 + \frac{y}{2} - \frac{3}{2}\right) dy$$
$$= \int_1^3 \left(\frac{5}{2}y - \frac{5}{2}\right) dy$$
$$= \left[\frac{5}{4}(y-1)^2\right]_1^3$$
$$= \mathbf{5}$$

(7) $\displaystyle\iint_D x^3 e^{y^3}\,dx\,dy$
$$= \int_0^1 \int_{x^2}^1 x^3 e^{y^3}\,dy\,dx$$
$$= \int_0^1 \int_0^{\sqrt{y}} x^3 e^{y^3}\,dx\,dy$$
$$= \int_0^1 \left[\frac{1}{4}x^4 e^{y^3}\right]_0^{\sqrt{y}} dy$$
$$= \int_0^1 \frac{1}{4}y^2 e^{y^3}\,dy$$
$$= \left[\frac{1}{12}e^{y^3}\right]_0^1$$
$$= \boldsymbol{\frac{1}{12}(e-1)}$$

250 **(1)** $\displaystyle\iint_D \sqrt{9 - x^2 - y^2}\,dx\,dy$
$$= \int_0^{2\pi} \int_0^{\sqrt{5}} \sqrt{9-r^2} \cdot r\,dr\,d\theta$$
$$= 2\pi \int_0^{\sqrt{5}} (9-r^2)^{\frac{1}{2}} r\,dr\,d\theta$$
$$= -\pi \int_0^{\sqrt{5}} (9-r^2)^{\frac{1}{2}} (-2r)\,dr\,d\theta$$
$$= -\pi \left[\frac{2}{3}(9-r^2)^{\frac{3}{2}}\right]_0^{\sqrt{5}}$$
$$= -\frac{2}{3}\pi \left((9-5)^{\frac{3}{2}} - (9-0)^{\frac{3}{2}}\right)$$
$$= -\frac{2}{3}\pi \left(4^{\frac{3}{2}} - 9^{\frac{3}{2}}\right)$$
$$= -\frac{2}{3}\pi \left(2^3 - 3^3\right)$$
$$= \boldsymbol{\frac{38}{3}\pi}$$

(2) $\displaystyle\iint_D \cos(x^2+y^2)\,dx\,dy$
$$= \int_0^{\frac{\pi}{2}} \int_0^1 \cos(r^2) \cdot r\,dr\,d\theta$$
$$= \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^1 r\cos(r^2)\,dr$$
$$= \frac{\pi}{2}\left[\frac{1}{2}\sin(r^2)\right]_0^1$$
$$= \boldsymbol{\frac{\pi}{4}\sin 1}$$

(3) $\displaystyle\iint_D \frac{2}{1+\sqrt{x^2+y^2}}\,dx\,dy$
$$= \int_\pi^{\frac{3}{2}\pi} \int_0^1 \frac{2}{1+r} \cdot r\,dr\,d\theta$$
$$= \int_\pi^{\frac{3}{2}\pi} d\theta \cdot \int_0^1 \frac{2r}{1+r}\,dr$$
$$= \frac{\pi}{2}\int_0^1 2\left(1 - \frac{1}{1+r}\right) dr$$
$$= \pi\Big[r - \log(1+r)\Big]_0^1$$
$$= \boldsymbol{\pi(1 - \log 2)}$$

(4) $\displaystyle\iint_D e^{-(x^2+y^2)}\,dx\,dy$
$$= \int_0^{\frac{\pi}{2}} \int_0^1 e^{-r^2} \cdot r\,dr\,d\theta$$
$$= \frac{\pi}{2}\int_0^1 re^{-r^2}\,dr$$
$$= \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^1$$
$$= \boldsymbol{\frac{\pi}{4}(1 - e^{-1})}$$

(5) $\displaystyle\iint_D \log(x^2+y^2+1)\,dx\,dy$

$\displaystyle = \int_0^{2\pi}\int_0^1 \log(r^2+1)\cdot r\,dr\,d\theta$

$\displaystyle = 2\pi\int_0^1 r\log(r^2+1)\,dr$

$\displaystyle = \pi\int_0^1 (2r)\log(r^2+1)\,dr$

ここで，$t=r^2+1$ と置換

$\displaystyle = \pi\int_1^2 \log t\,dt$

$\displaystyle = \pi\Big[t\log t - t\Big]_1^2$

$= \boldsymbol{\pi(2\log 2 - 1)}$

(6) $\displaystyle\iint_D \frac{1}{1+x^2+y^2}\,dx\,dy$

$\displaystyle = \int_0^{2\pi}\int_0^1 \frac{1}{1+r^2}\cdot r\,dr\,d\theta$

$\displaystyle = 2\pi\int_0^1 \frac{r}{1+r^2}\,dr$

$\displaystyle = \pi\Big[\log(1+r^2)\Big]_0^1$

$= \boldsymbol{\pi\log 2}$

(7) 領域 D は極座標変換で，領域

$D' = \{(r,\theta)\,|\,0 \leqq r \leqq 2\cos\theta,$

$-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}\}$ に写るので，

$\displaystyle\iint_D (x+y)\,dx\,dy$

$\displaystyle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r(\cos\theta + \sin\theta)r\,dr\,d\theta$

