

新版微分積分演習 解答

4章 積分法 II

1 節 定積分と不定積分

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$$\begin{aligned}(1) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \left\{ a + \frac{k(b-a)}{n} - 1 \right\} \frac{b-a}{n} &= \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n \frac{(b-a)(a-1)}{n} + \frac{(b-a)^2}{n^2} \sum_{k=1}^n k \right\} \\&= \lim_{n \rightarrow \infty} \left\{ \frac{(b-a)(a-1)}{n} \cdot n + \frac{(b-a)^2}{n^2} \cdot \frac{1}{2} n(n+1) \right\} = \lim_{n \rightarrow \infty} \left\{ (b-a)(a-1) + \frac{1}{2} (b-a)^2 \left(1 + \frac{1}{n} \right) \right\} \\&= (b-a)(a-1) + \frac{1}{2} (b-a)^2 = \frac{1}{2} (b-a)(b+a-2)\end{aligned}$$

$$\begin{aligned}(2) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \left\{ -2 \left(a + \frac{k(b-a)}{n} \right) \cdot \frac{b-a}{n} \right\} &= \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n \frac{-2a(b-a)}{n} - \frac{2(b-a)^2}{n^2} \sum_{k=1}^n k \right\} \\&= \lim_{n \rightarrow \infty} \left\{ \frac{-2a(b-a)}{n} \cdot n - \frac{2(b-a)^2}{n^2} \cdot \frac{1}{2} n(n+1) \right\} = \lim_{n \rightarrow \infty} \left\{ -2a(b-a) - (b-a)^2 \left(1 + \frac{1}{n} \right) \right\} \\&= -2a(b-a) - (b-a)^2 = a^2 - b^2\end{aligned}$$

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$$\begin{aligned}(1) \quad \lim_{n \rightarrow 0} \sum_{k=1}^n \left(\frac{2k}{n} \right)^2 \frac{2}{n} &= \lim_{n \rightarrow 0} \frac{8}{n^3} \sum_{k=1}^n k^2 = \lim_{n \rightarrow 0} \frac{8}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) = \lim_{n \rightarrow 0} \frac{4}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \\&= \frac{8}{3}\end{aligned}$$

$$\begin{aligned}(2) \quad \lim_{n \rightarrow 0} \sum_{k=1}^n \left\{ \left(\frac{2k}{n} \right)^2 + 1 \right\} \frac{2}{n} &= \lim_{n \rightarrow 0} \left\{ \frac{8}{n^3} \sum_{k=1}^n k^2 + \sum_{k=1}^n \frac{2}{n} \right\} = \lim_{n \rightarrow 0} \left\{ \frac{8}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) + \frac{2}{n} \cdot n \right\} \\&= \lim_{n \rightarrow 0} \left\{ \frac{4}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) + 2 \right\} = \frac{8}{3} + 2 = \frac{14}{3}\end{aligned}$$

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$$\begin{aligned}(1) \quad \lim_{n \rightarrow 0} \sum_{k=1}^n \left(1 + \frac{k}{n} \right)^2 \frac{1}{n} &= \lim_{n \rightarrow 0} \sum_{k=1}^n \left\{ \frac{1}{n} + \frac{2k}{n^2} + \frac{k^2}{n^3} \right\} = \lim_{n \rightarrow 0} \left\{ \sum_{k=1}^n \frac{1}{n} + \frac{2}{n^2} \sum_{k=1}^n k + \frac{1}{n^3} \sum_{k=1}^n k^2 \right\} \\&= \lim_{n \rightarrow 0} \left\{ \frac{1}{n} \cdot n + \frac{2}{n^2} \cdot \frac{1}{2} n(n+1) + \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) \right\} \\&= \lim_{n \rightarrow 0} \left\{ 1 + \left(1 + \frac{1}{n} \right) + \frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right\} = 1 + 1 + \frac{1}{3} = \frac{7}{3}\end{aligned}$$

$$\begin{aligned}(2) \quad \lim_{n \rightarrow 0} \sum_{k=1}^n \left\{ \chi + \frac{k}{n} - \chi \right\}^2 \frac{1}{n} &= \lim_{n \rightarrow 0} \sum_{k=1}^n \frac{k^2}{n^3} = \lim_{n \rightarrow 0} \frac{1}{n^3} \sum_{k=1}^n k^2 = \lim_{n \rightarrow 0} \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) \\&= \lim_{n \rightarrow 0} \frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) = \frac{1}{3}\end{aligned}$$

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$$(1) \quad 2x^2 - x \text{ を } x-1 \text{ で割ると, 商が } 2x+1, \text{ 余りが } 1 \text{ だから, } 2x^2 - x = (x-1)(2x+1) + 1$$

$$\int \frac{2x^2 - x}{x-1} dx = \int \left(2x+1 + \frac{1}{x-1} \right) dx = x^2 + x + \log|x-1| + C$$

$$(2) \quad x^2 - 2x + 1 \text{ を } x^2 + 1 \text{ で割ると, 商が } 1, \text{ 余りが } -2x \text{ だから, } x^2 - 2x + 1 = (x^2 + 1) \cdot 1 - 2x$$

$$\int \frac{x^2 - 2x + 1}{x^2 + 1} dx = \int \left(1 - \frac{2x}{x^2 + 1} \right) dx = x - \log(x^2 + 1) + C$$

$$(3) \quad 3x^3 + 5x^2 + 2x + 1 \text{ を } x+1 \text{ で割ると, 商が } 3x^2 + 2x, \text{ 余りが } 1 \text{ だから, } 3x^3 + 5x^2 + 2x + 1 = (x+1)(3x^2 + 2x) + 1$$

$$\int \frac{3x^3 + 5x^2 + 2x + 1}{x+1} dx = \int \left(3x^2 + 2x + \frac{1}{x+1} \right) dx = x^3 + x^2 + \log|x+1| + C$$

$$(4) \quad x^3 + x + 2 \text{ を } x^2 + 1 \text{ で割ると, 商が } x, \text{ 余りが } 2 \text{ だから, } x^3 + x + 2 = (x^2 + 1)x + 2$$

$$\int \frac{x^3 + x + 2}{x^2 + 1} dx = \int \left(x + \frac{2}{x^2 + 1} \right) dx = \frac{1}{2}x^2 + 2\tan^{-1} x + C$$

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$$(1) \int \frac{1}{x^2 + 3x + 2} dx = \int \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx = \log|x+1| - \log|x+2| + C = \log \left| \frac{x+1}{x+2} \right| + C$$

(2) $P(x) = x^3 - x^2 + x - 1$ とおくと, $P(1) = 1^3 - 1^2 + 1 - 1 = 0$ だから, 因数定理より, $P(x)$ は $x-1$ で割り切れ, $P(x) = (x-1)(x^2 + 1)$ となる。 $\frac{3x^2 - x}{x^3 - x^2 + x - 1} = \frac{a}{x-1} + \frac{bx+c}{x^2+1}$ とおき, 分母を払うと, $3x^2 - x = (a+b)x^2 + (-b+c)x + (a-c)$ だから, 両辺を比較して, $a=1, b=2, c=1$ を得る。よって, $\frac{3x^2 - x}{x^3 - x^2 + x - 1} = \frac{1}{x-1} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}$

$$\int \frac{3x^2 - x}{x^3 - x^2 + x - 1} dx = \int \left(\frac{1}{x-1} + \frac{2x}{x^2+1} + \frac{1}{x^2+1} \right) dx$$

$$= \log|x-1| + \log(x^2+1) + \tan^{-1} x + C$$

(3) $\frac{3x^2 + 7x + 5}{(x+1)(x^2 + 2x + 2)} = \frac{a}{x+1} + \frac{bx+c}{x^2+2x+2}$ とおき, 分母を払うと, $3x^2 + 7x + 5 = (a+b)x^2 + (2a+b+c)x + (2a+c)$ だから, 両辺を比較して, $a=1, b=2, c=3$ を得る。よって, $\frac{1}{x+1} + \frac{2x+3}{x^2+2x+2} = \frac{1}{x+1} + \frac{2x+2}{x^2+2x+2} + \frac{1}{(x+1)^2+1}$

