新版微分積分 II 演習 解答

2 章 積分法

1 節 定積分と不定積分

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(1) $\displaystyle \lim_{n\to\infty}\sum_{k=1}^{n}\left\{a+\frac{k(b-a)}{n}-1\right\}\frac{b-a}{n}=\lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{(b-a)(a-1)}{n}+\frac{(b-a)^2}{n^2}\sum_{k=1}^{n}k\right\}$

$\displaystyle =\lim_{n\to\infty}\left\{\frac{(b-a)(a-1)}{\cancel{n}}\cancel{n}+\frac{(b-a)^2}{n^2}\cdot\frac{1}{2}n(n+1)\right\}=\lim_{n\to\infty}\left\{(b-a)(a-1)+\frac{1}{2}(b-a)^2\left(1+\frac{1}{n}\right)\right\}$

$\displaystyle =(b-a)(a-1)+\frac{1}{2}(b-a)^2=\frac{1}{2}(b-a)(b+a-2)$

(2) $\displaystyle \lim_{n\to\infty}\sum_{k=1}^{n}\left\{-2\left(a+\frac{k(b-a)}{n}\right)\cdot\frac{b-a}{n}\right\}=\lim_{n\to\infty}\left\{\sum_{k=1}^{n}\frac{-2a(b-a)}{n}-\frac{2(b-a)^2}{n^2}\sum_{k=1}^{n}k\right\}$

$\displaystyle =\lim_{n\to\infty}\left\{\frac{-2a(b-a)}{n}\cdot n-\frac{\cancel{2}(b-a)^2}{n^2}\cdot\frac{1}{\cancel{2}}n(n+1)\right\}=\lim_{n\to\infty}\left\{-2a(b-a)-(b-a)^2\left(1+\frac{1}{n}\right)\right\}$

$\displaystyle =-2a(b-a)-(b-a)^2=a^2-b^2$

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(1) $\displaystyle \lim_{n\to 0}\sum_{k=1}^{n}\left(\frac{2k}{n}\right)^2\frac{2}{n}=\lim_{n\to 0}\frac{8}{n^3}\sum_{k=1}^{n}k^2=\lim_{n\to 0}\frac{8}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)=\lim_{n\to 0}\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$\displaystyle =\frac{8}{3}$

(2) $\displaystyle \lim_{n\to 0}\sum_{k=1}^{n}\left\{\left(\frac{2k}{n}\right)^2+1\right\}\frac{2}{n}=\lim_{n\to 0}\left\{\frac{8}{n^3}\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}\frac{2}{n}\right\}=\lim_{n\to 0}\left\{\frac{8}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)+\frac{2}{\cancel{n}}\cdot\cancel{n}\right\}$

$\displaystyle =\lim_{n\to 0}\left\{\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+2\right\}=\frac{8}{3}+2=\frac{14}{3}$

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(1) $\displaystyle \lim_{n\to 0}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^2\frac{1}{n}=\lim_{n\to 0}\sum_{k=1}^{n}\left\{\frac{1}{n}+\frac{2k}{n^2}+\frac{k^2}{n^3}\right\}=\lim_{n\to 0}\left\{\sum_{k=1}^{n}\frac{1}{n}+\frac{2}{n^2}\sum_{k=1}^{n}k+\frac{1}{n^3}\sum_{k=1}^{n}k^2\right\}$

$\displaystyle =\lim_{n\to 0}\left\{\frac{1}{\cancel{n}}\cdot\cancel{n}+\frac{\cancel{2}}{n^2}\cdot\frac{1}{\cancel{2}}n(n+1)+\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)\right\}$

$\displaystyle =\lim_{n\to 0}\left\{1+\left(1+\frac{1}{n}\right)+\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right\}=1+1+\frac{1}{3}=\frac{7}{3}$

(2) $\displaystyle \lim_{n\to 0}\sum_{k=1}^{n}\left\{\cancel{1}+\frac{k}{n}-\cancel{1}\right\}^2\frac{1}{n}=\lim_{n\to 0}\sum_{k=1}^{n}\frac{k^2}{n^3}=\lim_{n\to 0}\frac{1}{n^3}\sum_{k=1}^{n}k^2=\lim_{n\to 0}\frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)$

$\displaystyle =\lim_{n\to 0}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)=\frac{1}{3}$

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(1) $2x^2-x$ を $x-1$ で割ると，商が $2x+1$，余りが 1 だから，$2x^2-x=(x-1)(2x+1)+1$

$\displaystyle \therefore\ \int\frac{2x^2-x}{x-1}\,dx=\int\left(2x+1+\frac{1}{x-1}\right)dx=x^2+x+\log|x-1|+C$

(2) x^2-2x+1 を x^2+1 で割ると，商が 1，余りが $-2x$ だから，$x^2-2x+1=(x^2+1)\cdot 1-2x$

$\displaystyle \therefore\ \int\frac{x^2-2x+1}{x^2+1}\,dx=\int\left(1-\frac{2x}{x^2+1}\right)dx=x-\log(x^2+1)+C$

(3) $3x^3+5x^2+2x+1$ を $x+1$ で割ると，商が $3x^2+2x$，余りが 1 だから，$3x^3+5x^2+2x+1=(x+1)(3x^2+2x)+1$

$\displaystyle \therefore\ \int\frac{3x^2+5x^2+2x+1}{x+1}\,dx=\int\left(3x^2+2x+\frac{1}{x+1}\right)dx=x^3+x^2+\log|x+1|+C$

(4) x^3+x+2 を x^2+1 で割ると，商が x，余りが 2 だから，$x^3+x+2=(x^2+1)x+2$

$$\therefore \int \frac{x^3+x+2}{x^2+1}\,dx = \int \left(x+\frac{2}{x^2+1}\right)dx = \frac{1}{2}x^2+2\mathrm{Tan}^{-1}\,x+C$$

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(1) $\displaystyle\int \frac{1}{x^2+3x+2}\,dx = \int \left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx = \log|x+1|-\log|x+2|+C = \log\left|\frac{x+1}{x+2}\right|+C$

(2) $P(x)=x^3-x^2+x-1$ とおくと，$P(1)=1^3-1^2+1-1=0$ だから，因数定理より，$P(x)$ は $x-1$ で割り切れ，$P(x)=(x-1)(x^2+1)$ となる。$\dfrac{3x^2-x}{x^3-x^2+x-1}=\dfrac{a}{x-1}+\dfrac{bx+c}{x^2+1}$ とおき，分母を払うと，$3x^2-x=(a+b)x^2+(-b+c)x+(a-c)$ だから，両辺を比較して，$a=1$，$b=2$，$c=1$ を得る。よって，$\dfrac{3x^2-x}{x^3-x^2+x-1}=\dfrac{1}{x-1}+\dfrac{2x}{x^2+1}+\dfrac{1}{x^2+1}$

$$\therefore \int \frac{3x^2-x}{x^3-x^2+x-1}\,dx = \int \left(\frac{1}{x-1}+\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx$$

$$=\log|x-1|+\log(x^2+1)+\mathrm{Tan}^{-1}\,x+C$$

(3) $\dfrac{3x^2+7x+5}{(x+1)(x^2+2x+2)}=\dfrac{a}{x+1}+\dfrac{bx+c}{x^2+2x+2}$ とおき，分母を払うと，$3x^2+7x+5=(a+b)x^2+(2a+b+c)x+(2a+c)$ だから，両辺を比較して，$a=1$，$b=2$，$c=3$ を得る。

よって，$\dfrac{1}{x+1}+\dfrac{2x+3}{x^2+2x+2}=\dfrac{1}{x+1}+\dfrac{2x+2}{x^2+2x+2}+\dfrac{1}{(x+1)^2+1}$

$$\therefore \int \frac{3x^2+7x+5}{(x+1)(x^2+2x+2)}\,dx = \int \left(\frac{1}{x+1}+\frac{2x+2}{x^2+2x+2}+\frac{1}{(x+1)^2+1}\right)dx$$

$$=\log|x+1|+\log(x^2+2x+2)+\mathrm{Tan}^{-1}\,(x+1)+C$$

(4) $\dfrac{2x^2+7x+4}{x(x+2)^2}=\dfrac{a}{x}+\dfrac{b}{x+2}+\dfrac{c}{(x+2)^2}$ とおき，分母を払うと，