$\displaystyle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^{2\cos\theta} d\theta$

$\displaystyle = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta+\sin\theta)\cos^3\theta\,d\theta$

$\displaystyle = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^4\theta+\sin\theta\cos^3\theta)\,d\theta$

$\displaystyle = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta$

$\displaystyle = \frac{16}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$

Wallis の公式から

$\displaystyle = \frac{16}{3}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$

$= \boldsymbol{\pi}$

251 $\begin{cases} u = x-y \\ v = x+y \end{cases}$ とおく。このとき，領域 D は $D' = \{(u,v)\,|\,0 \leqq u \leqq \pi,\ 0 \leqq v \leqq \pi\}$ に写り，また，ヤコビ行列式の絶対値は $|J(u,v)| = \dfrac{1}{2}$ となるので

$\displaystyle\iint_D (x-y)\sin(x+y)\,dxdy$

$\displaystyle = \frac{1}{2}\int_0^{\pi}\int_0^{\pi} u\sin v\,dudv$

$\displaystyle = \frac{1}{2}\int_0^{\pi} u\,du\cdot\int_0^{\pi}\sin v\,dv$

$\displaystyle = \frac{1}{2}\cdot\left[\frac{u^2}{2}\right]_0^{\pi}\cdot\Big[-\cos v\Big]_0^{\pi}$

$\displaystyle = \frac{1}{2}\cdot\frac{\pi^2}{2}\cdot 2$

$\displaystyle = \boldsymbol{\frac{\pi^2}{2}}$

252 領域 D は極座標変換で，領域 $D' = \{(r,\theta)\,|\,0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq \frac{\pi}{2}\}$ に写るので，

$D'_\epsilon = \{(r,\theta)\,|\,0 < \epsilon \leqq r \leqq 1,\ 0 \leqq \theta \leqq \frac{\pi}{2}\}$

を考える。

$\displaystyle\iint_D \mathbf{Tan}^{-1}\left(\frac{y}{x}\right)dx\,dy$

$\displaystyle = \lim_{\epsilon\to+0}\int_0^{\frac{\pi}{2}}\int_\epsilon^1 \mathbf{Tan}^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right)\cdot r\,dr\,d\theta$

$\displaystyle = \lim_{\epsilon\to+0}\int_0^{\frac{\pi}{2}}\int_\epsilon^1 \theta\cdot r\,dr\,d\theta$

$\displaystyle = \lim_{\epsilon\to+0}\int_0^{\frac{\pi}{2}}\theta\,d\theta\cdot\int_\epsilon^1 r\,dr$

$\displaystyle = \lim_{\epsilon\to+0}\left[\frac{\theta^2}{2}\right]_0^{\frac{\pi}{2}}\cdot\left[\frac{r^2}{2}\right]_\epsilon^1$

$\displaystyle = \lim_{\epsilon\to+0}\frac{\pi^2}{8}\cdot\frac{1^2-\epsilon^2}{2}$

$\displaystyle = \boldsymbol{\frac{\pi^2}{16}}$

253 領域 D は極座標変換で，領域 $D' = \{(r,\theta)\,|\,0 \leqq r \leqq 1,\ 0 \leqq \theta \leqq \frac{\pi}{2}\}$ に写るので，

$D'_\epsilon = \{(r,\theta)\,|\,0 < \epsilon \leqq r \leqq 1,\ 0 \leqq \theta \leqq \frac{\pi}{2}\}$

を考える。

$\displaystyle\iint_D \log\sqrt{x^2+y^2}\,dx\,dy$

$\displaystyle = \lim_{\epsilon\to+0}\int_0^{\frac{\pi}{2}}\int_\epsilon^1 \log\sqrt{r}\cdot r\,dr\,d\theta$