$$\int \frac{3x^2 + 7x + 5}{(x+1)(x^2 + 2x + 2)} dx = \int \left(\frac{1}{x+1} + \frac{2x+2}{x^2+2x+2} + \frac{1}{(x+1)^2+1} \right) dx$$

$$= \log|x+1| + \log(x^2+2x+2) + \tan^{-1}(x+1) + C$$

(4) $\frac{2x^2 + 7x + 4}{x(x+2)^2} = \frac{a}{x} + \frac{b}{x+2} + \frac{c}{(x+2)^2}$ とおき, 分母を払うと, $2x^2 + 7x + 4 = (a+b)x^2 + (4a+2b+c)x + 4a$ だから, 両辺を比較して, $a=b=c=1$ を得る。よって, $\frac{2x^2 + 7x + 4}{x(x+2)^2} = \frac{1}{x} + \frac{1}{x+2} + \frac{1}{(x+2)^2}$

$$\int \frac{2x^2 + 7x + 4}{x(x+2)^2} dx = \int \left(\frac{1}{x} + \frac{1}{x+2} + \frac{1}{(x+2)^2} \right) dx = \log|x| + \log|x+2| - \frac{1}{x+2} + C$$

$$= \log|x(x+2)| - \frac{1}{x+2} + C$$

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(1) $t = \tan \frac{x}{2}$ とおくと, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2} dt$, $\frac{x}{2} = \tan^{-1} t$ だから,

$$\int \frac{1 - \cos x}{1 + \cos x} dx = \int \frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{(1+t^2) - (1-t^2)}{(1+t^2) + (1-t^2)} \cdot \frac{2}{1+t^2} dt$$

$$= \int \frac{2t^2}{2} \cdot \frac{2}{1+t^2} dt = 2 \int \frac{(1+t^2) - 1}{1+t^2} dt = 2 \int \left(1 - \frac{1}{1+t^2} \right) dt = 2t - 2\tan^{-1} t + C$$

$$= 2 \cdot \tan \frac{x}{2} - 2 \cdot \frac{x}{2} + C = 2 \tan \frac{x}{2} - x + C$$

[別解]

$$\int \frac{1 - \cos x}{1 + \cos x} dx = \int \frac{\frac{1 - \cos x}{2}}{\frac{1 + \cos x}{2}} dx = \int \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} dx = \int \frac{1 - \cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} dx$$

$$= \int \left(\frac{1}{\cos^2 \frac{x}{2}} - 1 \right) dx = 2 \tan \frac{x}{2} - x + C$$

(2) $t = \tan \frac{x}{2}$ とおくと, $\tan x = \frac{2t}{1-t^2}$, $dx = \frac{2}{1+t^2} dt$ だから,

$$\begin{aligned} \int \left(\frac{1}{\tan x} + \tan \frac{x}{2} \right) dx &= \int \left(\frac{1}{\frac{2t}{1-t^2}} + t \right) \frac{2}{1+t^2} dt = \int \left(\frac{1-t^2}{2t} + \frac{2t^2}{2t} \right) \frac{2}{1+t^2} dt \\ &= \int \frac{1+t^2}{2t} \cdot \frac{2}{1+t^2} dt = \int \frac{1}{t} dt = \log |t| + C = \log \left| \tan \frac{x}{2} \right| + C \end{aligned}$$

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$$(1) \int \frac{1}{\sqrt{4x-x^2}} dx = \int \frac{1}{\sqrt{4-(x^2-4x+4)}} dx = \int \frac{1}{\sqrt{2^2-(x-2)^2}} dx = \text{Sin}^{-1} \frac{x-2}{2} + C$$

$$(2) \int \frac{1}{\sqrt{8-6x-9x^2}} dx = \int \frac{1}{\sqrt{9-(9x^2+6x+1)}} dx = \int \frac{1}{\sqrt{3^2-(3x+1)^2}} dx \\ = \frac{1}{3} \text{Sin}^{-1} \frac{3x+1}{3} + C$$

$$(3) \int \sqrt{16-6x-x^2} dx = \int \sqrt{25-(x^2+6x+9)} dx = \int \sqrt{5^2-(x+3)^2} dx \\ = \frac{1}{2} \left\{ (x+3)\sqrt{16-6x-x^2} + 25 \text{Sin}^{-1} \frac{x+3}{5} \right\} + C$$

$$(4) \int \sqrt{6x-9x^2} dx = \int \sqrt{1-(9x^2-6x+1)} dx = \int \sqrt{1-(3x-1)^2} dx \\ = \frac{1}{6} \left\{ (3x-1)\sqrt{6x-9x^2} + \text{Sin}^{-1} (3x-1) \right\} + C$$

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$$(1) \int \frac{1}{\sqrt{x^2+4x+5}} dx = \int \frac{1}{\sqrt{(x+2)^2+1}} dx = \log \left| x+2+\sqrt{x^2+4x+5} \right| + C$$

$$(2) \int \frac{1}{\sqrt{4x^2+4x+3}} dx = \int \frac{1}{\sqrt{(2x+1)^2+2}} dx = \frac{1}{2} \log \left| 2x+1+\sqrt{4x^2+4x+3} \right| + C$$

$$(3) \int \sqrt{x^2+6x+10} dx = \int \sqrt{(x+3)^2+1} dx \\ = \frac{1}{2} \left\{ (x+3)\sqrt{x^2+6x+10} + \log \left| x+3+\sqrt{x^2+6x+10} \right| \right\} + C$$

$$(4) \int \sqrt{4x^2-4x+7} dx = \int \sqrt{(2x-1)^2+6} dx \\ = \frac{1}{2} \left\{ (2x-1)\sqrt{4x^2-4x+7} + 6 \log \left| (2x-1) + \sqrt{4x^2-4x+7} \right| \right\} \cdot \frac{1}{2} + C \\ = \frac{1}{4} \left\{ (2x-1)\sqrt{4x^2-4x+7} + 6 \log \left| 2x-1 + \sqrt{4x^2-4x+7} \right| \right\} + C$$

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$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{k}{n} \right)^3 \cdot \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1}{n^4} \sum_{k=1}^n k^3 = \lim_{n \rightarrow \infty} \frac{1}{n^4} \cdot \frac{1}{4} n^2 (n+1)^2 = \frac{1}{4} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^2 = \frac{1}{4}$$

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$$(1) f(x) \text{ は偶関数, } \sin x \text{ は奇関数だから, } f(x) \sin x \text{ は奇関数である。よって } \int_{-\pi}^{\pi} f(x) \sin x dx = 0$$

$$(2) f(x), \cos x \text{ はともに偶関数だから, } f(x) \cos x \text{ は偶関数である。よって} \\ \int_{-\pi}^{\pi} f(x) \cos x dx = 2 \int_0^{\pi} (\pi-x) \cos x dx = 2 \left[(\pi-x) \sin x \right]_0^{\pi} - \int_0^{\pi} (-1) \sin x dx = 2 \int_0^{\pi} \sin x dx \\ = 2 \cdot \left[-\cos x \right]_0^{\pi} = 4$$

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$$(1) \int \frac{x^2+2x+2}{x^2+1} dx = \int \left(1 + \frac{2x}{x^2+1} + \frac{1}{x^2+1} \right) dx = x + \log(x^2+1) + \text{Tan}^{-1} x + C$$

$$(2) \int \frac{1}{\sqrt{2-x-x^2}} dx = \int \frac{1}{\sqrt{\frac{9}{4} - (x^2+x+\frac{1}{4})}} dx = \int \frac{1}{\sqrt{(\frac{3}{2})^2 - (x+\frac{1}{2})^2}} dx \\ = \text{Sin}^{-1} \frac{x+\frac{1}{2}}{\frac{3}{2}} + C = \text{Sin}^{-1} \frac{2x+1}{3} + C$$

