

1 節 重積分

$$110 \quad (1) \quad \int_0^3 \int_0^1 (4 - y^2) dy dx$$

$$= \int_0^3 \left[4y - \frac{y^3}{3} \right]_0^1 dx$$

$$= \int_0^3 \left(4 - \frac{1}{3} \right) dx$$

$$= \frac{11}{3} \int_0^3 dx$$

$$= 11$$

$$(2) \quad \int_{-1}^1 \int_{-1}^0 (x + y + 1) dy dx$$

$$= \int_{-1}^1 \left[(x + 1)y + \frac{y^2}{2} \right]_{-1}^0 dx$$

$$= \int_{-1}^1 \left(x + \frac{1}{2} \right) dx$$

$$= 2 \int_0^1 \frac{1}{2} dx$$

$$= \int_0^1 dx$$

$$= 1$$

$$(3) \quad \int_{-2}^{-1} \int_0^1 (\sin(\pi y) + \cos(\pi x)) dy dx$$

$$= \int_{-2}^{-1} \left[-\frac{1}{\pi} \cos(\pi y) \right.$$

$$\left. + y \cos(\pi x) \right]_0^1 dx$$

$$= \int_{-2}^{-1} \left(\frac{2}{\pi} + \cos(\pi x) \right) dx$$

$$= \left[\frac{2}{\pi} x + \frac{1}{\pi} \sin(\pi x) \right]_{-2}^{-1}$$

$$= \frac{2}{\pi}$$

$$(4) \quad \int_{-1}^2 \int_0^1 x e^{xy} dy dx$$

$$= \int_{-1}^2 \left[e^{xy} \right]_0^1 dx$$

$$= \int_{-1}^2 (e^x - 1) dx$$

$$= \left[e^x - x \right]_{-1}^2$$

$$= e^2 - e^{-1} - 3$$

$$111 \quad (1) \quad \int_{-2}^0 \int_0^{x+2} dy dx$$

$$= \int_{-2}^0 \left[y \right]_0^{x+2} dx$$

$$= \int_{-2}^0 (x + 2) dx$$

$$= \left[\frac{(x + 2)^2}{2} \right]_{-2}^0$$

$$= 2$$

$$(2) \quad \int_0^1 \int_0^{1-x} (x^2 + y^2) dy dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left(x^2(1-x) + \frac{(1-x)^3}{3} \right) dx$$

$$= \int_0^1 \left(x^2 - x^3 + \frac{(1-x)^3}{3} \right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12} \right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4} \right) - \left(-\frac{1}{12} \right)$$

$$= \frac{1}{6}$$

$$(3) \quad \int_1^2 \int_x^{2x} \frac{x}{y} dy dx$$

$$= \int_1^2 \left[x \log y \right]_x^{2x} dx$$

$$= (\log 2) \int_1^2 x dx$$

$$= \log 2 \cdot \left[\frac{x^2}{2} \right]_1^2$$

$$= \frac{3}{2} \log 2$$

$$(4) \quad \int_0^\pi \int_0^{\sin x} 2y dy dx$$

$$= \int_0^\pi \left[y^2 \right]_0^{\sin x} dx$$

$$= \int_0^\pi \sin^2 x dx$$

$$= \frac{1}{2} \int_0^\pi (1 - \cos 2x) dx$$

$$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^\pi$$

$$= \frac{\pi}{2}$$

$$112 \quad (1) \quad \int_0^1 \int_{y^2}^y dx dy$$

$$= \int_0^1 \left[x \right]_{y^2}^y dy$$

$$= \int_0^1 (y - y^2) dy$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1$$