$\displaystyle = \lim_{\epsilon\to+0}\frac{\pi}{4}\int_\epsilon^1 r\log r\,dr$

$$= \lim_{\epsilon \to +0} \frac{\pi}{4} \left[\frac{r^2}{2} \log r - \frac{r^2}{4} \right]_\epsilon^1$$
$$= \lim_{\epsilon \to +0} \frac{\pi}{4} \left(\frac{1}{2} \log 1 - \frac{1}{4} - \frac{\epsilon^2}{2} \log \epsilon + \frac{\epsilon^2}{4} \right)$$
ここで $\lim_{\epsilon \to +0} \epsilon^2 \log \epsilon = 0$ であるから
$$= -\boldsymbol{\frac{\pi}{8}}$$

２節 重積分の応用

254 **(1)** $V = \int_0^1 \int_0^1 (x+y)\, dx\, dy$
$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^1 dx$$
$$= \int_0^1 \left(x + \frac{1}{2} \right) dx$$
$$= \left[\frac{x^2 + x}{2} \right]_0^1$$
$$= \mathbf{1}$$

(2) $V = \int_0^1 \int_0^1 x^2\, dx\, dy$
$$= \int_0^1 dy \cdot \int_0^1 x^2\, dx$$
$$= 1 \cdot \left[\frac{x^3}{3} \right]_0^1$$
$$= \boldsymbol{\frac{1}{3}}$$

(3) $V = \int_0^1 \int_0^{x^2} (x+y)\, dx\, dy$
$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{x^2} dx$$
$$= \int_0^1 \left(x^3 + \frac{x^4}{2} \right) dx$$
$$= \left[\frac{x^4}{4} + \frac{x^5}{10} \right]_0^1$$
$$= \frac{1}{4} + \frac{1}{10}$$
$$= \boldsymbol{\frac{7}{20}}$$

255 **(1)** 曲面 $z = x^2 + y^2$ と 平面 $z = 1$ との交わりは $x^2 + y^2 = 1$ となる。

∴求める体積 V と積分領域 D は
$$V = \iint_D \Big(1 - (x^2 + y^2) \Big)\, dxdy,$$
$$D = \Big\{ (x,\, y) \,|\, x^2 + y^2 \leqq 1 \Big\}$$
極座標に変換すれば
$$V = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, drd\theta$$
$$= 2\pi \int_0^1 (r - r^3)\, dr$$
$$= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1$$
$$= 2\pi \left(\frac{1}{2} - \frac{1}{4} \right)$$
$$= \boldsymbol{\frac{\pi}{2}}$$

(2) 曲面 $z = x^2 + y^2$ と 平面 $z = 2x$ との交わりは $x^2 + y^2 = 2x$ より $(x-1)^2 + y^2 = 1$ であるから，求める体積 V と積分領域 D は
$$V = \iint_D \Big(2x - (x^2 + y^2) \Big)\, dxdy,$$
$$D = \Big\{ (x,\, y) \,|\, (x-1)^2 + y^2 \leqq 1 \Big\}$$
極座標に変換すると D は
$$D' = \{ (r,\, \theta) \,|\, 0 \leqq r \leqq 2\cos\theta,\; -\tfrac{\pi}{2} \leqq \theta \leqq \tfrac{\pi}{2} \}$$
に写るので
$$V = \iint_D \Big(2x - (x^2 + y^2) \Big)\, dxdy,$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (2r\cos\theta - r^2) \cdot r\, dr\, d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (2r^2 \cos\theta - r^3)\, dr$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2}{3} r^3 \cos\theta - \frac{r^4}{4} \right]_0^{2\cos\theta} d\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{16}{3} \cos^4\theta - 4\cos^4\theta \right) d\theta$$
$$= \frac{4}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$
$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$
$$= \frac{8}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$
$$= \boldsymbol{\frac{\pi}{2}}$$

(3) $x^2 + y^2 + z^2 = 2^2 = 4$ より $z = \pm\sqrt{4 - x^2 - y^2}$ であるから，求める体積 V と積分領域 D は
$$V = \iint_D \left\{ \sqrt{4 - x^2 - y^2} - (-\sqrt{4 - x^2 - y^2}) \right\} dxdy$$
$$= 2 \iint_D \sqrt{4 - x^2 - y^2}\, dxdy$$
$$D = \Big\{ (x,\, y) \,|\, x^2 + y^2 \leqq 1 \Big\}$$
極座標に変換して
$$V = 2 \int_0^{2\pi} \int_0^1 (4 - r^2)^{\frac{1}{2}}\, r\, dr\, d\theta$$

$$= 4\pi\int_0^1 (4-r^2)^{\frac{1}{2}} r\,dr$$
$$= 4\pi\left[-\frac{1}{3}(4-r^2)^{\frac{3}{2}}\right]_0^1$$
$$= \boldsymbol{\frac{4}{3}(8-3\sqrt{3})\pi}$$