(3) $t = \tan \frac{x}{2}$ とおくと ,

$$\begin{aligned} \int \frac{1}{\sin x + \sin^2 x} dx &= \int \frac{1}{\frac{2t}{1+t^2} + \frac{4t^2}{(1+t^2)^2}} \cdot \frac{2}{1+t^2} dt = \int \frac{1}{t + \frac{2t^2}{1+t^2}} dt = \int \frac{t^2+1}{t^3+2t^2+t} dt \\ &= \int \frac{t^2+1}{t(t+1)^2} dt = \int \left\{ \frac{1}{t} - \frac{2}{(t+1)^2} \right\} dt = \log |t| + \frac{2}{t+1} + C = \log \left| \tan \frac{x}{2} \right| + \frac{2}{\tan \frac{x}{2} + 1} + C \end{aligned}$$

(4) $\int \sqrt{x^2+x+1} dx = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx$
 $= \frac{1}{2} \left\{ \left(x+\frac{1}{2}\right) \sqrt{x^2+x+1} + \frac{3}{4} \log \left| \left(x+\frac{1}{2}\right) + \sqrt{x^2+x+1} \right| \right\} + C$

(5) $\int \frac{2x+1}{\sqrt{x^2+1}} dx = \int \left(\frac{2x}{\sqrt{x^2+1}} + \frac{1}{\sqrt{x^2+1}} \right) dx = 2\sqrt{x^2+1} + \log \left| x + \sqrt{x^2+1} \right| + C$

(6) $\int (2x+1)\sqrt{1-x^2} dx = \int (2x\sqrt{1-x^2} + \sqrt{1-x^2}) dx$
 $= -\frac{2}{3}\sqrt{(1-x^2)^3} + \frac{1}{2} \left\{ x\sqrt{1-x^2} + \sin^{-1} x \right\} + C$

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(1) $\int_{-1}^0 \frac{3x^3+4x^2-4x+1}{x+2} dx = \int_{-1}^0 \left(3x^2-2x+\frac{1}{x+2} \right) dx = \left[x^3-x^2+\log|x+2| \right]_{-1}^0 = 2 + \log 2$

(2) $\int_1^2 \frac{1}{x^2(x+1)} dx = \int_1^2 \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right) dx = \left[-\log|x| - \frac{1}{x} + \log|x+1| \right]_1^2$
 $= -\log 2 - \frac{1}{2} + \log 3 + 1 - \log 2 = \frac{1}{2} + \log \frac{3}{4}$

(3) $\int_0^{\frac{\pi}{3}} \frac{1}{\cos x} dx = \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2} dt = -\int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(t-1)(t+1)} dt = -\int_0^{\frac{1}{\sqrt{3}}} \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt$
 $= -\left[\log|t-1| - \log|t+1| \right]_0^{\frac{1}{\sqrt{3}}} = -\log \left| \frac{1}{\sqrt{3}} - 1 \right| + \log \left| \frac{1}{\sqrt{3}} + 1 \right| + \log 1 - \log 1$
 $= \log \left| \frac{\frac{1}{\sqrt{3}}+1}{\frac{1}{\sqrt{3}}-1} \right| = \log \left| \frac{1+\sqrt{3}}{1-\sqrt{3}} \right| = \log \left| \frac{4+2\sqrt{3}}{-2} \right| = \log(2+\sqrt{3})$

(4) $\int_0^1 \sqrt{2-x^2} dx = \left[\frac{1}{2} \left\{ x\sqrt{2-x^2} + 2\sin^{-1} \frac{x}{\sqrt{2}} \right\} \right]_0^1 = \frac{1}{2} \left\{ 1 + 2\sin^{-1} \frac{1}{\sqrt{2}} - 0 - 2\sin^{-1} 0 \right\}$
 $= \frac{1}{2} \left\{ 1 + 2 \cdot \frac{\pi}{4} \right\} = \frac{2+\pi}{4}$

(5) $\int_0^4 \frac{1}{\sqrt{x^2+9}} dx = \left[\log \left| x + \sqrt{x^2+9} \right| \right]_0^4 = \log \left| 4 + \sqrt{25} \right| - \log \left| 0 + \sqrt{9} \right| = \log 9 - \log 3 = \log 3$

(6) $\int_0^3 \sqrt{x^2+16} dx = \left[\frac{1}{2} \left\{ x\sqrt{x^2+16} + 16 \log \left| x + \sqrt{x^2+16} \right| \right\} \right]_0^3$
 $= \frac{1}{2} \left\{ 3\sqrt{25} + 16 \log \left| 3 + \sqrt{25} \right| - 0 - 16 \log \sqrt{16} \right\} = \frac{1}{2} \left\{ 15 + 16 \log 8 - 16 \log 4 \right\} = \frac{15}{2} + 8 \log 2$

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(1) $t = \frac{\pi}{2} - x$ とおくと , $\frac{dt}{dx} = -1$ $dx = -dt$ $\frac{x}{t} \left| \begin{array}{l} 0 \rightarrow \frac{\pi}{2} \\ \frac{\pi}{2} \rightarrow 0 \end{array} \right.$
 $\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f\left(\cos\left(\frac{\pi}{2}-x\right)\right) dx = \int_{\frac{\pi}{2}}^0 f(\cos t) \cdot (-1) dt = \int_0^{\frac{\pi}{2}} f(\cos t) dt = \int_0^{\frac{\pi}{2}} f(\cos x) dx$

(2) $t = Lx$ とおくと , $\frac{dt}{dx} = L$ $dx = \frac{1}{L} dt$, $\frac{x}{t} \left| \begin{array}{l} -1 \rightarrow 1 \\ -L \rightarrow L \end{array} \right.$

$$\int_{-1}^1 f(Lx) dx = \int_{-L}^L f(t) \frac{1}{L} dt = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$(3) \quad s = xt \text{ とおくと, } \frac{ds}{dt} = x \quad dt = \frac{1}{x} ds, \quad \begin{array}{c|c} t & 0 \rightarrow 1 \\ s & 0 \rightarrow x \end{array}$$

$$\frac{d}{dx} \left\{ x \int_0^1 f(xt) dt \right\} = \frac{d}{dx} \left\{ x \int_0^x f(s) \cdot \frac{1}{x} ds \right\} = \frac{d}{dx} \left\{ \int_0^x f(s) ds \right\} = f(x)$$

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$$(1) \quad (\text{与式}) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{n^2 + k^2}} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{\sqrt{1 + \left(\frac{k}{n}\right)^2}} \cdot \frac{1}{n} = \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$

$$= \left[\log \left| x + \sqrt{x^2 + 1} \right| \right]_0^1 = \log \left| 1 + \sqrt{2} \right| - \log \left| 0 + \sqrt{1} \right| = \log(1 + \sqrt{2})$$

$$(2) \quad (\text{与式}) = \lim_{n \rightarrow \infty} \frac{1}{n\sqrt{n}} \sum_{k=1}^n \sqrt{k} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{\frac{k}{n}} \cdot \frac{1}{n} = \int_0^1 \sqrt{x} dx = \left[\frac{2}{3} \sqrt{x^3} \right]_0^1 = \frac{2}{3}$$

2 節 定積分の応用

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$$(1) \quad \int_0^{\frac{\pi}{2}} |2(1 - \cos t) \cdot 2(1 - \cos t)| dt = 4 \int_0^{\frac{\pi}{2}} (1 - 2 \cos t + \cos^2 t) dt = 4 \int_0^{\frac{\pi}{2}} \left(1 - 2 \cos t + \frac{1 + \cos 2t}{2} \right) dt$$

$$= 4 \left[t - 2 \sin t + \frac{1}{2} \left(t + \frac{1}{2} \sin 2t \right) \right]_0^{\frac{\pi}{2}} = 4 \left(\frac{\pi}{2} - 2 \sin \frac{\pi}{2} + \frac{\pi}{4} + \frac{1}{4} \sin \pi \right) = 3\pi - 8$$

$$(2) \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |2 \sin t \cdot (-2 \sin t)| dt = 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2 \sin^2 t dt = 2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - \cos 2t) dt = 2 \left[t - \frac{1}{2} \sin 2t \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= 2 \left\{ \left(\frac{\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} \right) - \left(-\frac{\pi}{4} - \frac{1}{2} \sin \left(-\frac{\pi}{2} \right) \right) \right\} = 2 \left\{ \left(\frac{\pi}{4} - \frac{1}{2} \right) + \left(\frac{\pi}{4} - \frac{1}{2} \right) \right\} = \pi - 2$$