$$= \frac{1}{6}$$

$$(2) \quad \int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y) dx dy$$

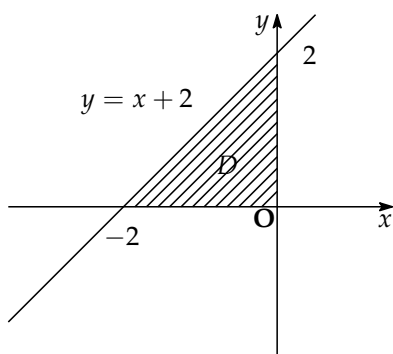
$$= \int_\pi^{2\pi} \left[-\cos x + x \cos y \right]_0^\pi dy$$

$$= \int_\pi^{2\pi} (\pi \cos y + 2) dy$$

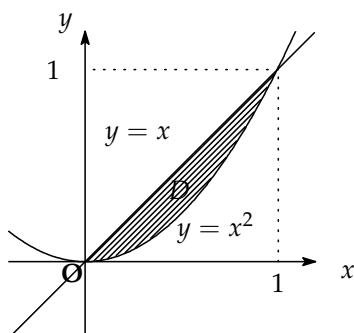
$$\begin{aligned}
&= \left[\pi \sin y + 2y \right]_{\pi}^{2\pi} \\
&= 2\pi \\
(3) \quad &\int_0^1 \int_0^{\pi} y \cos(xy) dx dy \\
&= \int_0^1 \left[\sin(xy) \right]_0^{\pi} dy \\
&= \int_0^1 \sin(\pi y) dy \\
&= \left[-\frac{1}{\pi} \cos(\pi y) \right]_0^1 \\
&= \frac{2}{\pi}
\end{aligned}$$

$$\begin{aligned}
(4) \quad &\int_0^1 \int_0^1 e^{x+y} dx dy \\
&= \int_0^1 e^x dx \cdot \int_0^1 e^y dy \\
&= \left[e^x \right]_0^1 \cdot \left[e^y \right]_0^1 \\
&= (e-1)^2
\end{aligned}$$

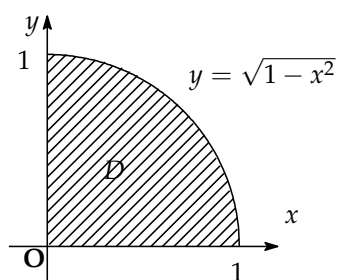
113 (1) $D = \{(x, y) \mid -2 \leq x \leq 0, 0 \leq y \leq x+2\}$



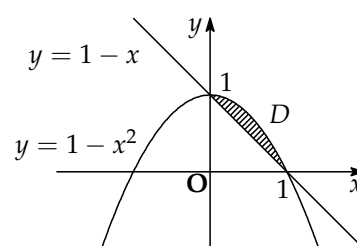
(2) $D = \{(x, y) \mid 0 \leq x \leq 1, x^2 \leq y \leq x\}$



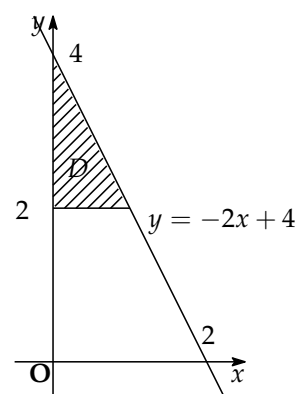
(3) $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}$



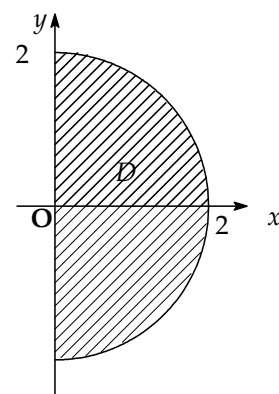
(4) $D = \{(x, y) \mid 0 \leq y \leq 1, 1-y \leq x \leq \sqrt{1-y}\}$