256 **(1)** $\displaystyle\int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$
$$= \int_0^{+\infty} e^{-x^2}\,dx \cdot \int_0^{+\infty} e^{-y^2}\,dy$$
$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\sqrt{\pi}}{2}$$
$$= \boldsymbol{\frac{\pi}{4}}$$

(2) 極座標に変換して
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x^2+y^2)e^{-(x^2+y^2)}\,dx\,dy$$
$$= \int_0^{2\pi}\int_0^{+\infty} r^2e^{-r^2}r\,dr\,d\theta$$
$$= 2\pi\int_0^{+\infty} r^2e^{-r^2}r\,dr$$ $t=r^2$ と置換して
$$= \pi\int_0^{+\infty} te^{-t}\,dt$$
$$= \pi\int_0^{+\infty} t(-e^{-t})'\,dt$$
$$= \pi\left(\left[-te^{-t}\right]_0^{+\infty} + \int_0^{+\infty} e^{-t}\,dt\right)$$
$$= \pi\left[-e^{-t}\right]_0^{+\infty}$$
$$= \boldsymbol{\pi}$$

257 質量 M は
$$M = \int_0^1\int_x^1 (x+y)\,dy\,dx$$
$$= \int_0^1\left[xy+\frac{y^2}{2}\right]_x^1 dx$$
$$= \int_0^1\left(\frac{1}{2}+x-\frac{x^3}{2}\right)\,dx$$
$$= \left[\frac{x}{2}+\frac{x^2}{2}-\frac{x^3}{2}\right]_0^1$$
$$= \frac{1}{2}$$
$$M_x = \int_0^1\int_x^1 x(x+y)\,dy\,dx$$
$$= \int_0^1\left[x^2y+x\frac{y^2}{2}\right]_x^1 dx$$
$$= \int_0^1\left(\frac{x}{2}+x^2-\frac{3x^3}{2}\right)\,dx$$
$$= \left[\frac{x^2}{4}+\frac{x^3}{3}-\frac{3x^4}{8}\right]_0^1$$
$$= \frac{5}{24}$$
$$M_y = \int_0^1\int_x^1 y(x+y)\,dy\,dx$$
$$= \int_0^1\left[\frac{xy^2}{2}+\frac{y^3}{3}\right]_x^1 dx$$
$$= \int_0^1\left(\frac{1}{3}+\frac{x}{2}-\frac{5x^3}{6}\right)\,dx$$
$$= \left[\frac{x}{3}+\frac{x^2}{4}-\frac{5x^4}{24}\right]_0^1$$
$$= \frac{9}{24}$$
以上から重心 G の座標は
$$G = \left(\frac{M_x}{M},\,\frac{M_y}{M}\right)$$
$$= \boldsymbol{\left(\frac{5}{12},\,\frac{3}{4}\right)}$$

258 **(1)** $x^2+y^2+z^2=a^2$ より
$z=\sqrt{a^2-x^2-y^2}$ であるから
$$z_x = -x(a^2-x^2-y^2)^{-\frac{1}{2}}$$
$$z_y = -y(a^2-x^2-y^2)^{-\frac{1}{2}}$$
$$\therefore\quad \sqrt{(z_x)^2+(z_y)^2+1} = a(a^2-x^2-y^2)^{-\frac{1}{2}}$$
従って，求める曲面積 S は
$$S = \iint_D a(a^2-x^2-y^2)^{-\frac{1}{2}}\,dxdy$$
$$D = \left\{(x,\,y)\mid x^2+y^2 \leqq a^2-b^2\right\}$$
となる。極座標に変換すると
$$S = \int_0^{2\pi}\int_0^{\sqrt{a^2-b^2}} a(a^2-r^2)^{-\frac{1}{2}}\cdot r\,dr\,d\theta$$
$$= 2\pi a\int_0^{\sqrt{a^2-b^2}}(a^2-r^2)^{-\frac{1}{2}}\cdot r\,dr$$
$$= -\pi a\int_0^{\sqrt{a^2-b^2}}(a^2-r^2)^{-\frac{1}{2}}\cdot(-2r)\,dr$$
$$= -\pi a\left[2(a^2-r^2)^{\frac{1}{2}}\right]_0^{\sqrt{a^2-b^2}}$$
$$= -2\pi a\left(\sqrt{b^2}-\sqrt{a^2}\right)$$
$$= \boldsymbol{2a(a-b)\pi}$$