(3) 曲線の $t = 0$ から $t = 1$ までの部分と x 軸および直線 $x = 1$ で囲まれた図形の面積を求めて 2 倍すればよい。

$$2 \int_0^1 |t^3 \cdot 2t| dt = 4 \int_0^1 t^4 dt = 4 \left[\frac{1}{5} t^5 \right]_0^1 = \frac{4}{5}$$

189

$$(1) \quad \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \sin \theta)^2 d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + 2 \sin \theta + \sin^2 \theta) d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(1 + 2 \sin \theta + \frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{2} \left[\theta - 2 \cos \theta + \frac{1}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right) \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left\{ \left(\frac{\pi}{2} - 2 \cos \frac{\pi}{2} + \frac{1}{2} \left(\frac{\pi}{2} - \frac{1}{2} \sin \pi \right) \right) - \left(0 - 2 \cos 0 + \frac{1}{2} (0 - 0) \right) \right\}$$

$$= \frac{1}{2} \left\{ \frac{\pi}{2} + \frac{\pi}{4} + 2 \right\} = \frac{3\pi}{8} + 1$$

(2) レムニスケートの $\theta = 0$ から $\theta = \frac{\pi}{4}$ までの部分と x 軸で囲まれた図形の面積を 4 倍すればよい。

$$4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} a^2 \cos 2\theta d\theta = 2a^2 \left[\frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}} = a^2 \left(\sin \frac{\pi}{2} - \sin 0 \right) = a^2$$

190

(1) $y = \frac{1}{4}x^2$ より $y' = \frac{1}{2}x$ だから

$$\int_0^1 \sqrt{1 + \frac{x^2}{4}} dx = \frac{1}{2} \int_0^1 \sqrt{x^2 + 4} dx = \frac{1}{2} \left[\frac{1}{2} \left\{ x \sqrt{x^2 + 4} + 4 \log \left| x + \sqrt{x^2 + 4} \right| \right\} \right]_0^1$$

$$= \frac{1}{4} \left\{ \sqrt{5} + 4 \log \left| 1 + \sqrt{5} \right| - 0 - 4 \log 2 \right\} = \frac{\sqrt{5}}{4} + \log \frac{1 + \sqrt{5}}{2}$$

(2) $y = \log |\cos x|$ より $y' = -\frac{\sin x}{\cos x}$ だから, 求める長さを L とすると

$$\int_0^{\frac{\pi}{3}} \sqrt{1 + \frac{\sin^2 x}{\cos^2 x}} dx = \int_0^{\frac{\pi}{3}} \sqrt{\frac{\cos^2 x + \sin^2 x}{\cos^2 x}} dx = \int_0^{\frac{\pi}{3}} \frac{1}{\cos x} dx$$

ここで, $t = \tan \frac{x}{2}$ とおくと, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2} dt$, $\left. \begin{matrix} x & 0 & \rightarrow & \frac{\pi}{3} \\ t & 0 & \rightarrow & \frac{\sqrt{3}}{3} \end{matrix} \right\}$ だから

$$\begin{aligned} L &= \int_0^{\frac{\sqrt{3}}{3}} \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2} dt = - \int_0^{\frac{\sqrt{3}}{3}} \frac{2}{t^2-1} dt = - \int_0^{\frac{\sqrt{3}}{3}} \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt \\ &= - \left[\log |t-1| - \log |t+1| \right]_0^{\frac{\sqrt{3}}{3}} = \left[\log \left| \frac{t+1}{t-1} \right| \right]_0^{\frac{\sqrt{3}}{3}} = \log \left| \frac{\frac{\sqrt{3}}{3}+1}{\frac{\sqrt{3}}{3}-1} \right| - \log \left| \frac{1}{1} \right| = \log \left| \frac{1+\sqrt{3}}{1-\sqrt{3}} \right| \\ &= \log \left| \frac{4+2\sqrt{3}}{2} \right| = \log(2+\sqrt{3}) \end{aligned}$$

191

(1) $x = 2t^2$, $y = t^3$ より $\frac{dx}{dt} = 4t$, $\frac{dy}{dt} = 3t^2$ だから, 求める曲線の長さを L とすると

$$\int_0^1 \sqrt{16t^2 + 9t^4} dt = \int_0^1 t \sqrt{16 + 9t^2} dt$$

ここで, $s = 16 + 9t^2$ とおくと, $\frac{ds}{dt} = 18t$ より, $t dt = \frac{1}{18} ds$, $\left. \begin{matrix} t & 0 & \rightarrow & 1 \\ s & 16 & \rightarrow & 25 \end{matrix} \right\}$ だから

$$L = \int_{16}^{25} \sqrt{s} \cdot \frac{1}{18} ds = \frac{1}{18} \left[\frac{2}{3} \sqrt{s^3} \right]_{16}^{25} = \frac{1}{27} (5^3 - 4^3) = \frac{61}{27}$$

(2) $x = a(t - \sin t)$, $y = a(1 - \cos t)$ より $\frac{dx}{dt} = a(1 - \cos t)$, $\frac{dy}{dt} = a \sin t$ だから

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} dt &= a \int_0^{\frac{\pi}{2}} \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} dt = a \int_0^{\frac{\pi}{2}} \sqrt{2(1 - \cos t)} dt \\ &= a \int_0^{\frac{\pi}{2}} \sqrt{4 \sin^2 \frac{t}{2}} dt = 2a \int_0^{\frac{\pi}{2}} \sin \frac{t}{2} dt = 2a \left[-2 \cos \frac{t}{2} \right]_0^{\frac{\pi}{2}} = -4a \left(\cos \frac{\pi}{4} - \cos 0 \right) \\ &= -4a \left(\frac{\sqrt{2}}{2} - 1 \right) = 2a(2 - \sqrt{2}) \end{aligned}$$

(3) $r = e^\theta$ より $\frac{dr}{d\theta} = e^\theta$ だから

$$\int_0^\pi \sqrt{e^{2\theta} + e^{2\theta}} d\theta = \int_0^\pi \sqrt{2e^{2\theta}} d\theta = \sqrt{2} \int_0^\pi e^\theta d\theta = \sqrt{2} [e^\theta]_0^\pi = \sqrt{2}(e^\pi - 1)$$

(4) $r = a(1 + \cos \theta)$ より $\frac{dr}{d\theta} = -a \sin \theta$ だから

$$\begin{aligned} \int_{\frac{\pi}{2}}^\pi \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta &= a \int_{\frac{\pi}{2}}^\pi \sqrt{1 + 2 \cos \theta + \cos^2 \theta + \sin^2 \theta} d\theta = a \int_{\frac{\pi}{2}}^\pi \sqrt{2(1 + \cos \theta)} d\theta \\ &= a \int_{\frac{\pi}{2}}^\pi \sqrt{4 \cos^2 \frac{\theta}{2}} d\theta = 2a \int_{\frac{\pi}{2}}^\pi \cos \frac{\theta}{2} d\theta = 2a \left[2 \sin \frac{\theta}{2} \right]_{\frac{\pi}{2}}^\pi = 2a \left(2 \sin \frac{\pi}{2} - 2 \sin \frac{\pi}{4} \right) = 2a(2 - \sqrt{2}) \end{aligned}$$

192

(1) $\pi \int_0^{\frac{\pi}{4}} \tan^2 x dx = \pi \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - 1 \right) dx = \pi [\tan x - x]_0^{\frac{\pi}{4}} = \pi \left(1 - \frac{\pi}{4} - 0 + 0 \right) = \frac{\pi(4 - \pi)}{4}$

(2) $\pi \int_1^e (\log x)^2 dx = \pi \left\{ \left[x(\log x)^2 \right]_1^e - \int_1^e x \cdot 2 \log x \cdot \frac{1}{x} dx \right\} = \pi \left\{ e - 0 - 2 \int_1^e \log x dx \right\}$
 $= \pi \left\{ e - 2 \left(\left[x \log x \right]_1^e - \int_1^e x \cdot \frac{1}{x} dx \right) \right\} = \pi \left\{ e - 2 \left(e - 0 - \left[x \right]_1^e \right) \right\} = \pi \left\{ e - 2(e - (e - 1)) \right\}$
 $= \pi(e - 2)$