(5) $D = \{(x, y) \mid 2 \leq y \leq 4, x \geq 0, y \leq 4 - 2x\}$



(6) $D = \{(x, y) \mid -2 \leq y \leq 2, 0 \leq x \leq \sqrt{4-y^2}\}$



$$\begin{aligned}
114 \quad (1) \quad & \int_0^\pi \int_x^\pi \frac{\cos y}{y} dy dx \\
&= \int_0^\pi \frac{\cos y}{y} \left(\int_0^y dx \right) dy \\
&= \int_0^\pi \frac{\cos y}{y} \cdot y dy \\
&= \int_0^\pi \cos y dy \\
&= [\sin y]_0^\pi \\
&= \sin \pi - \sin 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
(2) \quad & \int_0^1 \int_x^1 y^2 \sin(\pi xy) dy dx \\
&= \int_0^1 \left(\int_0^y y^2 \sin(\pi xy) dx \right) dy \\
&= \int_0^1 \left[-\frac{y}{\pi} \cos(\pi xy) \right]_0^y dy \\
&= \frac{1}{\pi} \int_0^\pi (y - y \cos(\pi y^2)) dy \\
&= \frac{1}{\pi} \left[\frac{y^2}{2} - \frac{1}{2\pi} \sin(\pi y^2) \right]_0^1 \\
&= \frac{1}{\pi} \left(\frac{1}{2} - \frac{1}{2\pi} \sin(\pi) \right) \\
&= \frac{1}{2\pi}
\end{aligned}$$

$$\begin{aligned}
(3) \quad & \int_0^1 \int_y^1 x^2 e^{xy} dx dy \\
&= \int_0^1 \left(\int_0^x x^2 e^{xy} dy \right) dx \\
&= \int_0^1 [x e^{xy}]_0^x dx \\
&= \int_0^1 (x e^{x^2} - x) dx \\
&= \left[\frac{e^{x^2}}{2} - \frac{x^2}{2} \right]_0^1 \\
&= \frac{e}{2} - \frac{1}{2} - \frac{1}{2} \\
&= \frac{e}{2} - 1
\end{aligned}$$

$$\begin{aligned}
(4) \quad & \int_0^2 \int_{\frac{y}{2}}^1 e^{x^2} dx dy \\
&= \int_0^1 e^{x^2} \left(\int_0^{2x} dy \right) dx \\
&= \int_0^1 2x e^{x^2} dx \\
&= [e^{x^2}]_0^1 \\
&= e - 1
\end{aligned}$$

$$\begin{aligned}
115 \quad (1) \quad & \iint_D \sqrt{x^2 + y^2} dx dy \\
&= \int_0^1 \int_0^{2\pi} r \cdot r dr d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^1 r^2 dr \\
&= 2\pi \left[\frac{r^3}{3} \right]_0^1 \\
&= \frac{2}{3} \pi
\end{aligned}$$

$$\begin{aligned}
(2) \quad & \iint_D x dx dy \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r \cos \theta \cdot r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta \cdot \int_0^1 r^2 dr \\
&= [\sin \theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot \left[\frac{r^3}{3} \right]_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
116 \quad (1) \quad & \iint_D 6xy^2 dx dy \\
&= \int_2^4 \int_1^2 6xy^2 dy dx \\
&= \int_2^4 2x dx \cdot \int_1^2 3y^2 dy \\
&= [x^2]_2^4 \cdot [y^3]_1^2 \\
&= (16 - 4) \cdot (8 - 1) \\
&= 84
\end{aligned}$$

$$\begin{aligned}
(2) \quad & \iint_D \frac{1}{(2x + 3y)^2} dx dy \\
&= \int_0^1 \int_1^2 (2x + 3y)^{-2} dy dx \\
&= \int_0^1 \left[-\frac{1}{3} (2x + 3y)^{-1} \right]_1^2 dx \\
&= \frac{1}{3} \int_0^1 \left(\frac{1}{2x + 3} - \frac{1}{2x + 6} \right) dx \\
&= \frac{1}{6} [\log(2x + 3) - \log(2x + 6)]_0^1 \\
&= \frac{1}{6} (\log 5 - \log 8 - \log 3 + \log 6) \\
&= \frac{1}{6} \log \left(\frac{5}{4} \right)
\end{aligned}$$