(2) $z=\mathbf{Tan}^{-1}\left(\dfrac{y}{x}\right)$ より
$$z_x = \frac{-y}{x^2+y^2}$$
$$z_y = \frac{x}{x^2+y^2}$$
$$\therefore\quad \sqrt{(z_x)^2+(z_y)^2+1} = \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}$$
従って，求める曲面積 S は
$$S = \iint_D \frac{\sqrt{x^2+y^2+1}}{\sqrt{x^2+y^2}}\,dxdy$$

$D = \{(x, y) \mid x^2 + y^2 < 1,$
$x > 0, y > 0\}$

となる。極座標に変換すると

$$S = \int_0^{\frac{\pi}{2}} \int_0^1 \frac{\sqrt{r^2+1}}{\sqrt{r^2}} \cdot r\, dr\, d\theta$$
$$= \frac{\pi}{2} \int_0^1 \sqrt{r^2+1}\, dr$$
$$= \frac{\pi}{2} \left[\frac{1}{2} r\sqrt{r^2+1} + \frac{1}{2}\log(r + \sqrt{r^2+1}) \right]_0^1$$
$$= \boldsymbol{\frac{\pi}{4}\left(\sqrt{2} + \log(1+\sqrt{2})\right)}$$

６章の問題

A-1 **(1)** $\displaystyle\int_0^1 \int_0^2 xy^2\, dx\, dy$
$$= \int_0^1 y^2\, dy \cdot \int_0^2 x\, dx$$
$$= \left[\frac{y^3}{3}\right]_0^1 \cdot \left[\frac{x^2}{2}\right]_0^2$$
$$= \frac{1}{3} \cdot \frac{4}{2}$$
$$= \mathbf{\frac{2}{3}}$$

(2) $\displaystyle\int_1^2 \int_0^2 (2x+y)\, dy\, dx$
$$= \int_1^2 \left[2xy + \frac{y^2}{2}\right]_0^2 dx$$
$$= \int_1^2 (4x+2)\ dx$$
$$= \left[2x^2 + 2x\right]_1^2$$
$$= \mathbf{8}$$

A-2 **(1)** $\displaystyle\int_{-1}^0 \int_0^{x+1} f(x, y)\, dy\, dx + \int_0^1 \int_0^{1-x} f(x, y)\, dy\, dx$

または

$$\int_0^1 \int_{y-1}^{1-y} f(x, y)\, dx\, dy$$

(2) $\displaystyle\int_0^1 \int_{1-x}^1 f(x, y)\, dy\, dx$

または

$$\int_0^1 \int_{1-y}^1 f(x, y)\, dx\, dy$$

(3) $\displaystyle\int_0^1 \int_0^{2-2x} f(x, y)\, dy\, dx$

または

$$\int_0^2 \int_0^{\frac{2-y}{2}} f(x, y)\, dx\, dy$$

A-3 **(1)** $\displaystyle\int_0^2 \int_1^2 xy^2\, dx\, dy$
$$= \int_0^2 y^2\, dy \cdot \int_1^2 x\, dx$$
$$= \left[\frac{y^3}{3}\right]_0^2 \cdot \left[\frac{x^2}{2}\right]_1^2$$
$$= \frac{8}{3} \cdot \frac{3}{2}$$
$$= \mathbf{4}$$

(2) $\displaystyle\int_2^4 \int_1^3 (x+y)\, dx\, dy$
$$= \int_2^4 \left[\frac{x^2}{2} + xy\right]_1^3 dy$$
$$= \int_2^4 (4+2y)\, dy$$
$$= \left[4y + y^2\right]_2^4 = \mathbf{20}$$

A-4 **(1)** $\displaystyle\int_1^2 \int_3^{2x+1} f(x, y)\, dy\, dx$
$$= \int_3^5 \int_{\frac{y-1}{2}}^2 f(x, y)\, dx\, dy$$

(2) $\displaystyle\int_0^1 \int_y^{\sqrt{y}} f(x, y)\, dx\, dy$
$$= \int_0^1 \int_{x^2}^x f(x, y)\, dy\, dx$$

(3) $\displaystyle\int_0^{\sqrt{2}} \int_{y^2}^2 f(x, y)\, dx\, dy$
$$= \int_0^2 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$$