193

- (1) $\int_1^2 \frac{1}{\sqrt{x-1}} dx = \lim_{\varepsilon \rightarrow +0} \int_{1+\varepsilon}^2 \frac{1}{\sqrt{x-1}} dx = \lim_{\varepsilon \rightarrow +0} \left[2\sqrt{x-1} \right]_{1+\varepsilon}^2 = \lim_{\varepsilon \rightarrow +0} (2 - 2\sqrt{\varepsilon}) = 2$
- (2) $\int_0^2 \frac{1}{\sqrt{4-x^2}} dx = \lim_{\varepsilon \rightarrow +0} \int_0^{2-\varepsilon} \frac{1}{\sqrt{4-x^2}} dx = \lim_{\varepsilon \rightarrow +0} \left[\sin^{-1} \frac{x}{2} \right]_0^{2-\varepsilon} = \lim_{\varepsilon \rightarrow +0} \left(\sin^{-1} \frac{2-\varepsilon}{2} - \sin^{-1} 0 \right)$
 $= \sin^{-1} 1 = \frac{\pi}{2}$
- (3) $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} dx = \lim_{\substack{\varepsilon \rightarrow +0 \\ \varepsilon' \rightarrow +0}} \int_{-1+\varepsilon}^{1-\varepsilon'} \frac{1}{\sqrt{1-x^2}} dx = \lim_{\substack{\varepsilon \rightarrow +0 \\ \varepsilon' \rightarrow +0}} \left[\sin^{-1} x \right]_{-1+\varepsilon}^{1-\varepsilon'}$
 $= \lim_{\substack{\varepsilon \rightarrow +0 \\ \varepsilon' \rightarrow +0}} \left\{ \sin^{-1} (1-\varepsilon') - \sin^{-1} (-1+\varepsilon) \right\} = \sin^{-1} 1 - \sin^{-1} (-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi$

194

- (1) $\int_1^\infty \frac{1}{x^4} dx = \lim_{K \rightarrow \infty} \int_1^K \frac{1}{x^4} dx = \lim_{K \rightarrow \infty} \left[-\frac{1}{3x^3} \right]_1^K = -\frac{1}{3} \lim_{K \rightarrow \infty} \left(\frac{1}{K^3} - 1 \right) = \frac{1}{3}$
- (2) $\int_{-\infty}^0 e^x dx = \lim_{K \rightarrow \infty} \int_{-K}^0 e^x dx = \lim_{K \rightarrow \infty} \left[e^x \right]_{-K}^0 = \lim_{K \rightarrow \infty} (1 - e^{-K}) = 1$
- (3) $\int_{-\infty}^\infty \frac{1}{1+x^2} dx = \lim_{\substack{K \rightarrow \infty \\ K' \rightarrow \infty}} \int_{-K'}^K \frac{1}{1+x^2} dx = \lim_{\substack{K \rightarrow \infty \\ K' \rightarrow \infty}} \left[\tan^{-1} x \right]_{-K'}^K = \lim_{\substack{K \rightarrow \infty \\ K' \rightarrow \infty}} (\tan^{-1} K - \tan^{-1} K')$
 $= \frac{\pi}{2} - \left(-\frac{\pi}{2} \right) = \pi$

195

- (1) $x = \log(t^2 + 1)$, $y = 6t^3$ より, $\frac{dx}{dt} = \frac{2t}{t^2 + 1}$ だから, 求める面積を S とすると

$$S = \int_0^1 \left| 6t^3 \cdot \frac{2t}{t^2 + 1} \right| dt = 12 \int_0^1 \frac{t^4}{t^2 + 1} dt$$

t^4 を $t^2 + 1$ で割ると, 商が $t^2 - 1$, 余りが 1 だから, $t^4 = (t^2 + 1)(t^2 - 1) + 1$

$$\frac{t^4}{t^2 + 1} = \frac{(t^2 + 1)(t^2 - 1) + 1}{t^2 + 1} = t^2 - 1 + \frac{1}{t^2 + 1}$$

$$S = 12 \int_0^1 \left(t^2 - 1 + \frac{1}{t^2 + 1} \right) dt = 12 \left[\frac{1}{3} t^3 - t + \tan^{-1} t \right]_0^1 = 12 \left\{ \left(\frac{1}{3} - 1 + \frac{\pi}{4} \right) - (0 - 0 + 0) \right\}$$

 $= 3\pi - 8$

- (2) $x = \tan t$, $y = \sin t$ より, $\frac{dx}{dt} = \frac{1}{\cos^2 t}$ だから, 求める面積を S とすると

$$S = \int_0^{\frac{\pi}{3}} \left| \sin t \cdot \frac{1}{\cos^2 t} \right| dt$$

ここで, $u = \cos t$ とおくと, $du = -\sin t dt$, $\frac{t}{u} \Big|_0^{\frac{\pi}{3}} \rightarrow \frac{\frac{\pi}{3}}{\frac{1}{2}}$ だから,

$$S = \int_1^{\frac{1}{2}} \frac{1}{u^2} (-1) du = \int_{\frac{1}{2}}^1 \frac{1}{u^2} du = \left[-\frac{1}{u} \right]_{\frac{1}{2}}^1 = -1 + \frac{1}{\frac{1}{2}} = -1 + 2 = 1$$

- (3) $x = \tan t$, $y = \cos t$ より, $\frac{dx}{dt} = \frac{1}{\cos^2 t}$ だから, 求める面積を S とすると

$$S = \int_0^{\frac{\pi}{3}} \left| \cos t \cdot \frac{1}{\cos^2 t} \right| dt = \int_0^{\frac{\pi}{3}} \frac{1}{\cos t} dt$$

ここで, $u = \tan \frac{t}{2}$ とおくと, $\cos t = \frac{1-u^2}{1+u^2}$, $dt = \frac{2}{1+u^2} du$, $\frac{t}{u} \Big|_0^{\frac{\pi}{3}} \rightarrow \frac{\frac{\pi}{3}}{\frac{1}{\sqrt{3}}}$ だから,

$$S = \int_0^{\frac{1}{\sqrt{3}}} \frac{1+u^2}{1-u^2} \cdot \frac{2}{1+u^2} du = - \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(u-1)(u+1)} du = - \int_0^{\frac{1}{\sqrt{3}}} \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du$$

$$\begin{aligned}
&= -\left[\log|u-1| - \log|u+1|\right]_0^{\frac{1}{\sqrt{3}}} = \left[\log\left|\frac{u+1}{u-1}\right|\right]_0^{\frac{1}{\sqrt{3}}} = \log\left|\frac{\frac{1}{\sqrt{3}}+1}{\frac{1}{\sqrt{3}}-1}\right| - \log\left|\frac{1}{-1}\right| \\
&= \log\left|\frac{1+\sqrt{3}}{1-\sqrt{3}}\right| - \log 1 = \log\frac{\sqrt{3}+1}{\sqrt{3}-1} = \log\frac{4+2\sqrt{3}}{3-1} = \log(2+\sqrt{3})
\end{aligned}$$

196

- (1) $r \cos \theta = \cos 2\theta$ より, $r = \frac{\cos 2\theta}{\cos \theta}$ だから,

$$\begin{aligned}
r^2 &= \left(\frac{\cos 2\theta}{\cos \theta}\right)^2 = \frac{(2\cos^2 \theta - 1)^2}{\cos^2 \theta} = \frac{4\cos^4 \theta - 4\cos^2 \theta + 1}{\cos^2 \theta} = 4\cos^2 \theta - 4 + \frac{1}{\cos^2 \theta} \\
&= 4 \cdot \frac{1+\cos 2\theta}{2} - 4 + \frac{1}{\cos^2 \theta} = 2\cos 2\theta - 2 + \frac{1}{\cos^2 \theta}
\end{aligned}$$