$$\begin{aligned}
(3) \quad & \iint_D x \cos^2 y dx dy \\
&= \int_{-2}^3 x dx \cdot \int_0^{\frac{\pi}{2}} \cos^2 y dy \\
&= \int_{-2}^3 x dx \cdot \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2y}{2} dy \\
&= \left[\frac{x^2}{2} \right]_{-2}^3 \cdot \frac{1}{2} \left[y + \frac{1}{2} \sin 2y \right]_0^{\frac{\pi}{2}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{9-4}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\
&= \frac{5}{8}\pi \\
(4) \quad &\iint_D e^{\frac{x}{y}} dx dy \\
&= \int_1^2 \int_y^{y^3} e^{\frac{x}{y}} dx dy \\
&= \int_1^2 \left[ye^{\frac{x}{y}} \right]_y^{y^3} dy \\
&= \int_1^2 (ye^{y^2} - ey) dy \\
&= \left[\frac{1}{2}e^{y^2} - \frac{e}{2}y^2 \right]_1^2 \\
&= \frac{e^4}{2} - 2e \\
(5) \quad &\iint_D 4xy dx dy \\
&= \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy dy dx \\
&= \int_0^1 \left[2xy^2 \right]_{x^3}^{\sqrt{x}} dx \\
&= \int_0^1 (2x^2 - 2x^7) dx \\
&= \left[\frac{2}{3}x^2 - \frac{2}{8}x^8 \right]_0^1 \\
&= \frac{2}{3} - \frac{2}{8} \\
&= \frac{5}{12} \\
(6) \quad &\iint_D dx dy \\
&= \int_1^3 \int_{-\frac{y}{2} + \frac{3}{2}}^{2y-1} dx dy \\
&= \int_1^3 \left(2y - 1 + \frac{y}{2} - \frac{3}{2} \right) dy \\
&= \int_1^3 \left(\frac{5}{2}y - \frac{5}{2} \right) dy \\
&= \left[\frac{5}{4}(y-1)^2 \right]_1^3 \\
&= 5 \\
(7) \quad &\iint_D x^3 e^{y^3} dx dy \\
&= \int_0^1 \int_{x^2}^1 x^3 e^{y^3} dy dx \\
&= \int_0^1 \int_0^{\sqrt{y}} x^3 e^{y^3} dx dy \\
&= \int_0^1 \left[\frac{1}{4}x^4 e^{y^3} \right]_0^{\sqrt{y}} dy \\
&= \int_0^1 \frac{1}{4}y^2 e^{y^3} dy \\
&= \left[\frac{1}{12}e^{y^3} \right]_0^1
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{12}(e-1) \\
117 (1) \quad &\iint_D \sqrt{9-x^2-y^2} dx dy \\
&= \int_0^{2\pi} \int_0^{\sqrt{5}} \sqrt{9-r^2} \cdot r dr d\theta \\
&= 2\pi \int_0^{\sqrt{5}} (9-r^2)^{\frac{1}{2}} r dr d\theta \\
&= -\pi \int_0^{\sqrt{5}} (9-r^2)^{\frac{1}{2}} (-2r) dr d\theta \\
&= -\pi \left[\frac{2}{3}(9-r^2)^{\frac{3}{2}} \right]_0^{\sqrt{5}} \\
&= -\frac{2}{3}\pi \left((9-5)^{\frac{3}{2}} - (9-0)^{\frac{3}{2}} \right) \\
&= -\frac{2}{3}\pi \left(4^{\frac{3}{2}} - 9^{\frac{3}{2}} \right) \\
&= -\frac{2}{3}\pi \left(2^3 - 3^3 \right) \\
&= \frac{38}{3}\pi \\
(2) \quad &\iint_D \cos(x^2+y^2) dx dy \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \cos(r^2) \cdot r dr d\theta \\
&= \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^1 r \cos(r^2) dr \\
&= \frac{\pi}{2} \left[\frac{1}{2} \sin(r^2) \right]_0^1 \\
&= \frac{\pi}{4} \sin 1 \\
(3) \quad &\iint_D \frac{2}{1+\sqrt{x^2+y^2}} dx dy \\
&= \int_{-\frac{3}{2}\pi}^{\frac{3}{2}\pi} \int_0^1 \frac{2}{1+r} \cdot r dr d\theta \\
&= \int_{-\frac{3}{2}\pi}^{\frac{3}{2}\pi} d\theta \cdot \int_0^1 \frac{2r}{1+r} dr \\
&= \frac{\pi}{2} \int_0^1 2 \left(1 - \frac{1}{1+r} \right) dr \\
&= \pi \left[r - \log(1+r) \right]_0^1 \\
&= \pi(1 - \log 2) \\
(4) \quad &\iint_D e^{-(x^2+y^2)} dx dy \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 e^{-r^2} \cdot r dr d\theta \\
&= \frac{\pi}{2} \int_0^1 re^{-r^2} dr \\
&= \frac{\pi}{2} \left[-\frac{1}{2}e^{-r^2} \right]_0^1 \\
&= \frac{\pi}{4}(1 - e^{-1})
\end{aligned}$$