$2x^2+7x+4=(a+b)x^2+(4a+2b+c)x+4a$ だから，両辺を比較して，$a=b=c=1$ を得る。

よって，$\dfrac{2x^2+7x+4}{x(x+2)^2}=\dfrac{1}{x}+\dfrac{1}{x+2}+\dfrac{1}{(x+2)^2}$

$$\therefore \int \frac{2x^2+7x+4}{x(x+2)^2}\,dx = \int \left(\frac{1}{x}+\frac{1}{x+2}+\frac{1}{(x+2)^2}\right)dx = \log|x|+\log|x+2|-\frac{1}{x+2}+C$$

$$=\log|x(x+2)|-\frac{1}{x+2}+C$$

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(1) $t=\tan\dfrac{x}{2}$ とおくと，$\cos x=\dfrac{1-t^2}{1+t^2}$，$dx=\dfrac{2}{1+t^2}\,dt$，$\dfrac{x}{2}=\mathrm{Tan}^{-1}\,t$ だから，

$$\int \frac{1-\cos x}{1+\cos x}\,dx = \int \frac{1-\dfrac{1-t^2}{1+t^2}}{1+\dfrac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{(\cancel{1}+t^2)-(\cancel{1}-t^2)}{(1+\cancel{t^2})+(1-\cancel{t^2})}\cdot\frac{2}{1+t^2}\,dt$$

$$=\int \frac{\cancel{2}t^2}{\cancel{2}}\cdot\frac{2}{1+t^2}\,dt = 2\int \frac{(1+t^2)-1}{1+t^2}\,dt = 2\int\left(1-\frac{1}{1+t^2}\right)dt = 2t-2\mathrm{Tan}^{-1}\,t+C$$

$$=2\cdot\tan\frac{x}{2}-\cancel{2}\cdot\frac{x}{\cancel{2}}+C = 2\tan\frac{x}{2}-x+C$$

［別解］

$$\int \frac{1-\cos x}{1+\cos x}\,dx = \int \frac{\dfrac{1-\cos x}{2}}{\dfrac{1+\cos x}{2}}\,dx = \int \frac{\sin^2\dfrac{x}{2}}{\cos^2\dfrac{x}{2}}\,dx = \int \frac{1-\cos^2\dfrac{x}{2}}{\cos^2\dfrac{x}{2}}\,dx$$

$$=\int\left(\frac{1}{\cos^2\dfrac{x}{2}}-1\right)dx = 2\tan\frac{x}{2}-x+C$$

(2) $t=\tan\dfrac{x}{2}$ とおくと，$\tan x=\dfrac{2t}{1-t^2}$，$dx=\dfrac{2}{1+t^2}\,dt$ だから，

$$\int\left(\frac{1}{\tan x}+\tan\frac{x}{2}\right)dx=\int\left(\frac{1}{\frac{2t}{1-t^2}}+t\right)\frac{2}{1+t^2}\,dt=\int\left(\frac{1-t^2}{2t}+\frac{2t^2}{2t}\right)\frac{2}{1+t^2}\,dt$$

$$=\int\frac{\cancel{1+t^2}}{\cancel{2}t}\cdot\frac{\cancel{2}}{\cancel{1+t^2}}\,dt=\int\frac{1}{t}\,dt=\log|t|+C=\log\left|\tan\frac{x}{2}\right|+C$$

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(1) $\displaystyle\int\frac{1}{\sqrt{4x-x^2}}\,dx=\int\frac{1}{\sqrt{4-(x^2-4x+4)}}\,dx=\int\frac{1}{\sqrt{2^2-(x-2)^2}}\,dx=\mathrm{Sin}^{-1}\,\frac{x-2}{2}+C$

(2) $\displaystyle\int\frac{1}{\sqrt{8-6x-9x^2}}\,dx=\int\frac{1}{\sqrt{9-(9x^2+6x+1)}}\,dx=\int\frac{1}{\sqrt{3^2-(3x+1)^2}}\,dx$

$\displaystyle=\frac{1}{3}\mathrm{Sin}^{-1}\,\frac{3x+1}{3}+C$

(3) $\displaystyle\int\sqrt{16-6x-x^2}\,dx=\int\sqrt{25-(x^2+6x+9)}\,dx=\int\sqrt{5^2-(x+3)^2}\,dx$

$\displaystyle=\frac{1}{2}\left\{(x+3)\sqrt{16-6x-x^2}+25\,\mathrm{Sin}^{-1}\,\frac{x+3}{5}\right\}+C$

(4) $\displaystyle\int\sqrt{6x-9x^2}\,dx=\int\sqrt{1-(9x^2-6x+1)}\,dx=\int\sqrt{1-(3x-1)^2}\,dx$

$\displaystyle=\frac{1}{6}\left\{(3x-1)\sqrt{6x-9x^2}+\mathrm{Sin}^{-1}\,(3x-1)\right\}+C$

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(1) $\displaystyle\int\frac{1}{\sqrt{x^2+4x+5}}\,dx=\int\frac{1}{\sqrt{(x+2)^2+1}}\,dx=\log\left|x+2+\sqrt{x^2+4x+5}\right|+C$

(2) $\displaystyle\int\frac{1}{\sqrt{4x^2+4x+3}}\,dx=\int\frac{1}{\sqrt{(2x+1)^2+2}}\,dx=\frac{1}{2}\log\left|2x+1+\sqrt{4x^2+4x+3}\right|+C$

(3) $\displaystyle\int\sqrt{x^2+6x+10}\,dx=\int\sqrt{(x+3)^2+1}\,dx$

$\displaystyle=\frac{1}{2}\left\{(x+3)\sqrt{x^2+6x+10}+\log\left|x+3+\sqrt{x^2+6x+10}\right|\right\}+C$

(4) $\displaystyle\int\sqrt{4x^2-4x+7}\,dx=\int\sqrt{(2x-1)^2+6}\,dx$

$\displaystyle=\frac{1}{2}\left\{(2x-1)\sqrt{4x^2-4x+7}+6\log\left|(2x-1)+\sqrt{4x^2-4x+7}\right|\right\}\cdot\frac{1}{2}+C$

$\displaystyle=\frac{1}{4}\left\{(2x-1)\sqrt{4x^2-4x+7}+6\log\left|2x-1+\sqrt{4x^2-4x+7}\right|\right\}+C$

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$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3\cdot\frac{1}{n}=\lim_{n\to\infty}\frac{1}{n^4}\sum_{k=1}^{n}k^3=\lim_{n\to\infty}\frac{1}{n^4}\cdot\frac{1}{4}n^2(n+1)^2=\frac{1}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2=\frac{1}{4}$$

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(1) $f(x)$ は偶関数，$\sin x$ は奇関数だから，$f(x)\sin x$ は奇関数である。よって $\displaystyle\int_{-\pi}^{\pi}f(x)\sin x\,dx=0$

(2) $f(x)$，$\cos x$ はともに偶関数だから，$f(x)\cos x$ は偶関数である。よって

$\displaystyle\int_{-\pi}^{\pi}f(x)\cos x\,dx=2\int_0^{\pi}(\pi-x)\cos x\,dx=2\Big[(\pi-x)\sin x\Big]_0^{\pi}-\int_0^{\pi}(-1)\sin x\,dx=2\int_0^{\pi}\sin x\,dx$

$\displaystyle=2\cdot\Big[-\cos x\Big]_0^{\pi}=4$

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(1) $\displaystyle\int\frac{x^2+2x+2}{x^2+1}\,dx=\int\left(1+\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)dx=x+\log(x^2+1)+\mathrm{Tan}^{-1}\,x+C$

(2) $\displaystyle\int\frac{1}{\sqrt{2-x-x^2}}\,dx=\int\frac{1}{\sqrt{\frac{9}{4}-\left(x^2+x+\frac{1}{4}\right)}}\,dx=\int\frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}}\,dx$