A-5 **(1)** $\displaystyle\int_0^{\pi} \int_0^2 r^2\, dr\, d\theta$

(2) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin\theta\, dr\, d\theta$

B-1 **(1)** $\displaystyle\iint_D (1+x+y)\, dx\, dy$
$$= \int_0^1 \int_0^{1-x} (1+x+y)\, dydx$$
$$= \int_0^1 \left[y + xy + \frac{y^2}{2}\right]_0^{1-x} dx$$
$$= \int_0^1 \left((1-x) + x(1-x) + \frac{(1-x)^2}{2}\right) dx$$
$$= \int_0^1 \left(1 - x^2 + \frac{1}{2}(x-1)^2\right) dx$$
$$= \left[x - \frac{1}{3}x^3 + \frac{1}{6}(x-1)^3\right]_0^1$$
$$= 1 - \frac{1}{3} - \left(\frac{-1}{6}\right)$$
$$= \mathbf{\frac{5}{6}}$$

(2) $\displaystyle\iint_D \frac{1}{\sqrt{1+x}}\,dx\,dy$

$\displaystyle\int_0^1 \int_{x^2}^{x} \frac{1}{\sqrt{1+x}}\,dydx$

$\displaystyle= \int_0^1 \frac{1}{\sqrt{1+x}}\left[y\right]_{x^2}^{x} dx$

$\displaystyle= \int_0^1 \frac{1}{\sqrt{1+x}}(x-x^2)\,dx$

ここで，$t = \sqrt{1+x}$ と置換すれば，

$x = t^2-1$，$dx = 2tdt$，$x : 0 \to 1$ の

とき，$t : 1 \to \sqrt{2}$ であるから

$\displaystyle= \int_1^{\sqrt{2}} \frac{1}{t}\left(t^2-1-(t^2-1)^2\right)2t\,dt$

$\displaystyle= 2\int_1^{\sqrt{2}} \left(t^2-1-t^4+2t^2-1\right)\,dt$

$\displaystyle= 2\int_1^{\sqrt{2}} \left(-t^4+3t^2-2\right)\,dt$

$\displaystyle= 2\left[-\frac{1}{5}t^5+t^3-2t\right]_1^{\sqrt{2}}$

$\displaystyle= 2\left(-\frac{4\sqrt{2}}{5}+2\sqrt{2}-2\sqrt{2}\frac{1}{5}-1+2\right)$

$\displaystyle= 2\left(-\frac{4\sqrt{2}}{5}+\frac{6}{5}\right)$

$\displaystyle= \boldsymbol{\frac{12-8\sqrt{2}}{5}}$

(3) $\displaystyle\iint_D x^2y\,dx\,dy$

$\displaystyle= \int_0^1 \int_0^y x^2y\,dxdy$

$\displaystyle= \int_0^1 \left[\frac{x^3y}{3}\right]_0^y dy$

$\displaystyle= \int_0^1 \frac{y^4}{3}\,dy$

$\displaystyle= \left[\frac{y^5}{15}\right]_0^1$

$\displaystyle= \boldsymbol{\frac{1}{15}}$

(4) $\displaystyle\iint_D e^{-(x^2+y^2)}\,dx\,dy$

$\displaystyle= \int_0^{2\pi}\int_0^1 e^{-r^2}\,r\,drd\theta$

$\displaystyle= \int_0^{2\pi} d\theta \cdot \int_0^1 e^{-r^2}\,r\,dr$

$\displaystyle= 2\pi \cdot \int_0^1 e^{-r^2}\,r\,dr$

$\displaystyle= -\pi\int_0^1 e^{-r^2}(-r^2)'\,dr$

$\displaystyle= -\pi\left[e^{-r^2}\right]_0^1$

$= -\pi\left(e^{-1}-e^0\right)$

$= \boldsymbol{\pi\left(1-e^{-1}\right)}$

(5) $\displaystyle\iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$

$\displaystyle\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}}\,r\,drd\theta$

$\displaystyle= 2\pi\int_0^1 (1-r^2)^{-\frac{1}{2}}r\,dr$

$\displaystyle= -\pi\int_0^1 (1-r^2)^{-\frac{1}{2}}(-2r)\,dr$

$\displaystyle= -\pi\int_0^1 (1-r^2)^{-\frac{1}{2}}(1-r^2)'\,dr$

$\displaystyle= -\pi\left[2(1-r^2)^{\frac{1}{2}}\right]_0^1$

$= -\pi(1-2)$

$= \boldsymbol{2\pi}$

(6) $\displaystyle I = \iint_D \{(x-y)^2+(x+2y)^2\}\,dx\,dy$

ここで $\begin{cases} u = x-y \\ v = x+2y \end{cases}$

と変換すると

$dxdy = \dfrac{1}{3}dudv$ かつ $-2 \leqq u \leqq 2$，$-1 \leqq v \leqq 1$ となる。

$\displaystyle I = \frac{1}{3}\int_{-2}^{2}\left(\int_{-1}^{1}(u^2+v^2)\,dv\right)du$