$$\begin{aligned}
(5) \quad & \iint_D \log(x^2 + y^2 + 1) dx dy \\
&= \int_0^{2\pi} \int_0^1 \log(r^2 + 1) \cdot r dr d\theta \\
&= 2\pi \int_0^1 r \log(r^2 + 1) dr \\
&= \pi \int_0^1 (2r) \log(r^2 + 1) dr \\
&\text{ここで, } t = r^2 + 1 \text{ と置換} \\
&= \pi \int_1^2 \log t dt
\end{aligned}$$

$$\begin{aligned}
&= \pi \left[t \log t - t \right]_1^2 \\
&= \pi(2 \log 2 - 1) \\
(6) \quad & \iint_D \frac{1}{1 + x^2 + y^2} dx dy \\
&= \int_0^{2\pi} \int_0^1 \frac{1}{1 + r^2} \cdot r dr d\theta \\
&= 2\pi \int_0^1 \frac{r}{1 + r^2} dr \\
&= \pi \left[\log(1 + r^2) \right]_0^1 \\
&= \pi \log 2
\end{aligned}$$

(7) 領域 D は極座標変換で, 領域 $D' = \{(r, \theta) \mid 0 \leq r \leq 2 \cos \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ に写るので,

$$\begin{aligned}
& \iint_D (x + y) dx dy \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2 \cos \theta} r(\cos \theta + \sin \theta) r dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \theta + \sin \theta) \left[\frac{r^3}{3} \right]_0^{2 \cos \theta} d\theta \\
&= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \theta + \sin \theta) \cos^3 \theta d\theta \\
&= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^4 \theta + \sin \theta \cos^3 \theta) d\theta \\
&= \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \theta d\theta \\
&= \frac{16}{3} \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta
\end{aligned}$$

Wallis の公式から

$$\begin{aligned}
&= \frac{16}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\
&= \pi
\end{aligned}$$

118 $\begin{cases} u = x - y \\ v = x + y \end{cases}$ とおく。このとき, 領域 D は $D' = \{(u, v) \mid 0 \leq u \leq \pi, 0 \leq v \leq \pi\}$ に写り, また, ヤコビ行列式の絶対値は $|J(u, v)| = \frac{1}{2}$ となるので

$$\begin{aligned}
& \iint_D (x - y) \sin(x + y) dx dy \\
&= \frac{1}{2} \int_0^\pi \int_0^\pi u \sin v du dv \\
&= \frac{1}{2} \int_0^\pi u du \cdot \int_0^\pi \sin v dv \\
&= \frac{1}{2} \cdot \left[\frac{u^2}{2} \right]_0^\pi \cdot \left[-\cos v \right]_0^\pi \\
&= \frac{1}{2} \cdot \frac{\pi^2}{2} \cdot 2 \\
&= \frac{\pi^2}{2}
\end{aligned}$$