$\displaystyle=\mathrm{Sin}^{-1}\,\frac{x+\frac{1}{2}}{\frac{3}{2}}+C=\mathrm{Sin}^{-1}\,\frac{2x+1}{3}+C$

(3) $t=\tan\dfrac{x}{2}$ とおくと,

$$\int\frac{1}{\sin x+\sin^2 x}\,dx=\int\frac{1}{\dfrac{2t}{1+t^2}+\dfrac{4t^2}{(1+t^2)^2}}\cdot\frac{2}{1+t^2}\,dt=\int\frac{1}{t+\dfrac{2t^2}{1+t^2}}\,dt=\int\frac{t^2+1}{t^3+2t^2+t}\,dt$$

$$=\int\frac{t^2+1}{t(t+1)^2}\,dt=\int\left\{\frac{1}{t}-\frac{2}{(t+1)^2}\right\}dt=\log|t|+\frac{2}{t+1}+C=\log\left|\tan\frac{x}{2}\right|+\frac{2}{\tan\dfrac{x}{2}+1}+C$$

(4) $$\int\sqrt{x^2+x+1}\,dx=\int\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$=\frac{1}{2}\left\{\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}+\frac{3}{4}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|\right\}+C$$

(5) $$\int\frac{2x+1}{\sqrt{x^2+1}}\,dx=\int\left(\frac{2x}{\sqrt{x^2+1}}+\frac{1}{\sqrt{x^2+1}}\right)dx=2\sqrt{x^2+1}+\log\left|x+\sqrt{x^2+1}\right|+C$$

(6) $$\int(2x+1)\sqrt{1-x^2}\,dx=\int\left(2x\sqrt{1-x^2}+\sqrt{1-x^2}\right)dx$$

$$=-\frac{2}{3}\sqrt{(1-x^2)^3}+\frac{1}{2}\left\{x\sqrt{1-x^2}+\mathrm{Sin}^{-1}x\right\}+C$$

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(1) $$\int_{-1}^{0}\frac{3x^3+4x^2-4x+1}{x+2}\,dx=\int_{-1}^{0}\left(3x^2-2x+\frac{1}{x+2}\right)dx=\Big[x^3-x^2+\log|x+2|\Big]_{-1}^{0}=2+\log 2$$

(2) $$\int_{1}^{2}\frac{1}{x^2(x+1)}\,dx=\int_{1}^{2}\left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx=\left[-\log|x|-\frac{1}{x}+\log|x+1|\right]_{1}^{2}$$

$$=-\log 2-\frac{1}{2}+\log 3+1-\log 2=\frac{1}{2}+\log\frac{3}{4}$$

(3) $$\int_{0}^{\frac{\pi}{3}}\frac{1}{\cos x}\,dx=\int_{0}^{\frac{1}{\sqrt3}}\frac{\cancel{1+t^2}}{1-t^2}\cdot\frac{2}{\cancel{1+t^2}}\,dt=-\int_{0}^{\frac{1}{\sqrt3}}\frac{2}{(t-1)(t+1)}\,dt=-\int_{0}^{\frac{1}{\sqrt3}}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=-\Big[\log|t-1|-\log|t+1|\Big]_{0}^{\frac{1}{\sqrt3}}=-\log\left|\frac{1}{\sqrt3}-1\right|+\log\left|\frac{1}{\sqrt3}+1\right|+\cancel{\log 1}-\cancel{\log 1}$$

$$=\log\left|\frac{\dfrac{1}{\sqrt3}+1}{\dfrac{1}{\sqrt3}-1}\right|=\log\left|\frac{1+\sqrt3}{1-\sqrt3}\right|=\log\left|\frac{4+2\sqrt3}{-2}\right|=\log(2+\sqrt3)$$

(4) $$\int_{0}^{1}\sqrt{2-x^2}\,dx=\left[\frac{1}{2}\left\{x\sqrt{2-x^2}+2\mathrm{Sin}^{-1}\frac{x}{\sqrt2}\right\}\right]_{0}^{1}=\frac{1}{2}\left\{1+2\mathrm{Sin}^{-1}\frac{1}{\sqrt2}-0-\cancel{2\mathrm{Sin}^{-1}0}\right\}$$

$$=\frac{1}{2}\left\{1+2\cdot\frac{\pi}{4}\right\}=\frac{2+\pi}{4}$$

(5) $$\int_{0}^{4}\frac{1}{\sqrt{x^2+9}}\,dx=\Big[\log\left|x+\sqrt{x^2+9}\right|\Big]_{0}^{4}=\log\left|4+\sqrt{25}\right|-\log\left|0+\sqrt9\right|=\log 9-\log 3=\log 3$$

(6) $$\int_{0}^{3}\sqrt{x^2+16}\,dx=\left[\frac{1}{2}\left\{x\sqrt{x^2+16}+16\log\left|x+\sqrt{x^2+16}\right|\right\}\right]_{0}^{3}$$

$$=\frac{1}{2}\left\{3\sqrt{25}+16\log\left|3+\sqrt{25}\right|-0-16\log\sqrt{16}\right\}=\frac{1}{2}\left\{15+16\log 8-16\log 4\right\}=\frac{15}{2}+8\log 2$$

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(1) $t=\dfrac{\pi}{2}-x$ とおくと, $\dfrac{dt}{dx}=-1\quad\therefore\ dx=-dt$

x	0	$\to$	$\frac{\pi}{2}$
t	$\frac{\pi}{2}$	$\to$	0

$$\int_{0}^{\frac{\pi}{2}}f(\sin x)\,dx=\int_{0}^{\frac{\pi}{2}}f\left(\cos\left(\frac{\pi}{2}-x\right)\right)dx=\int_{\frac{\pi}{2}}^{0}f(\cos t)\cdot(-1)dt=\int_{0}^{\frac{\pi}{2}}f(\cos t)\,dt=\int_{0}^{\frac{\pi}{2}}f(\cos x)\,dx$$

(2) $t=Lx$ とおくと, $\dfrac{dt}{dx}=L\quad\therefore\ dx=\dfrac{1}{L}dt,$

x	-1	$\to$	1
t	$-L$	$\to$	L

$$\therefore \int_{-1}^{1} f(Lx)\,dx = \int_{-L}^{L} f(t)\frac{1}{L}\,dt = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

(3) $s = xt$ とおくと, $\dfrac{ds}{dt} = x \quad \therefore dt = \dfrac{1}{x}ds,\quad \begin{array}{c|ccc} t & 0 & \to & 1 \\ \hline s & 0 & \to & x \end{array}$

$$\therefore \frac{d}{dx}\left\{x\int_0^1 f(xt)\,dt\right\} = \frac{d}{dx}\left\{x\int_0^x f(s)\cdot\frac{1}{\cancel{x}}\,ds\right\} = \frac{d}{dx}\left\{\int_0^x f(s)\,ds\right\} = f(x)$$

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(1) $(与式) = \displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\left(\frac{k}{n}\right)^2}}\cdot\frac{1}{n} = \int_0^1\frac{1}{\sqrt{1+x^2}}\,dx$

$$= \left[\log\left|x+\sqrt{x^2+1}\right|\right]_0^1 = \log\left|1+\sqrt{2}\right| - \cancel{\log\left|0+\sqrt{1}\right|} = \log(1+\sqrt{2})$$

(2) $(与式) = \displaystyle\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\sum_{k=1}^{n}\sqrt{k} = \lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}\cdot\frac{1}{n} = \int_0^1\sqrt{x}\,dx = \left[\frac{2}{3}\sqrt{x^3}\right]_0^1 = \frac{2}{3}$

2 節 定積分の応用

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(1) $\displaystyle\int_0^{\frac{\pi}{2}}\left|2(1-\cos t)\cdot 2(1-\cos t)\right|dt = 4\int_0^{\frac{\pi}{2}}(1-2\cos t+\cos^2 t)\,dt = 4\int_0^{\frac{\pi}{2}}\left(1-2\cos t+\frac{1+\cos 2t}{2}\right)dt$

$$= 4\left[t-2\sin t+\frac{1}{2}\left(t+\frac{1}{2}\sin 2t\right)\right]_0^{\frac{\pi}{2}} = 4\left(\frac{\pi}{2}-2\sin\frac{\pi}{2}+\frac{\pi}{4}+\frac{1}{4}\cancel{\sin\pi}\right) = 3\pi-8$$