よって, 求める面積は

$$\begin{aligned}
&\frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{\cos 2\theta}{\cos \theta}\right)^2 d\theta = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(2\cos 2\theta - 2 + \frac{1}{\cos^2 \theta}\right) d\theta = \frac{1}{2} \left[\sin 2\theta - 2\theta + \tan \theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{1}{2} \left\{ \left(\sin \frac{\pi}{2} - \frac{\pi}{2} + \tan \frac{\pi}{4}\right) - \left(\sin \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} + \tan \left(-\frac{\pi}{4}\right)\right) \right\} \\
&= \frac{1}{2} \left\{ 1 - \frac{\pi}{2} + 1 + 1 - \frac{\pi}{2} + 1 \right\} = 2 - \frac{\pi}{2}
\end{aligned}$$

- (2) $r = 1 + 2\cos \theta$ より,

$$r^2 = (1 + 2\cos \theta)^2 = 1 + 4\cos \theta + 4\cos^2 \theta = 1 + 4\cos \theta + 4 \cdot \frac{1+\cos 2\theta}{2} = 3 + 4\cos \theta + 2\cos 2\theta$$

よって, 求める面積は

$$\begin{aligned}
&\frac{1}{2} \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} (1 + 2\cos \theta)^2 d\theta = \frac{1}{2} \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} (3 + 4\cos \theta + 2\cos 2\theta) d\theta = \frac{1}{2} \left[3\theta + 4\sin \theta + \sin 2\theta\right]_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \\
&= \frac{1}{2} \left\{ \left(2\pi + 4\sin \frac{2\pi}{3} + \sin \frac{4\pi}{3}\right) - \left(-2\pi + 4\sin \left(-\frac{2\pi}{3}\right) + \sin \left(-\frac{4\pi}{3}\right)\right) \right\} \\
&= \frac{1}{2} \left\{ 2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2} + 2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2} \right\} = 2\pi + \frac{3\sqrt{3}}{2}
\end{aligned}$$

197

- (1) $y = \log|\sin x|$ より, $y' = \frac{\cos x}{\sin x}$ だから,

$$\sqrt{1+(y')^2} = \sqrt{1 + \frac{\cos^2 x}{\sin^2 x}} = \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} = \frac{1}{\sin x}$$

$$t = \tan \frac{x}{2} \text{ とおくと, } \sin x = \frac{2t}{1+t^2}, dx = \frac{2}{1+t^2} dt, \left. \frac{x}{t} \right|_{\frac{1}{\sqrt{3}}}^{\frac{\pi}{3}} \rightarrow \frac{\frac{\pi}{2}}{1} \text{ だから, 求める長さは}$$

$$\begin{aligned}
&\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1+(y')^2} dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{\sin x} dx = \int_{\frac{1}{\sqrt{3}}}^1 \frac{\cancel{1+t^2}}{2t} \cdot \frac{2}{\cancel{1+t^2}} dt = \int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{t} dt = \left[\log t\right]_{\frac{1}{\sqrt{3}}}^1 \\
&= \log 1 - \log \frac{1}{\sqrt{3}} = -\log 3^{-\frac{1}{2}} = \frac{1}{2} \log 3
\end{aligned}$$

- (2) $x = e^t \cos t, y = e^t \sin t$ より, $\frac{dx}{dt} = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t),$

$$\frac{dy}{dt} = e^t \sin t + e^t \cos t = e^t(\cos t + \sin t) \text{ だから,}$$

$$\begin{aligned}
&\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t) + e^{2t}(\cos^2 t + 2\cos t \sin t + \sin^2 t) \\
&= 2e^{2t}(\cos^2 t + \sin^2 t) = 2e^{2t}
\end{aligned}$$

よって, 求める長さは

$$\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{2} \int_0^1 e^t dt = \sqrt{2} \left[e^t\right]_0^1 = \sqrt{2}(e-1)$$

- (3) $x = t \cos t$, $y = t \sin t$ より, $\frac{dx}{dt} = \cos t - t \sin t$, $\frac{dy}{dt} = \sin t + t \cos t$ だから,

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (\cos^2 t - 2t \cos t \sin t + t^2 \sin^2 t) + (\sin^2 t + 2t \sin t \cos t + t^2 \cos^2 t) \\ &= (\cos^2 t + \sin^2 t) + t^2(\sin^2 t + \cos^2 t) = 1 + t^2 \end{aligned}$$

よって, 求める長さは

$$\begin{aligned} \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt &= \int_0^1 \sqrt{1+t^2} dt = \left[\frac{1}{2} \left\{ t\sqrt{t^2+1} + \log|t + \sqrt{t^2+1}| \right\} \right]_0^1 \\ &= \frac{1}{2} \left\{ (\sqrt{2} + \log|1 + \sqrt{2}|) - (0 + \log 1) \right\} = \frac{1}{2} (\sqrt{2} + \log|1 + \sqrt{2}|) \end{aligned}$$

- (4) $x = \sin^2 t$, $y = \sin t \cos t$ より, $\frac{dx}{dt} = 2 \sin t \cos t = \sin 2t$, $\frac{dy}{dt} = \cos^2 t - \sin^2 t = \cos 2t$ だから, 求める長さは

$$\int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^\pi \sqrt{\sin^2 2t + \cos^2 2t} dt = \int_0^\pi dt = [t]_0^\pi = \pi$$

198

- (1) $x = \sin t$, $y = \sqrt{t}$ より, $\frac{dx}{dt} = \cos t$ だから, 求める体積は

$$\begin{aligned} \pi \int_0^{\frac{\pi}{2}} (\sqrt{t})^2 \cos t dt &= \pi \int_0^{\frac{\pi}{2}} t \cos t dt = \pi \left\{ [t \sin t]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin t dt \right\} \\ &= \pi \left\{ \frac{\pi}{2} \sin \frac{\pi}{2} - 0 - [-\cos t]_0^{\frac{\pi}{2}} \right\} = \pi \left\{ \frac{\pi}{2} + \cos \frac{\pi}{2} - \cos 0 \right\} = \frac{\pi(\pi-2)}{2} \end{aligned}$$

- (2) $x = t^2$, $y = -\cos t$ より, $\frac{dx}{dt} = 2t$ だから, 求める体積は

$$\begin{aligned} \pi \int_0^\pi \cos^2 t \cdot 2t dt &= \pi \int_0^\pi \frac{1+\cos 2t}{2} \cdot 2t dt = \pi \left\{ \int_0^\pi t dt + \int_0^\pi t \cos 2t dt \right\} \\ &= \pi \left\{ \left[\frac{1}{2} t^2 \right]_0^\pi + \left[t \cdot \frac{1}{2} \sin 2t \right]_0^\pi - \int_0^\pi \frac{1}{2} \sin 2t dt \right\} = \pi \left\{ \left(\frac{\pi^2}{2} - 0 \right) + (0-0) - \frac{1}{2} \left[-\frac{1}{2} \cos 2t \right]_0^\pi \right\} \\ &= \pi \left\{ \frac{\pi^2}{2} + \frac{1}{4} (\cos 2\pi - \cos 0) \right\} = \frac{\pi^3}{2} \end{aligned}$$

- (3) $x = \cos^2 t$, $y = \sin t \cos t$ より, $\frac{dx}{dt} = 2 \cos t \sin t$ だから, 求める体積を V とすると

$$\begin{aligned} \pi \int_0^{\frac{\pi}{2}} (\sin t \cos t)^2 \cdot 2 \cos t \sin t dt &= 2\pi \int_0^{\frac{\pi}{2}} \sin^3 t \cos^3 t dt = 2\pi \int_0^{\frac{\pi}{2}} \sin^3 t (1 - \sin^2 t) \cos t dt \\ &= 2\pi \int_0^{\frac{\pi}{2}} (\sin^3 t - \sin^5 t) \cos t dt \end{aligned}$$

$$\text{ここで, } u = \sin t \text{ とおくと, } \frac{du}{dt} = \cos t \quad du = \cos t dt \quad \begin{array}{c|c} t & 0 \rightarrow \frac{\pi}{2} \\ u & 0 \rightarrow 1 \end{array} \text{ だから}$$

$$V = 2\pi \int_0^1 (u^3 - u^5) du = 2\pi \left[\frac{1}{4} u^4 - \frac{1}{6} u^6 \right]_0^1 = 2\pi \left(\frac{1}{4} - \frac{1}{6} \right) = \frac{\pi}{6}$$