$\displaystyle= \frac{4}{3}\int_0^2\left(\int_0^1(u^2+v^2)\,dv\right)du$

$\displaystyle= \frac{4}{3}\int_0^2\left[u^2v+\frac{1}{3}v^3\right]_0^1 du$

$\displaystyle= \frac{4}{3}\int_0^2\left(u^2+\frac{1}{3}\right)du$

$\displaystyle= \frac{4}{3}\left[\frac{1}{3}u^3+\frac{1}{3}u\right]_0^2$

$\displaystyle= \frac{4}{9}\left[u^3+u\right]_0^2$

$\displaystyle= \frac{4}{9}\cdot(8+2)$

$\displaystyle= \boldsymbol{\frac{40}{9}}$

(7) $\displaystyle I = \iint_D (x+y)e^{x-y}\,dx\,dy$

ここで $\begin{cases} u = x+y \\ v = x-y \end{cases}$

と変換すると

$dxdy = \dfrac{1}{2}dudv$ かつ $0 \leqq u \leqq 1$，$0 \leqq v \leqq 1$ となる。

$$I=\frac{1}{2}\int_0^1\int_0^1 ue^v\,du\,dv$$
$$=\frac{1}{2}\int_0^1 u\,du\cdot\int_0^1 e^v\,dv$$
$$=\frac{1}{2}\left[\frac{u^2}{2}\right]_0^1\cdot\left[e^v\right]_0^1$$
$$=\frac{1}{2}\cdot\frac{1}{2}\cdot(e-1)$$
$$=\boldsymbol{\frac{e-1}{4}}$$

(8) $\displaystyle\iint_D (x^2+y^2)\,dx\,dy$

$$\begin{cases} u=\frac{x}{a} \\ v=\frac{y}{b}\end{cases}$$
と変換すると

$$\begin{pmatrix} du \\ dv\end{pmatrix}=\begin{pmatrix}\frac{1}{a} & 0 \\ 0 & \frac{1}{b}\end{pmatrix}\begin{pmatrix} dx \\ dy\end{pmatrix}$$

$\therefore\quad dxdy=ab\,dudv$ かつ
積分領域 D は $D=\{(u,v)\mid u^2+v^2\leqq 1\}$ となるから
$$I=\iint_D((au)^2+(bv)^2)ab\,du\,dv$$
$$=\iint_D(a^3bu^2+ab^3v^2)\,du\,dv$$
極座標を導入して

$$I=4\int_0^{\frac{\pi}{2}}\int_0^1(a^3br^2\cos^2\theta+ab^3r^2\sin^2\theta)r\,dr\,d\theta$$
$$=4\int_0^{\frac{\pi}{2}}(a^3b\cos^2\theta+ab^3\sin^2\theta)\left(\int_0^1 r^3\,dr\right)d\theta$$
$$=4\int_0^{\frac{\pi}{2}}(a^3b\cos^2\theta+ab^3\sin^2\theta)\cdot\frac{1}{4}\,d\theta$$
$$=\int_0^{\frac{\pi}{2}}(a^3b\cos^2\theta+ab^3\sin^2\theta)\,d\theta$$
$$=a^3b\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta+ab^3\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta$$
$$=a^3b\times\frac{\pi}{4}+ab^3\times\frac{\pi}{4}$$
$$=\boldsymbol{\frac{ab}{4}(a^2+b^2)\pi}$$

(9) $\displaystyle\iint_D\frac{x+y}{x^2+y^2}\,dx\,dy$
$$=\int_0^1\int_0^x\frac{x+y}{x^2+y^2}\,dydx$$

ここで，まず $\displaystyle\int_0^x\frac{x+y}{x^2+y^2}\,dy$ を計算する
$$\int_0^x\frac{x+y}{x^2+y^2}\,dy$$
$$=\int_0^x\frac{x}{x^2+y^2}\,dy+\int_0^x\frac{y}{x^2+y^2}\,dy$$
$$=x\int\frac{1}{x^2+y^2}\,dy+\frac{1}{2}\int_0^x\frac{2y}{x^2+y^2}\,dy$$
$$=x\left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x+\left[\frac{1}{2}\log(x^2+y^2)\right]_0^x$$
$$=\tan^{-1}(1)-\tan^{-1}(0)+\frac{1}{2}\Big(\log(2x^2)-\log(x^2)\Big)$$
$$=\frac{\pi}{4}+\frac{1}{2}\cdot\log\left(\frac{2x^2}{x^2}\right)$$
$$=\frac{\pi}{4}+\frac{1}{2}\log 2$$
したがって，
$$\int_0^1\int_0^x\frac{x+y}{x^2+y^2}\,dydx$$
$$=\int_0^1\left(\frac{\pi}{4}+\frac{1}{2}\log 2\right)dx$$
$$=\left(\frac{\pi}{4}+\frac{1}{2}\log 2\right)\int_0^1 dx$$
$$=\left(\frac{\pi}{4}+\frac{1}{2}\log 2\right)\Big[x\Big]_0^1$$
$$=\left(\frac{\pi}{4}+\frac{1}{2}\log 2\right)(1-0)$$
$$=\boldsymbol{\frac{\pi}{4}+\frac{1}{2}\log 2}$$