(2) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left|2\sin t\cdot(-2\sin t)\right|dt = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}2\sin^2 t\,dt = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-\cos 2t)\,dt = 2\left[t-\frac{1}{2}\sin 2t\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$$= 2\left\{\left(\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}\right)-\left(-\frac{\pi}{4}-\frac{1}{2}\sin\left(-\frac{\pi}{2}\right)\right)\right\} = 2\left\{\left(\frac{\pi}{4}-\frac{1}{2}\right)+\left(\frac{\pi}{4}-\frac{1}{2}\right)\right\} = \pi-2$$

(3) 曲線の $t=0$ から $t=1$ までの部分と x 軸および直線 $x=1$ で囲まれた図形の面積を求めて 2 倍すればよい。

$$2\int_0^1\left|t^3\cdot 2t\right|dt = 4\int_0^1 t^4\,dt = 4\left[\frac{1}{5}t^5\right]_0^1 = \frac{4}{5}$$

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(1) $\displaystyle\frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+2\sin\theta+\sin^2\theta)\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$

$$= \frac{1}{2}\left[\theta-2\cos\theta+\frac{1}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left\{\left(\frac{\pi}{2}-\cancel{2\cos\frac{\pi}{2}}+\frac{1}{2}\left(\frac{\pi}{2}-\frac{1}{2}\cancel{\sin\pi}\right)\right)-\left(0-2\cos 0+\frac{1}{2}\cancel{(0-0)}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{\pi}{2}+\frac{\pi}{4}+2\right\} = \frac{3\pi}{8}+1$$

(2) レムニスケートの $\theta=0$ から $\theta=\dfrac{\pi}{4}$ までの部分と x 軸で囲まれた図形の面積を 4 倍すればよい。

$$4\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}}a^2\cos 2\theta\,d\theta = \cancel{2}a^2\left[\frac{1}{\cancel{2}}\sin 2\theta\right]_0^{\frac{\pi}{4}} = a^2\left(\sin\frac{\pi}{2}-\cancel{\sin 0}\right) = a^2$$

57

(1) $y = \dfrac{1}{4}x^2$ より $y' = \dfrac{1}{2}x$ だから

$$\int_0^1\sqrt{1+\frac{x^2}{4}}\,dx = \frac{1}{2}\int_0^1\sqrt{x^2+4}\,dx = \frac{1}{2}\left[\frac{1}{2}\left\{x\sqrt{x^2+4}+4\log\left|x+\sqrt{x^2+4}\right|\right\}\right]_0^1$$

$$= \frac{1}{4}\left\{\sqrt{5}+4\log\left|1+\sqrt{5}\right|-0-4\log 2\right\} = \frac{\sqrt{5}}{4}+\log\frac{1+\sqrt{5}}{2}$$

(2) $y=\log|\cos x|$ より $y'=-\dfrac{\sin x}{\cos x}$ だから，求める長さを L とすると

$$\int_0^{\frac{\pi}{3}}\sqrt{1+\frac{\sin^2 x}{\cos^2 x}}\,dx=\int_0^{\frac{\pi}{3}}\sqrt{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}\,dx=\int_0^{\frac{\pi}{3}}\frac{1}{\cos x}\,dx$$

ここで，$t=\tan\dfrac{x}{2}$ とおくと，$\cos x=\dfrac{1-t^2}{1+t^2}$，$dx=\dfrac{2}{1+t^2}dt$，$\begin{array}{c|ccc} x & 0 & \to & \frac{\pi}{3} \\ \hline t & 0 & \to & \frac{\sqrt{3}}{3}\end{array}$ だから

$$L=\int_0^{\frac{\sqrt{3}}{3}}\frac{\cancel{1+t^2}}{1-t^2}\cdot\frac{2}{\cancel{1+t^2}}\,dt=-\int_0^{\frac{\sqrt{3}}{3}}\frac{2}{t^2-1}\,dt=-\int_0^{\frac{\sqrt{3}}{3}}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=-\Big[\log|t-1|-\log|t+1|\Big]_0^{\frac{\sqrt{3}}{3}}=\left[\log\left|\frac{t+1}{t-1}\right|\right]_0^{\frac{\sqrt{3}}{3}}=\log\left|\frac{\frac{\sqrt{3}}{3}+1}{\frac{\sqrt{3}}{3}-1}\right|-\cancel{\log\left|\frac{1}{1}\right|}=\log\left|\frac{1+\sqrt{3}}{1-\sqrt{3}}\right|$$

$$=\log\left|\frac{4+2\sqrt{3}}{2}\right|=\log\left(2+\sqrt{3}\right)$$

58

(1) $x=2t^2$，$y=t^3$ より $\dfrac{dx}{dt}=4t$，$\dfrac{dy}{dt}=3t^2$ だから，求める曲線の長さを L とすると

$$\int_0^1\sqrt{16t^2+9t^4}\,dt=\int_0^1 t\sqrt{16+9t^2}\,dt$$

ここで，$s=16+9t^2$ とおくと，$\dfrac{ds}{dt}=18t$ より，$tdt=\dfrac{1}{18}ds$，$\begin{array}{c|ccc} t & 0 & \to & 1 \\ \hline s & 16 & \to & 25\end{array}$ だから

$$L=\int_{16}^{25}\sqrt{s}\cdot\frac{1}{18}\,ds=\frac{1}{18}\left[\frac{2}{3}\sqrt{s^3}\right]_{16}^{25}=\frac{1}{27}\left(5^3-4^3\right)=\frac{61}{27}$$

(2) $x=a(t-\sin t)$，$y=a(1-\cos t)$ より $\dfrac{dx}{dt}=a(1-\cos t)$，$\dfrac{dy}{dt}=a\sin t$ だから

$$\int_0^{\frac{\pi}{2}}\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\,dt=a\int_0^{\frac{\pi}{2}}\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt=a\int_0^{\frac{\pi}{2}}\sqrt{2(1-\cos t)}\,dt$$

$$=a\int_0^{\frac{\pi}{2}}\sqrt{4\sin^2\frac{t}{2}}\,dt=2a\int_0^{\frac{\pi}{2}}\sin\frac{t}{2}\,dt=2a\left[-2\cos\frac{t}{2}\right]_0^{\frac{\pi}{2}}=-4a\left(\cos\frac{\pi}{4}-\cos 0\right)$$

$$=-4a\left(\frac{\sqrt{2}}{2}-1\right)=2a\left(2-\sqrt{2}\right)$$

(3) $r=e^\theta$ より $\dfrac{dr}{d\theta}=e^\theta$ だから

$$\int_0^\pi\sqrt{e^{2\theta}+e^{2\theta}}\,d\theta=\int_0^\pi\sqrt{2e^{2\theta}}\,d\theta=\sqrt{2}\int_0^\pi e^\theta\,d\theta=\sqrt{2}\Big[e^\theta\Big]_0^\pi=\sqrt{2}\,(e^\pi-1)$$

(4) $r=a(1+\cos\theta)$ より $\dfrac{dr}{d\theta}=-a\sin\theta$ だから

$$\int_{\frac{\pi}{2}}^\pi\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta=a\int_{\frac{\pi}{2}}^\pi\sqrt{1+2\cos\theta+\cos^2\theta+\sin^2\theta}\,d\theta=a\int_{\frac{\pi}{2}}^\pi\sqrt{2(1+\cos\theta)}\,d\theta$$

$$=a\int_{\frac{\pi}{2}}^\pi\sqrt{4\cos^2\frac{\theta}{2}}\,d\theta=2a\int_{\frac{\pi}{2}}^\pi\cos\frac{\theta}{2}\,d\theta=2a\left[2\sin\frac{\theta}{2}\right]_{\frac{\pi}{2}}^\pi=2a\left(2\sin\frac{\pi}{2}-2\sin\frac{\pi}{4}\right)=2a\left(2-\sqrt{2}\right)$$