119 領域 D は極座標変換で, 領域

$D' = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$ に写るので,

$D'_\epsilon = \{(r, \theta) \mid 0 < \epsilon \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$ を考える。

$$\begin{aligned}
& \iint_D \mathbf{Tan}^{-1} \left(\frac{y}{x} \right) dx dy \\
&= \lim_{\epsilon \rightarrow +0} \int_0^{\frac{\pi}{2}} \int_\epsilon^1 \mathbf{Tan}^{-1} \left(\frac{r \sin \theta}{r \cos \theta} \right) \cdot r dr d\theta \\
&= \lim_{\epsilon \rightarrow +0} \int_0^{\frac{\pi}{2}} \int_\epsilon^1 \theta \cdot r dr d\theta \\
&= \lim_{\epsilon \rightarrow +0} \int_0^{\frac{\pi}{2}} \theta d\theta \cdot \int_\epsilon^1 r dr \\
&= \lim_{\epsilon \rightarrow +0} \left[\frac{\theta^2}{2} \right]_0^{\frac{\pi}{2}} \cdot \left[\frac{r^2}{2} \right]_\epsilon^1 \\
&= \lim_{\epsilon \rightarrow +0} \frac{\pi^2}{8} \cdot \frac{1^2 - \epsilon^2}{2} \\
&= \frac{\pi^2}{16}
\end{aligned}$$

120 領域 D は極座標変換で, 領域

$D' = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$ に写るので,

$D'_\epsilon = \{(r, \theta) \mid 0 < \epsilon \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\}$ を考える。

$$\begin{aligned}
& \iint_D \log \sqrt{x^2 + y^2} dx dy \\
&= \lim_{\epsilon \rightarrow +0} \int_0^{\frac{\pi}{2}} \int_\epsilon^1 \log \sqrt{r} \cdot r dr d\theta \\
&= \lim_{\epsilon \rightarrow +0} \frac{\pi}{4} \int_\epsilon^1 r \log r dr
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\epsilon \rightarrow +0} \frac{\pi}{4} \left[\frac{r^2}{2} \log r - \frac{r^2}{4} \right]_{\epsilon}^1 \\
&= \lim_{\epsilon \rightarrow +0} \frac{\pi}{4} \left(\frac{1}{2} \log 1 - \frac{1}{4} - \frac{\epsilon^2}{2} \log \epsilon + \frac{\epsilon^2}{4} \right) \\
&\text{ここで } \lim_{\epsilon \rightarrow +0} \epsilon^2 \log \epsilon = 0 \text{ であるから} \\
&= -\frac{\pi}{8}
\end{aligned}$$

2節 重積分の応用

$$\begin{aligned}
121 \quad (1) \quad V &= \int_0^1 \int_0^1 (x+y) \, dx \, dy \\
&= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^1 dx \\
&= \int_0^1 \left(x + \frac{1}{2} \right) dx \\
&= \left[\frac{x^2}{2} + x \right]_0^1 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
(2) \quad V &= \int_0^1 \int_0^1 x^2 \, dx \, dy \\
&= \int_0^1 dy \cdot \int_0^1 x^2 \, dx \\
&= 1 \cdot \left[\frac{x^3}{3} \right]_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
(3) \quad V &= \int_0^1 \int_0^{x^2} (x+y) \, dx \, dy \\
&= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{x^2} dx \\
&= \int_0^1 \left(x^3 + \frac{x^4}{2} \right) dx \\
&= \left[\frac{x^4}{4} + \frac{x^5}{10} \right]_0^1 \\
&= \frac{1}{4} + \frac{1}{10} \\
&= \frac{7}{20}
\end{aligned}$$