(10) 極座標に変換する
$$\iint_D\frac{1}{(x^2+y^2)^2}\,dx\,dy$$
$$=\int_0^{2\pi}\int_1^{+\infty}\frac{1}{r^4}\cdot r\,dr\,d\theta$$
$$=2\pi\int_1^{+\infty}r^{-3}\,dr$$
$$=\lim_{R\to+\infty}2\pi\int_1^R r^{-3}\,dr$$
$$=\lim_{R\to+\infty}2\pi\left[-\frac{1}{2r^2}\right]_1^R$$
$$=\lim_{R\to+\infty}2\pi\left(-\frac{1}{2R^2}+\frac{1}{2}\right)$$
$$=\boldsymbol{\pi}$$

B-2　まず $z=-a$ によって切られる

$\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}+z^2=1$ の切り口の方程式を求めると

$\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}+(-a)^2=1$

$\Longrightarrow x^2+y^2=a^2-a^4=a^2(1-a^2)$

であるから，切り口は半径 $a\sqrt{1-a^2}$ の円であることが分かる.

$\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}+z^2=1$ を z について解けば

$$z=\pm\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{a^2}}$$
$$=\pm\sqrt{\frac{a^2-(x^2+y^2)}{a^2}}$$
$$=\pm\frac{\sqrt{a^2-(x^2+y^2)}}{a}$$

であるから，求める体積 V は

$$V=\iint_D\left\{-a-\left(-\frac{\sqrt{a^2-(x^2+y^2)}}{a}\right)\right\}dxdy$$

$D=\{(x,y)\,|\,x^2+y^2\leqq a^2(1-a^2)\}$

ゆえに V は

$$V=\iint_D\left(\frac{\sqrt{a^2-(x^2+y^2)}}{a}-a\right)dxdy$$

極座標を導入すれば

$$V=4\int_0^{\frac{\pi}{2}}\int_0^{a\sqrt{1-a^2}}\left(\frac{\sqrt{a^2-r^2}}{a}-a\right)r\,dr\,d\theta$$
$$=4\int_0^{\frac{\pi}{2}}d\theta\cdot\int_0^{a\sqrt{1-a^2}}\left(\frac{1}{a}r(a^2-r^2)^{\frac{1}{2}}-ar\right)dr$$
$$=4\cdot\frac{\pi}{2}\cdot\left[-\frac{1}{3a}(a^2-r^2)^{\frac{3}{2}}-\frac{a}{2}r^2\right]_0^{a\sqrt{1-a^2}}$$
$$=2\pi\cdot\frac{a^5-3a^3+2a^2}{6}$$
$$=\boldsymbol{\frac{\pi}{3}(a^5-3a^3+2a^2)}$$

V を最大にする a を求めるために $g(a)=a^5-3a^3+2a^2$ とおく.

$g'(a)=5a^4-9a^2+4a$

$=a(5a^3-9a+4)$

$=a(a-1)(5a^2+5a-4)$

極値を求めるために $g'(a)=0$ なる a を求める. $a(a-1)(5a^2+5a-4)=0$ から，まず $a=0$, $a=1$, 次に $5a^2+5a-4=0$ を解いて

$$a=\frac{-5\pm\sqrt{25-4\times5\times(-4)}}{10}$$
$$=\frac{-5\pm\sqrt{105}}{10}$$

$0<a\leqq1$ が a の満たすべき条件なので，結局 $g'(a)=0$ なる a は $a=1$ と $a=\dfrac{-5+\sqrt{105}}{10}$ （$\sqrt{105}\fallingdotseq 10.2$）となる. 増減表を書いて

a	0	$\cdots$	$\frac{-5+\sqrt{105}}{10}$	$\cdots$	1
$g'(a)$	$\times$	$+$	0	$-$	0
$g(a)$	$\times$	$\nearrow$	最大	$\searrow$	0

増減表から（最大値自体を求める必要はない）V を最大にする a の値は $a=\boldsymbol{\dfrac{-5+\sqrt{105}}{10}}$ である。