59

(1) $\pi\displaystyle\int_0^{\frac{\pi}{4}}\tan^2 x\,dx=\pi\int_0^{\frac{\pi}{4}}\left(\frac{1}{\cos^2 x}-1\right)dx=\pi\Big[\tan x-x\Big]_0^{\frac{\pi}{4}}=\pi\left(1-\frac{\pi}{4}-0+0\right)=\frac{\pi(4-\pi)}{4}$

(2) $\pi\displaystyle\int_1^e(\log x)^2\,dx=\pi\left\{\Big[x(\log x)^2\Big]_1^e-\int_1^e \cancel{x}\cdot 2\log x\cdot\frac{1}{\cancel{x}}\,dx\right\}=\pi\left\{e-0-2\int_1^e\log x\,dx\right\}$

$$=\pi\left\{e-2\left(\Big[x\log x\Big]_1^e-\int_1^e \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx\right)\right\}=\pi\left\{e-2\left(e-0-\Big[x\Big]_1^e\right)\right\}=\pi\left\{e-2(\cancel{e}-(\cancel{e}-1))\right\}$$

$$=\pi(e-2)$$

60

(1) $\displaystyle\int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{\varepsilon\to+0}\int_{1+\varepsilon}^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{\varepsilon\to+0}\left[2\sqrt{x-1}\right]_{1+\varepsilon}^2 = \lim_{\varepsilon\to+0}\left(2-2\sqrt{\varepsilon}\right) = 2$

(2) $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{\varepsilon\to+0}\int_0^{2-\varepsilon} \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{\varepsilon\to+0}\left[\mathrm{Sin}^{-1}\frac{x}{2}\right]_0^{2-\varepsilon} = \lim_{\varepsilon\to+0}\left(\mathrm{Sin}^{-1}\frac{2-\varepsilon}{2} - \cancel{\mathrm{Sin}^{-1}0}\right)$

$\displaystyle = \mathrm{Sin}^{-1}1 = \frac{\pi}{2}$

(3) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\substack{\varepsilon\to+0\\ \varepsilon'\to+0}}\int_{-1+\varepsilon}^{1-\varepsilon'} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\substack{\varepsilon\to+0\\ \varepsilon'\to+0}}\left[\mathrm{Sin}^{-1}x\right]_{-1+\varepsilon}^{1-\varepsilon'}$

$\displaystyle = \lim_{\substack{\varepsilon\to+0\\ \varepsilon'\to+0}}\left\{\mathrm{Sin}^{-1}(1-\varepsilon') - \mathrm{Sin}^{-1}(-1+\varepsilon)\right\} = \mathrm{Sin}^{-1}1 - \mathrm{Sin}^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

61

(1) $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx = \lim_{K\to\infty}\int_1^K \frac{1}{x^4}\,dx = \lim_{K\to\infty}\left[-\frac{1}{3x^3}\right]_1^K = -\frac{1}{3}\lim_{K\to\infty}\left(\frac{1}{K^3}-1\right) = \frac{1}{3}$

(2) $\displaystyle\int_{-\infty}^0 e^x\,dx = \lim_{K\to\infty}\int_{-K}^0 e^x\,dx = \lim_{K\to\infty}\left[e^x\right]_{-K}^0 = \lim_{K\to\infty}\left(1-e^{-K}\right) = 1$

(3) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \lim_{\substack{K\to\infty\\ K'\to\infty}}\int_{-K'}^K \frac{1}{1+x^2}\,dx = \lim_{\substack{K\to\infty\\ K'\to\infty}}\left[\mathrm{Tan}^{-1}x\right]_{-K'}^K = \lim_{\substack{K\to\infty\\ K'\to\infty}}\left(\mathrm{Tan}^{-1}K - \mathrm{Tan}^{-1}K'\right)$

$\displaystyle = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

62

(1) $x=\log(t^2+1)$, $y=6t^3$ より，$\dfrac{dx}{dt} = \dfrac{2t}{t^2+1}$ だから，求める面積を S とすると

$\displaystyle S = \int_0^1 \left|6t^3\cdot\frac{2t}{t^2+1}\right|dt = 12\int_0^1 \frac{t^4}{t^2+1}\,dt$

t^4 を t^2+1 で割ると，商が t^2-1，余りが 1 だから，$t^4=(t^2+1)(t^2-1)+1$

$\displaystyle \therefore\ \frac{t^4}{t^2+1} = \frac{(t^2+1)(t^2-1)+1}{t^2+1} = t^2-1+\frac{1}{t^2+1}$

$\displaystyle \therefore\ S = 12\int_0^1\left(t^2-1+\frac{1}{t^2+1}\right)dt = 12\left[\frac{1}{3}t^3 - t + \mathrm{Tan}^{-1}t\right]_0^1 = 12\left\{\left(\frac{1}{3}-1+\frac{\pi}{4}\right)-(0-0+0)\right\}$

$= 3\pi - 8$

(2) $x=\tan t$, $y=\sin t$ より，$\dfrac{dx}{dt} = \dfrac{1}{\cos^2 t}$ だから，求める面積を S とすると

$\displaystyle S = \int_0^{\frac{\pi}{3}}\left|\sin t\cdot\frac{1}{\cos^2 t}\right|dt$

ここで，$u=\cos t$ とおくと，$du = -\sin t\,dt$，$\begin{array}{c|ccc} t & 0 & \to & \frac{\pi}{3} \\ \hline u & 1 & \to & \frac{1}{2}\end{array}$ だから，

$\displaystyle S = \int_1^{\frac{1}{2}} \frac{1}{u^2}(-1)du = \int_{\frac{1}{2}}^1 \frac{1}{u^2}\,du = \left[-\frac{1}{u}\right]_{\frac{1}{2}}^1 = -1 + \frac{1}{\frac{1}{2}} = -1+2 = 1$

(3) $x=\tan t$, $y=\cos t$ より，$\dfrac{dx}{dt} = \dfrac{1}{\cos^2 t}$ だから，求める面積を S とすると

$\displaystyle S = \int_0^{\frac{\pi}{3}}\left|\cos t\cdot\frac{1}{\cos^2 t}\right|dt = \int_0^{\frac{\pi}{4}}\frac{1}{\cos t}\,dt$

ここで，$u=\tan\dfrac{t}{2}$ とおくと，$\cos t = \dfrac{1-u^2}{1+u^2}$，$dt = \dfrac{2}{1+u^2}du$，$\begin{array}{c|ccc} t & 0 & \to & \frac{\pi}{3} \\ \hline u & 0 & \to & \frac{1}{\sqrt{3}}\end{array}$ だから，

$\displaystyle S = \int_0^{\frac{1}{\sqrt{3}}} \frac{\cancel{1+u^2}}{1-u^2}\cdot\frac{2}{\cancel{1+u^2}}\,du = -\int_0^{\frac{1}{\sqrt{3}}}\frac{2}{(u-1)(u+1)}\,du = -\int_0^{\frac{1}{\sqrt{3}}}\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du$

$$= -\Big[\log|u-1| - \log|u+1|\Big]_0^{\frac{1}{\sqrt{3}}} = \left[\log\left|\frac{u+1}{u-1}\right|\right]_0^{\frac{1}{\sqrt{3}}} = \log\left|\frac{\frac{1}{\sqrt{3}}+1}{\frac{1}{\sqrt{3}}-1}\right| - \log\left|\frac{1}{-1}\right|$$

$$= \log\left|\frac{1+\sqrt{3}}{1-\sqrt{3}}\right| - \cancel{\log 1} = \log\frac{\sqrt{3}+1}{\sqrt{3}-1} = \log\frac{4+2\sqrt{3}}{3-1} = \log(2+\sqrt{3})$$

63

(1) $r\cos\theta = \cos 2\theta$ より，$r = \dfrac{\cos 2\theta}{\cos\theta}$ だから，

$$r^2 = \left(\frac{\cos 2\theta}{\cos\theta}\right)^2 = \frac{(2\cos^2\theta - 1)^2}{\cos^2\theta} = \frac{4\cos^4\theta - 4\cos^2\theta + 1}{\cos^2\theta} = 4\cos^2\theta - 4 + \frac{1}{\cos^2\theta}$$

$$= 4\cdot\frac{1+\cos 2\theta}{2} - 4 + \frac{1}{\cos^2\theta} = 2\cos 2\theta - 2 + \frac{1}{\cos^2\theta}$$