$$\begin{aligned}
122 \quad (1) \quad &\text{曲面 } z = x^2 + y^2 \text{ と平面 } z = 1 \text{ との} \\
&\text{交わりは } x^2 + y^2 = 1 \text{ となる。} \\
&\therefore \text{求める体積 } V \text{ と積分領域 } D \text{ は} \\
V &= \iint_D (1 - (x^2 + y^2)) \, dx \, dy, \\
D &= \{(x, y) \mid x^2 + y^2 \leq 1\} \\
&\text{極座標に変換すれば} \\
V &= \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr \, d\theta \\
&= 2\pi \int_0^1 (r - r^3) \, dr
\end{aligned}$$

$$\begin{aligned}
&= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 \\
&= 2\pi \left(\frac{1}{2} - \frac{1}{4} \right) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
(2) \quad &\text{曲面 } z = x^2 + y^2 \text{ と平面 } z = 2x \text{ との} \\
&\text{交わりは } x^2 + y^2 = 2x \text{ より } (x-1)^2 + y^2 = 1 \text{ であるから, 求める体} \\
&\text{積 } V \text{ と積分領域 } D \text{ は} \\
V &= \iint_D (2x - (x^2 + y^2)) \, dx \, dy, \\
D &= \{(x, y) \mid (x-1)^2 + y^2 \leq 1\} \\
&\text{極座標に変換すると } D \text{ は} \\
D' &= \{(r, \theta) \mid 0 \leq r \leq 2 \cos \theta, \\
&\quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}
\end{aligned}$$

に写るので

$$\begin{aligned}
V &= \iint_D (2x - (x^2 + y^2)) \, dx \, dy, \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2 \cos \theta} (2r \cos \theta - r^2) \cdot r \, dr \, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2 \cos \theta} (2r^2 \cos \theta - r^3) \, dr \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{2}{3} r^3 \cos \theta - \frac{r^4}{4} \right]_0^{2 \cos \theta} d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{16}{3} \cos^4 \theta - 4 \cos^4 \theta \right) d\theta \\
&= \frac{4}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \theta \, d\theta \\
&= \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta \\
&= \frac{8}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
(3) \quad &x^2 + y^2 + z^2 = 2^2 = 4 \text{ より } z = \\
&\pm \sqrt{4 - x^2 - y^2} \text{ であるから, 求める} \\
&\text{体積 } V \text{ と積分領域 } D \text{ は} \\
V &= \iint_D \left\{ \sqrt{4 - x^2 - y^2} - (-\sqrt{4 - x^2 - y^2}) \right\} \, dx \, dy \\
&= 2 \iint_D \sqrt{4 - x^2 - y^2} \, dx \, dy \\
D &= \{(x, y) \mid x^2 + y^2 \leq 1\} \\
&\text{極座標に変換して} \\
V &= 2 \int_0^{2\pi} \int_0^1 (4 - r^2)^{\frac{1}{2}} r \, dr \, d\theta
\end{aligned}$$

$$= 4\pi \int_0^1 (4-r^2)^{\frac{1}{2}} r dr$$

$$= 4\pi \left[-\frac{1}{3}(4-r^2)^{\frac{3}{2}} \right]_0^1$$

$$= \frac{4}{3}(8-3\sqrt{3})\pi$$

$$123 \quad (1) \quad \int_0^{+\infty} \int_0^{+\infty} e^{-(x^2+y^2)} dx dy$$

$$= \int_0^{+\infty} e^{-x^2} dx \cdot \int_0^{+\infty} e^{-y^2} dy$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2}$$

$$= \frac{\pi}{4}$$

(2) 極座標に変換して

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x^2+y^2)e^{-(x^2+y^2)} dx dy$$

$$= \int_0^{2\pi} \int_0^{+\infty} r^2 e^{-r^2} r dr d\theta$$

$$= 2\pi \int_0^{+\infty} r^2 e^{-r^2} r dr = r^2 \text{ と置換して}$$

$$= \pi \int_0^{+\infty} t e^{-t} dt$$

$$= \pi \int_0^{+\infty} t(-e^{-t})' dt$$

$$= \pi \left([-te^{-t}]_0^{+\infty} + \int_0^{+\infty} e^{-t} dt \right)$$

$$= \pi [-e^{-t}]_0^{+\infty}$$

$$= \pi$$

124 質量 M は

$$M = \int_0^1 \int_x^1 (x+y) dy dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_x^1 dx$$