- (4) $x = \cos t$, $y = e^t$ より, $\frac{dx}{dt} = -\sin t$ だから, 求める体積を V とすると

$$\begin{aligned} V &= \pi \int_0^\pi (e^t)^2 |-\sin t| dt = \pi \int_0^\pi e^{2t} \sin t dt = \pi \left\{ [e^{2t}(-\cos t)]_0^\pi - \int_0^\pi 2e^{2t}(-\cos t) dt \right\} \\ &= \pi \left\{ -e^{2\pi} \cos \pi + e^0 \cos 0 + 2 \int_0^\pi e^{2t} \cos t dt \right\} = \pi \left\{ e^{2\pi} + 1 + 2 \left([e^{2t} \sin t]_0^\pi - \int_0^\pi 2e^{2t} \sin t dt \right) \right\} \\ &= \pi \left\{ e^{2\pi} + 1 + 2 \left((e^{2\pi} \sin \pi - e^0 \sin 0) - 2 \int_0^\pi e^{2t} \sin t dt \right) \right\} = \pi(e^{2\pi} + 1) - 4V \\ 5V &= \pi(e^{2\pi} + 1) \quad V = \frac{\pi(e^{2\pi} + 1)}{5} \end{aligned}$$

199

- (1) $\int_1^2 \frac{1}{\sqrt{x^2-1}} dx = \lim_{\varepsilon \rightarrow +0} \int_{1+\varepsilon}^2 \frac{1}{\sqrt{x^2-1}} dx = \lim_{\varepsilon \rightarrow +0} \left[\log|x + \sqrt{x^2-1}| \right]_{1+\varepsilon}^2$

$$\begin{aligned}
&= \lim_{\varepsilon \rightarrow +0} \left\{ \log \left| 2 + \sqrt{3} \right| - \log \left| 1 + \varepsilon + \sqrt{(1 + \varepsilon)^2 - 1} \right| \right\} = \log \left| 2 + \sqrt{3} \right| - \log(1) = \log(2 + \sqrt{3}) \\
(2) \quad &\int_0^\infty x e^{-x^2} dx = \lim_{K \rightarrow \infty} \int_0^K x e^{-x^2} dx = \lim_{K \rightarrow \infty} \left[-\frac{1}{2} e^{-x^2} \right]_0^K = -\frac{1}{2} \lim_{K \rightarrow \infty} (e^{-K^2} - e^0) = -\frac{1}{2} (0 - 1) \\
&= \frac{1}{2} \\
(3) \quad &I = \int_0^K e^{-x} \cos x dx \text{ とおくと} \\
&I = \left[e^{-x} \sin x \right]_0^K - \int_0^K (-e^{-x}) \sin x dx = e^{-K} \sin K - e^0 \sin 0 + \int_0^K e^{-x} \sin x dx \\
&= e^{-K} \sin K + \left[e^{-x} (-\cos x) \right]_0^K - \int_0^K (-e^{-x}) (-\cos x) dx \\
&= e^{-K} \sin K - e^{-K} \cos K + e^0 \cos 0 - \int_0^K (-e^{-x}) (-\cos x) dx = e^{-K} \sin K - e^{-K} \cos K + 1 - I \\
&2I = e^{-K} \sin K - e^{-K} \cos K + 1 \quad I = \frac{1}{2} e^{-K} \sin K - \frac{1}{2} e^{-K} \cos K + \frac{1}{2} \\
&\text{ここで, } -1 \leq \sin K \leq 1 \text{ より, } -e^{-K} \leq e^{-K} \sin K \leq e^{-K} \text{ だから} \\
&0 = \lim_{K \rightarrow \infty} (-e^{-K}) \leq \lim_{K \rightarrow \infty} e^{-K} \sin K \leq \lim_{K \rightarrow \infty} e^{-K} = 0 \quad \lim_{K \rightarrow \infty} e^{-K} \sin K = 0 \\
&\text{同様にして, } \lim_{K \rightarrow \infty} e^{-K} \cos K = 0 \\
&\int_0^\infty e^{-x} \cos x dx = \lim_{K \rightarrow \infty} \int_0^K e^{-x} \cos x dx = \lim_{K \rightarrow \infty} \left\{ \frac{1}{2} e^{-K} \sin K - \frac{1}{2} e^{-K} \cos K + \frac{1}{2} \right\} \\
&= 0 + \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

4 章の問題

1

(1) ② (2) ③

2

③

3

(1) $\frac{2x-9}{(x-2)(x+3)} = \frac{a}{x-2} + \frac{b}{x+3}$ とおき, 分母を払うと, $2x-9 = (a+b)x + (3a-2b)$ だから, 両辺を比較して, $a = -1, b = 3$ を得る。よって

$$\begin{aligned}
\int \frac{2x-9}{(x-2)(x+3)} dx &= \int \left(-\frac{1}{x-2} + \frac{3}{x+3} \right) dx = -\log|x-2| + 3\log|x+3| + C \\
&= \log \left| \frac{(x+3)^3}{x-2} \right| + C
\end{aligned}$$

(2) $\int \frac{3-2x^2}{\sqrt{1-x^2}} dx = \int \frac{1+2(1-x^2)}{\sqrt{1-x^2}} dx = \int \left(\frac{1}{\sqrt{1-x^2}} + 2\sqrt{1-x^2} \right) dx$

$$= \sin^{-1} x + 2 \cdot \frac{1}{2} \left\{ x\sqrt{1-x^2} + \sin^{-1} x \right\} + C = x\sqrt{1-x^2} + 2\sin^{-1} x + C$$

4

(1) $\sin(\pi-x) = \sin x$ より, $\int_{\frac{\pi}{2}}^{\pi} f(\sin x) dx = \int_{\frac{\pi}{2}}^{\pi} f(\sin(\pi-x)) dx$

ここで, $t = \pi - x$ とおくと, $\frac{dt}{dx} = -1 \quad dx = -dt$, $\frac{x}{t} \left| \begin{array}{c} \frac{\pi}{2} \rightarrow \pi \\ \frac{\pi}{2} \rightarrow 0 \end{array} \right.$ だから $\int_{\frac{\pi}{2}}^{\pi} f(\sin x) dx =$

$$\int_{\frac{\pi}{2}}^0 f(\sin t) \cdot (-1) dt = \int_0^{\frac{\pi}{2}} f(\sin t) dt = \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$\int_0^{\pi} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\sin x) dx + \int_{\frac{\pi}{2}}^{\pi} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\sin x) dx + \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$= 2 \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

$$(2) \quad \sin(x - 2\pi) = \sin x \text{ より, } \int_{2\pi-c}^{2\pi} f(\sin x) dx = \int_{2\pi-c}^{2\pi} f(\sin(x - 2\pi)) dx$$

$$\text{ここで, } t = x - 2\pi \text{ とおくと, } \frac{dt}{dx} = 1 \quad dt = dx, \quad \begin{array}{c|c} x & 2\pi - c \rightarrow 2\pi \\ t & -c \rightarrow 0 \end{array} \text{ だから}$$

$$\int_{2\pi-c}^{2\pi} f(\sin x) dx = \int_{-c}^0 f(\sin t) dt = \int_{-c}^0 f(\sin x) dx$$

$$\begin{aligned} \int_0^{2\pi} f(\sin x) dx &= \int_0^{2\pi-c} f(\sin x) dx + \int_{2\pi-c}^{2\pi} f(\sin x) dx = \int_0^{2\pi-c} f(\sin x) dx + \int_{-c}^0 f(\sin x) dx \\ &= \int_{-c}^{2\pi-c} f(\sin x) dx \end{aligned}$$

$$(3) \quad \text{合成公式より, } a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \alpha) \text{ と表せる。ただし, } \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}},$$

$$\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}} \text{ である。ここで, } t = x + \alpha \text{ とおくと, } dt = dx, \quad \begin{array}{c|c} x & 0 \rightarrow 2\pi \\ t & \alpha \rightarrow 2\pi + \alpha \end{array} \text{ だから}$$

$$\begin{aligned} \int_0^{2\pi} f(a \cos x + b \sin x) dx &= \int_0^{2\pi} f(\sqrt{a^2 + b^2} \sin(x + \alpha)) dx = \int_{\alpha}^{2\pi+\alpha} f(\sqrt{a^2 + b^2} \sin t) dt \\ &= \int_{-\pi}^{\pi} f(\sqrt{a^2 + b^2} \sin t) dt = \int_{-\pi}^0 f(\sqrt{a^2 + b^2} \sin t) dt + \int_0^{\pi} f(\sqrt{a^2 + b^2} \sin t) dt \end{aligned}$$