よって，求める面積は

$$\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{\cos 2\theta}{\cos\theta}\right)^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(2\cos 2\theta - 2 + \frac{1}{\cos^2\theta}\right)d\theta = \frac{1}{2}\Big[\sin 2\theta - 2\theta + \tan\theta\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left\{\left(\sin\frac{\pi}{2} - \frac{\pi}{2} + \tan\frac{\pi}{4}\right) - \left(\sin\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} + \tan\left(-\frac{\pi}{4}\right)\right)\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{\pi}{2} + 1 + 1 - \frac{\pi}{2} + 1\right\} = 2 - \frac{\pi}{2}$$

(2) $r = 1 + 2\cos\theta$ より，

$$r^2 = (1+2\cos\theta)^2 = 1 + 4\cos\theta + 4\cos^2\theta = 1 + 4\cos\theta + 4\cdot\frac{1+\cos 2\theta}{2} = 3 + 4\cos\theta + 2\cos 2\theta$$

よって，求める面積は

$$\frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}(1+2\cos\theta)^2 d\theta = \frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}(3 + 4\cos\theta + 2\cos 2\theta)\,d\theta = \frac{1}{2}\Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}}$$

$$= \frac{1}{2}\left\{\left(2\pi + 4\sin\frac{2\pi}{3} + \sin\frac{4\pi}{3}\right) - \left(-2\pi + 4\sin\left(-\frac{2\pi}{3}\right) + \sin\left(-\frac{4\pi}{3}\right)\right)\right\}$$

$$= \frac{1}{2}\left\{2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2} + 2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right\} = 2\pi + \frac{3\sqrt{3}}{2}$$

64

(1) $y = \log|\sin x|$ より，$y' = \dfrac{\cos x}{\sin x}$ だから，

$$\sqrt{1+(y')^2} = \sqrt{1 + \frac{\cos^2 x}{\sin^2 x}} = \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} = \frac{1}{\sin x}$$

$t = \tan\dfrac{x}{2}$ とおくと，$\sin x = \dfrac{2t}{1+t^2}$，$dx = \dfrac{2}{1+t^2}dt$，$\begin{array}{c|ccc} x & \frac{\pi}{3} & \to & \frac{\pi}{2} \\ \hline t & \frac{1}{\sqrt{3}} & \to & 1 \end{array}$ だから，求める長さは

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sqrt{1+(y')^2}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{\sin x}\,dx = \int_{\frac{1}{\sqrt{3}}}^{1}\frac{\cancel{1+t^2}}{\cancel{2}t}\cdot\frac{\cancel{2}}{\cancel{1+t^2}}\,dt = \int_{\frac{1}{\sqrt{3}}}^{1}\frac{1}{t}\,dt = \Big[\log t\Big]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= \cancel{\log 1} - \log\frac{1}{\sqrt{3}} = -\log 3^{-\frac{1}{2}} = \frac{1}{2}\log 3$$

(2) $x = e^t\cos t$，$y = e^t\sin t$ より，$\dfrac{dx}{dt} = e^t\cos t - e^t\sin t = e^t(\cos t - \sin t)$，

$\dfrac{dy}{dt} = e^t\sin t + e^t\cos t = e^t(\cos t + \sin t)$ だから，

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}(\cos^2 t - \cancel{2\cos t\sin t} + \sin^2 t) + e^{2t}(\cos^2 t + \cancel{2\cos t\sin t} + \sin^2 t)$$

$$= 2e^{2t}(\cos^2 t + \sin^2 t) = 2e^{2t}$$

よって，求める長さは

$$\int_0^1\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{2}\int_0^1 e^t\,dt = \sqrt{2}\Big[e^t\Big]_0^1 = \sqrt{2}(e-1)$$

(3) $x=t\cos t$，$y=t\sin t$ より，$\dfrac{dx}{dt}=\cos t-t\sin t$，$\dfrac{dy}{dt}=\sin t+t\cos t$ だから，

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=(\cos^2 t-\cancel{2t\cos t\sin t}+t^2\sin^2 t)+(\sin^2 t+\cancel{2t\sin t\cos t}+t^2\cos^2 t)$$

$$=(\cos^2 t+\sin^2 t)+t^2(\sin^2 t+\cos^2 t)=1+t^2$$

よって，求める長さは

$$\int_0^1\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^1\sqrt{1+t^2}\,dt=\left[\frac12\left\{t\sqrt{t^2+1}+\log\left|t+\sqrt{t^2+1}\right|\right\}\right]_0^1$$

$$=\frac12\left\{\left(\sqrt2+\log\left|1+\sqrt2\right|\right)-(0+\cancel{\log 1})\right\}=\frac12\left(\sqrt2+\log\left|1+\sqrt2\right|\right)$$

(4) $x=\sin^2 t$，$y=\sin t\cos t$ より，$\dfrac{dx}{dt}=2\sin t\cos t=\sin 2t$，$\dfrac{dy}{dt}=\cos^2 t-\sin^2 t=\cos 2t$ だから，求める長さは

$$\int_0^\pi\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^\pi\sqrt{\sin^2 2t+\cos^2 2t}\,dt=\int_0^\pi dt=\Big[t\Big]_0^\pi=\pi$$

65

(1) $x=\sin t$，$y=\sqrt{t}$ より，$\dfrac{dx}{dt}=\cos t$ だから，求める体積は

$$\pi\int_0^{\frac{\pi}{2}}\left(\sqrt{t}\right)^2\cos t\,dt=\pi\int_0^{\frac{\pi}{2}}t\cos t\,dt=\pi\left\{\Big[t\sin t\Big]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin t\,dt\right\}$$

$$=\pi\left\{\frac{\pi}{2}\sin\frac{\pi}{2}-0-\Big[-\cos t\Big]_0^{\frac{\pi}{2}}\right\}=\pi\left\{\frac{\pi}{2}+\cancel{\cos\frac{\pi}{2}}-\cos 0\right\}=\frac{\pi(\pi-2)}{2}$$

(2) $x=t^2$，$y=-\cos t$ より，$\dfrac{dx}{dt}=2t$ だから，求める体積は

$$\pi\int_0^\pi\cos^2 t\cdot 2t\,dt=\pi\int_0^\pi\frac{1+\cos 2t}{\cancel{2}}\cdot\cancel{2}t\,dt=\pi\left\{\int_0^\pi t\,dt+\int_0^\pi t\cos 2t\,dt\right\}$$

$$=\pi\left\{\left[\frac12t^2\right]_0^\pi+\left[t\cdot\frac12\sin 2t\right]_0^\pi-\int_0^\pi\frac12\sin 2t\,dt\right\}=\pi\left\{\left(\frac{\pi^2}{2}-0\right)+(0-0)-\frac12\left[-\frac12\cos 2t\right]_0^\pi\right\}$$

$$=\pi\left\{\frac{\pi^2}{2}+\frac14\cancel{(\cos 2\pi-\cos 0)}\right\}=\frac{\pi^3}{2}$$

(3) $x=\cos^2 t$，$y=\sin t\cos t$ より，$\dfrac{dx}{dt}=2\cos t\sin t$ だから，求める体積を V とすると

$$\pi\int_0^{\frac{\pi}{2}}(\sin t\cos t)^2\cdot 2\cos t\sin t\,dt=2\pi\int_0^{\frac{\pi}{2}}\sin^3 t\cos^3 t\,dt=2\pi\int_0^{\frac{\pi}{2}}\sin^3 t(1-\sin^2 t)\cos t\,dt$$

$$=2\pi\int_0^{\frac{\pi}{2}}(\sin^3 t-\sin^5 t)\cos t\,dt$$

ここで，$u=\sin t$ とおくと，$\dfrac{du}{dt}=\cos t$　$\therefore\ du=\cos t\,dt$　$\begin{array}{c|ccc} t & 0 & \to & \frac{\pi}{2} \\ \hline u & 0 & \to & 1 \end{array}$ だから

$$V=2\pi\int_0^1(u^3-u^5)\,du=2\pi\left[\frac14u^4-\frac16u^6\right]_0^1=2\pi\left(\frac14-\frac16\right)=\frac{\pi}{6}$$

(4) $x=\cos t$，$y=e^t$ より，$\dfrac{dx}{dt}=-\sin t$ だから，求める体積を V とすると

$$V=\pi\int_0^\pi(e^t)^2|-\sin t|\,dt=\pi\int_0^\pi e^{2t}\sin t\,dt=\pi\left\{\Big[e^{2t}(-\cos t)\Big]_0^\pi-\int_0^\pi 2e^{2t}(-\cos t)\,dt\right\}$$

$$=\pi\left\{-e^{2\pi}\cos\pi+e^0\cos 0+2\int_0^\pi e^{2t}\cos t\,dt\right\}=\pi\left\{e^{2\pi}+1+2\left(\Big[e^{2t}\sin t\Big]_0^\pi-\int_0^\pi 2e^{2t}\sin t\,dt\right)\right\}$$

$$=\pi\left\{e^{2\pi}+1+2\left(\left(e^{2\pi}\cancel{\sin\pi}-e^0\cancel{\sin 0}\right)-2\int_0^\pi e^{2t}\sin t\,dt\right)\right\}=\pi\left(e^{2\pi}+1\right)-4V$$

$$\therefore\ 5V=\pi\left(e^{2\pi}+1\right)\quad\therefore\ V=\frac{\pi\left(e^{2\pi}+1\right)}{5}$$

66

(1) $\displaystyle\int_1^2\frac{1}{\sqrt{x^2-1}}\,dx=\lim_{\varepsilon\to+0}\int_{1+\varepsilon}^2\frac{1}{\sqrt{x^2-1}}\,dx=\lim_{\varepsilon\to+0}\left[\log\left|x+\sqrt{x^2-1}\right|\right]_{1+\varepsilon}^2$

$= \lim_{\varepsilon \to +0} \left\{ \log\left|2+\sqrt{3}\right| - \log\left|1+\varepsilon+\sqrt{(1+\varepsilon)^2-1}\right| \right\} = \log\left|2+\sqrt{3}\right| - \cancel{\log(1-0)} = \log(2+\sqrt{3})$

(2) $\displaystyle\int_0^\infty xe^{-x^2}\,dx = \lim_{K\to\infty}\int_0^K xe^{-x^2}\,dx = \lim_{K\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^K = -\frac{1}{2}\lim_{K\to\infty}\left(e^{-K^2}-e^0\right) = -\frac{1}{2}(0-1)$

$= \dfrac{1}{2}$

(3) $I = \displaystyle\int_0^K e^{-x}\cos x\,dx$ とおくと

$I = \left[e^{-x}\sin x\right]_0^K - \displaystyle\int_0^K (-e^{-x})\sin x\,dx = e^{-K}\sin K - e^0\cancel{\sin 0} + \int_0^K e^{-x}\sin x\,dx$

$= e^{-K}\sin K + \left[e^{-x}(-\cos x)\right]_0^K - \displaystyle\int_0^K (-e^{-x})(-\cos x)\,dx$

$= e^{-K}\sin K - e^{-K}\cos K + e^0\cos 0 - \displaystyle\int_0^K (-e^{-x})(-\cos x)\,dx = e^{-K}\sin K - e^{-K}\cos K + 1 - I$

$\therefore\ 2I = e^{-K}\sin K - e^{-K}\cos K + 1 \quad \therefore\ I = \dfrac{1}{2}e^{-K}\sin K - \dfrac{1}{2}e^{-K}\cos K + \dfrac{1}{2}$

ここで，$-1 \leqq \sin K \leqq 1$ より，$-e^{-K} \leqq e^{-K}\sin K \leqq e^{-K}$ だから

$0 = \lim_{K\to\infty}\left(-e^{-K}\right) \leqq \lim_{K\to\infty} e^{-K}\sin K \leqq \lim_{K\to\infty} e^{-K} = 0 \quad \therefore\ \lim_{K\to\infty} e^{-K}\sin K = 0$

同様にして，$\lim_{K\to\infty} e^{-K}\cos K = 0$

$\therefore\ \displaystyle\int_0^\infty e^{-x}\cos x\,dx = \lim_{K\to\infty}\int_0^K e^{-x}\cos x\,dx = \lim_{K\to\infty}\left\{\frac{1}{2}e^{-K}\sin K - \frac{1}{2}e^{-K}\cos K) + \frac{1}{2}\right\}$

$= 0 + \dfrac{1}{2} = \dfrac{1}{2}$

2 章の問題

1

(1) ② (2) ③

2

③

3

(1) $\dfrac{2x-9}{(x-2)(x+3)} = \dfrac{a}{x-2} + \dfrac{b}{x+3}$ とおき，分母を払うと，$2x-9 = (a+b)x + (3a-2b)$ だから，両辺を比較して，$a=-1$，$b=3$ を得る。よって

$\displaystyle\int \frac{2x-9}{(x-2)(x+3)}\,dx = \int\left(-\frac{1}{x-2} + \frac{3}{x+3}\right)dx = -\log|x-2| + 3\log|x+3| + C$

$= \log\left|\dfrac{(x+3)^3}{x-2}\right| + C$

(2) $\displaystyle\int \frac{3-2x^2}{\sqrt{1-x^2}}\,dx = \int \frac{1+2(1-x^2)}{\sqrt{1-x^2}}\,dx = \int\left(\frac{1}{\sqrt{1-x^2}} + 2\sqrt{1-x^2}\right)dx$

$= \mathrm{Sin}^{-1}x + \cancel{2}\cdot\dfrac{1}{\cancel{2}}\left\{x\sqrt{1-x^2} + \mathrm{Sin}^{-1}x\right\} + C = x\sqrt{1-x^2} + 2\mathrm{Sin}^{-1}x + C$

4

(1) $\sin(\pi - x) = \sin x$ より，$\displaystyle\int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx = \int_{\frac{\pi}{2}}^{\pi} f(\sin(\pi-x))\,dx$

ここで，$t = \pi - x$ とおくと，$\dfrac{dt}{dx} = -1 \quad \therefore\ dx = -dt,$ $\begin{array}{c|ccc} x & \frac{\pi}{2} & \to & \pi \\ \hline t & \frac{\pi}{2} & \to & 0 \end{array}$ だから $\displaystyle\int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx =$

$\displaystyle\int_{\frac{\pi}{2}}^{0} f(\sin t)\cdot(-1)\,dt = \int_0^{\frac{\pi}{2}} f(\sin t)\,dt = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

$\therefore\ \displaystyle\int_0^{\pi} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx + \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

$= 2\int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

(2) $\sin(x-2\pi)=\sin x$ より，$\displaystyle\int_{2\pi-c}^{2\pi} f(\sin x)\,dx = \int_{2\pi-c}^{2\pi} f(\sin(x-2\pi))\,dx$

ここで，$t=x-2\pi$ とおくと，$\dfrac{dt}{dx}=1 \quad \therefore\ dt=dx,\quad \begin{array}{c|ccc} x & 2\pi-c & \to & 2\pi \\ \hline t & -c & \to & 0 \end{array}$ だから

$$\int_{2\pi-c}^{2\pi} f(\sin x)\,dx = \int_{-c}^{0} f(\sin t)\,dt = \int_{-c}^{0} f(\sin x)\,dx$$

$$\therefore \int_0^{2\pi} f(\sin x)\,dx = \int_0^{2\pi-c} f(\sin x)\,dx + \int_{2\pi-c}^{2\pi} f(\sin x)\,dx = \int_0^{2\pi-c} f(\sin x)\,dx + \int_{-c}^{0} f(\sin x)\,dx$$

$$= \int_{-c}^{2\pi-c} f(\sin x)\,dx$$

(3) 合成公式より，$a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\alpha)$ と表せる。ただし，$\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$，