ただし, 3 番目の等号は (2) を利用した ($c = \pi + \alpha$ とした)。

$$(1) \text{ より, } \int_0^{\pi} f(\sqrt{a^2 + b^2} \sin t) dt = 2 \int_0^{\frac{\pi}{2}} f(\sqrt{a^2 + b^2} \sin t) dt$$

$$\text{また, } u = -t \text{ とおくと, } du = -dt \quad \begin{array}{c|c} t & -\pi \rightarrow 0 \\ u & \pi \rightarrow 0 \end{array} \text{ だから, (1) より}$$

$$\begin{aligned} \int_{-\pi}^0 f(\sqrt{a^2 + b^2} \sin t) dt &= \int_{\pi}^0 f(\sqrt{a^2 + b^2} \sin(-u)) du = \int_0^{\pi} f(-\sqrt{a^2 + b^2} \sin u) du \\ &= 2 \int_0^{\frac{\pi}{2}} f(-\sqrt{a^2 + b^2} \sin u) du = 2 \int_{\frac{\pi}{2}}^0 f(\sqrt{a^2 + b^2} \sin(-u)) du = 2 \int_{-\frac{\pi}{2}}^0 f(\sqrt{a^2 + b^2} \sin t) dt \\ \int_0^{2\pi} f(a \cos x + b \sin x) dx &= 2 \int_{-\frac{\pi}{2}}^0 f(\sqrt{a^2 + b^2} \sin t) dt + 2 \int_0^{\frac{\pi}{2}} f(\sqrt{a^2 + b^2} \sin t) dt \\ &= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\sqrt{a^2 + b^2} \sin t) dt = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\sqrt{a^2 + b^2} \sin x) dx \end{aligned}$$

5

$$\begin{aligned} (1) \quad \int t \sin^2 t dt &= \int t \cdot \frac{1 - \cos 2t}{2} dt = \frac{1}{2} \left\{ \int t dt - \int t \cos 2t dt \right\} \\ &= \frac{1}{2} \left\{ \frac{1}{2} t^2 - \left(\frac{1}{2} t \sin 2t - \frac{1}{2} \int \sin 2t dt \right) \right\} = \frac{1}{4} \left\{ t^2 - t \sin 2t - \frac{1}{2} \cos 2t \right\} + C \\ &= \frac{1}{8} (2t^2 - 2t \sin 2t - \cos 2t) + C \end{aligned}$$

$$\begin{aligned} (2) \quad x(t) &= -\frac{t}{\pi} \cos t, y(t) = \sin t \text{ より, } \frac{dx}{dt} = -\frac{1}{\pi} \cos t + \frac{t}{\pi} \sin t \text{ だから, 求める面積は} \\ \int_{\frac{\pi}{2}}^{\pi} \left| \sin t \cdot \left(-\frac{1}{\pi} \cos t + \frac{t}{\pi} \sin t \right) \right| dt &= \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} (-\sin t \cos t + t \sin^2 t) dt \\ &= \frac{1}{\pi} \left[-\frac{1}{2} \sin^2 t + \frac{1}{8} (2t^2 - 2t \sin 2t - \cos 2t) \right]_{\frac{\pi}{2}}^{\pi} = -\frac{1}{8\pi} \left[4 \sin^2 t - 2t^2 + 2t \sin 2t + \cos 2t \right]_{\frac{\pi}{2}}^{\pi} \\ &= -\frac{1}{8\pi} \left\{ (0 - 2\pi^2 + 0 + 1) - \left(4 - \frac{\pi^2}{2} + 0 - 1 \right) \right\} = -\frac{1}{8\pi} \left\{ -\frac{3\pi^2}{2} - 2 \right\} = \frac{3\pi^2 + 4}{16\pi} \end{aligned}$$

6

$$(1) \quad S = \int_t^{2t} (3 - e^x) dx = \left[3x - e^x \right]_t^{2t} = (6t - e^{2t}) - (3t - e^t) = 3t - e^{2t} + e^t$$

$$(2) \quad \frac{dS}{dt} = 3 - 2e^{2t} + e^t = -(2e^t - 3)(e^t + 1) \text{ より, } \frac{dS}{dt} = 0 \text{ となるのは, } e^t = \frac{3}{2}, \text{ すなわち, } t = \log \frac{3}{2} \text{ のときである。}$$

t	0		$\log \frac{3}{2}$		$\frac{1}{2} \log 3$
$\frac{dS}{dt}$	+	+	0	-	-
S	0	\nearrow	$3 \log \frac{3}{2} - \frac{3}{4}$	\searrow	$\frac{3}{2} \log 3 - 3 + \sqrt{3}$

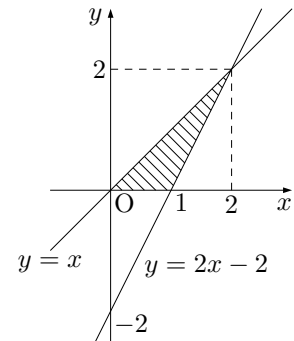
増減表より, $t = \log \frac{3}{2}$ のとき最大値 $3 \log \frac{3}{2} - \frac{3}{4}$

7

$$(1) \quad \pi \int_0^1 (\sqrt{x})^2 dx = \pi \int_0^1 x dx = \pi \left[\frac{1}{2} x^2 \right]_0^1 = \frac{\pi}{2}$$

(2) 直線 $y = x$ と $y = 2x - 2$ の交点は $(2, 2)$ であり, 右図の斜線部を回転して得られる回転体の体積を求めればよいから

$$\begin{aligned} & \pi \int_0^2 x^2 dx - \pi \int_1^2 (2x - 2)^2 dx = \pi \int_0^2 x^2 dx - 4\pi \int_1^2 (x^2 - 2x + 1) dx \\ &= \pi \left\{ \left[\frac{1}{3} x^3 \right]_0^2 - 4 \left[\frac{1}{3} x^3 - x^2 + x \right]_1^2 \right\} \\ &= \pi \left\{ \frac{8}{3} - 4 \left(\frac{8}{3} - 4 + 2 - \frac{1}{3} + 1 - 1 \right) \right\} = \pi \left\{ \frac{8}{3} - 4 \cdot \frac{1}{3} \right\} = \frac{4\pi}{3} \end{aligned}$$



8

$x = \tan^{-1} t$, $y = \frac{1}{t^2}$ より, $\frac{dx}{dt} = \frac{1}{1+t^2}$, $\frac{x}{t} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \rightarrow \frac{\pi}{2}$ だから, 求める面積は

$$\begin{aligned} & \int_1^\infty \left| \frac{1}{t^2} \cdot \frac{1}{1+t^2} \right| dt = \int_1^\infty \left(\frac{1}{t^2} - \frac{1}{1+t^2} \right) dt = \lim_{K \rightarrow \infty} \int_1^K \left(\frac{1}{t^2} - \frac{1}{1+t^2} \right) dt \\ &= \lim_{K \rightarrow \infty} \left[-\frac{1}{t} - \tan^{-1} t \right]_1^K = \lim_{K \rightarrow \infty} \left\{ -\frac{1}{K} - \tan^{-1} K + \frac{1}{1} + \tan^{-1} 1 \right\} = 0 - \frac{\pi}{2} + 1 + \frac{\pi}{4} \\ &= 1 - \frac{\pi}{4} \end{aligned}$$