$\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$ である。ここで，$t=x+\alpha$ とおくと，$dt=dx,\quad \begin{array}{c|ccc} x & 0 & \to & 2\pi \\ \hline t & \alpha & \to & 2\pi+\alpha \end{array}$ だから

$$\int_0^{2\pi} f(a\cos x + b\sin x)\,dx = \int_0^{2\pi} f\left(\sqrt{a^2+b^2}\sin(x+\alpha)\right)dx = \int_{\alpha}^{2\pi+\alpha} f\left(\sqrt{a^2+b^2}\sin t\right)dt$$

$$= \int_{-\pi}^{\pi} f\left(\sqrt{a^2+b^2}\sin t\right)dt = \int_{-\pi}^{0} f\left(\sqrt{a^2+b^2}\sin t\right)dt + \int_0^{\pi} f\left(\sqrt{a^2+b^2}\sin t\right)dt$$

ただし，3 番目の等号は (2) を利用した（$c=\pi+\alpha$ とした）。

(1) より，$\displaystyle\int_0^{\pi} f\left(\sqrt{a^2+b^2}\sin t\right)dt = 2\int_0^{\frac{\pi}{2}} f\left(\sqrt{a^2+b^2}\sin t\right)dt$

また，$u=-t$ とおくと，$du=dt\quad \begin{array}{c|ccc} t & -\pi & \to & 0 \\ \hline u & \pi & \to & 0 \end{array}$ だから，(1) より

$$\int_{-\pi}^{0} f\left(\sqrt{a^2+b^2}\sin t\right)dt = \int_{\pi}^{0} f\left(\sqrt{a^2+b^2}\sin(-u)\right)du = \int_0^{\pi} f\left(-\sqrt{a^2+b^2}\sin u\right)du$$

$$= 2\int_0^{\frac{\pi}{2}} f\left(-\sqrt{a^2+b^2}\sin u\right)du = 2\int_{\frac{\pi}{2}}^{0} f\left(\sqrt{a^2+b^2}\sin(-u)\right)du = 2\int_{-\frac{\pi}{2}}^{0} f\left(\sqrt{a^2+b^2}\sin t\right)dt$$

$$\therefore \int_0^{2\pi} f(a\cos x + b\sin x)\,dx = 2\int_{-\frac{\pi}{2}}^{0} f\left(\sqrt{a^2+b^2}\sin t\right)dt + 2\int_0^{\frac{\pi}{2}} f\left(\sqrt{a^2+b^2}\sin t\right)dt$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\left(\sqrt{a^2+b^2}\sin t\right)dt = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\left(\sqrt{a^2+b^2}\sin x\right)dx$$

5

(1) $\displaystyle\int t\sin^2 t\,dt = \int t\cdot\frac{1-\cos 2t}{2}\,dt = \frac{1}{2}\left\{\int t\,dt - \int t\cos 2t\,dt\right\}$

$$= \frac{1}{2}\left\{\frac{1}{2}t^2 - \left(\frac{1}{2}t\sin 2t - \frac{1}{2}\int \sin 2t\,dt\right)\right\} = \frac{1}{4}\left\{t^2 - t\sin 2t - \frac{1}{2}\cos 2t\right\} + C$$

$$= \frac{1}{8}(2t^2 - 2t\sin 2t - \cos 2t) + C$$

(2) $x(t) = -\dfrac{t}{\pi}\cos t$，$y(t)=\sin t$ より，$\dfrac{dx}{dt} = -\dfrac{1}{\pi}\cos t + \dfrac{t}{\pi}\sin t$ だから，求める面積は

$$\int_{\frac{\pi}{2}}^{\pi} \left|\sin t\cdot\left(-\frac{1}{\pi}\cos t + \frac{t}{\pi}\sin t\right)\right| dt = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} (-\sin t\cos t + t\sin^2 t)\,dt$$

$$= \frac{1}{\pi}\left[-\frac{1}{2}\sin^2 t + \frac{1}{8}(2t^2 - 2t\sin 2t - \cos 2t)\right]_{\frac{\pi}{2}}^{\pi} = -\frac{1}{8\pi}\left[4\sin^2 t - 2t^2 + 2t\sin 2t + \cos 2t\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{1}{8\pi}\left\{(0 - 2\pi^2 + 0 + 1) - \left(4 - \frac{\pi^2}{2} + 0 - 1\right)\right\} = -\frac{1}{8\pi}\left\{-\frac{3\pi^2}{2} - 2\right\} = \frac{3\pi^2+4}{16\pi}$$

6

(1) $S=\displaystyle\int_t^{2t}\left(3-e^x\right)dx=\Big[3x-e^x\Big]_t^{2t}=\left(6t-e^{2t}\right)-\left(3t-e^t\right)=3t-e^{2t}+e^t$

(2) $\dfrac{dS}{dt}=3-2e^{2t}+e^t=-\left(2e^t-3\right)\left(e^t+1\right)$ より，$\dfrac{dS}{dt}=0$ となるのは，$e^t=\dfrac{3}{2}$，すなわち，$t=\log\dfrac{3}{2}$ のときである。

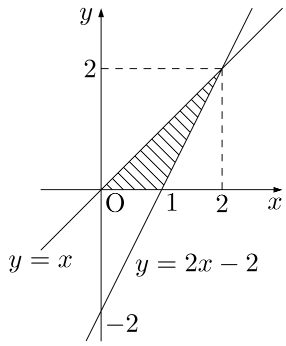
t	0		$\log\frac{3}{2}$		$\frac{1}{2}\log 3$
$\frac{dS}{dt}$	$+$	$+$	0	$-$	$-$
S	0	$\nearrow$	$3\log\frac{3}{2}-\frac{3}{4}$	$\searrow$	$\frac{3}{2}\log 3-3+\sqrt{3}$

増減表より，$t=\log\dfrac{3}{2}$ のとき最大値 $3\log\dfrac{3}{2}-\dfrac{3}{4}$

7

(1) $\pi\displaystyle\int_0^1\left(\sqrt{x}\right)^2dx=\pi\int_0^1 x\,dx=\pi\left[\frac{1}{2}x^2\right]_0^1=\frac{\pi}{2}$

(2) 直線 $y=x$ と $y=2x-2$ の交点は $(2,2)$ であり，右図の斜線部を回転して得られる回転体の体積を求めればよいから



$$\pi\int_0^2 x^2\,dx-\pi\int_1^2(2x-2)^2\,dx=\pi\int_0^2 x^2\,dx-4\pi\int_1^2\left(x^2-2x+1\right)dx$$

$$=\pi\left\{\left[\frac{1}{3}x^3\right]_0^2-4\left[\frac{1}{3}x^3-x^2+x\right]_1^2\right\}$$

$$=\pi\left\{\frac{8}{3}-4\left(\frac{8}{3}-4+2-\frac{1}{3}+1-1\right)\right\}=\pi\left\{\frac{8}{3}-4\cdot\frac{1}{3}\right\}=\frac{4\pi}{3}$$

8

$x=\mathrm{Tan}^{-1}\,t$，$y=\dfrac{1}{t^2}$ より，$\dfrac{dx}{dt}=\dfrac{1}{1+t^2}$，$\begin{array}{c|ccc} x & \frac{\pi}{4} & \to & \frac{\pi}{2} \\ \hline t & 1 & \to & \infty \end{array}$ だから，求める面積は

$$\int_1^\infty\left|\frac{1}{t^2}\cdot\frac{1}{1+t^2}\right|dt=\int_1^\infty\left(\frac{1}{t^2}-\frac{1}{1+t^2}\right)dt=\lim_{K\to\infty}\int_1^K\left(\frac{1}{t^2}-\frac{1}{1+t^2}\right)dt$$

$$=\lim_{K\to\infty}\left[-\frac{1}{t}-\mathrm{Tan}^{-1}\,t\right]_1^K=\lim_{K\to\infty}\left\{-\frac{1}{K}-\mathrm{Tan}^{-1}\,K+\frac{1}{1}+\mathrm{Tan}^{-1}\,1\right\}=0-\frac{\pi}{2}+1+\frac{\pi}{4}$$

$$=1-\frac{\pi}{4}$$