$$= \int_0^1 \left(\frac{1}{2} + x - \frac{x^3}{2} \right) dx$$

$$= \left[\frac{x}{2} + \frac{x^2}{2} - \frac{x^4}{8} \right]_0^1$$

$$= \frac{1}{2}$$

$$M_x = \int_0^1 \int_x^1 x(x+y) dy dx$$

$$= \int_0^1 \left[x^2 y + x \frac{y^2}{2} \right]_x^1 dx$$

$$= \int_0^1 \left(\frac{x}{2} + x^2 - \frac{3x^3}{2} \right) dx$$

$$= \left[\frac{x^2}{4} + \frac{x^3}{3} - \frac{3x^4}{8} \right]_0^1$$

$$= \frac{5}{24}$$

$$M_y = \int_0^1 \int_x^1 y(x+y) dy dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_x^1 dx$$

$$= \int_0^1 \left(\frac{1}{3} + \frac{x}{2} - \frac{5x^3}{6} \right) dx$$

$$= \left[\frac{x}{3} + \frac{x^2}{4} - \frac{5x^4}{24} \right]_0^1$$

$$= \frac{9}{24}$$

以上から重心 G の座標は

$$G = \left(\frac{M_x}{M}, \frac{M_y}{M} \right)$$

$$= \left(\frac{5}{12}, \frac{3}{4} \right)$$

125 (1) $x^2 + y^2 + z^2 = a^2$ より

$$z = \sqrt{a^2 - x^2 - y^2} \text{ であるから}$$

$$z_x = -x(a^2 - x^2 - y^2)^{-\frac{1}{2}}$$

$$z_y = -y(a^2 - x^2 - y^2)^{-\frac{1}{2}}$$

$$\therefore \sqrt{(z_x)^2 + (z_y)^2 + 1}$$

$$= a(a^2 - x^2 - y^2)^{-\frac{1}{2}}$$

従って、求める曲面積 S は

$$S = \iint_D a(a^2 - x^2 - y^2)^{-\frac{1}{2}} dx dy$$

$$D = \{(x, y) \mid x^2 + y^2 \leq a^2 - b^2\}$$

となる。極座標に変換すると

$$S =$$

$$\int_0^{2\pi} \int_0^{\sqrt{a^2-b^2}} a(a^2 - r^2)^{-\frac{1}{2}} \cdot r dr d\theta$$

$$= 2\pi a \int_0^{\sqrt{a^2-b^2}} (a^2 - r^2)^{-\frac{1}{2}} \cdot r dr$$

$$= -\pi a \int_0^{\sqrt{a^2-b^2}} (a^2 - r^2)^{-\frac{1}{2}}$$

$$\cdot (-2r) dr$$

$$= -\pi a \left[2(a^2 - r^2)^{\frac{1}{2}} \right]_0^{\sqrt{a^2-b^2}}$$

$$= -2\pi a (\sqrt{b^2} - \sqrt{a^2})$$

$$= 2a(a-b)\pi$$

(2) $z = \tan^{-1} \left(\frac{y}{x} \right)$ より

$$z_x = \frac{-y}{x^2 + y^2}$$

$$z_y = \frac{x}{x^2 + y^2}$$

$$\therefore \sqrt{(z_x)^2 + (z_y)^2 + 1}$$

$$= \frac{\sqrt{x^2 + y^2 + 1}}{\sqrt{x^2 + y^2}}$$

従って、求める曲面積 S は

$$S = \iint_D \frac{\sqrt{x^2 + y^2 + 1}}{\sqrt{x^2 + y^2}} dx dy$$

$$D = \{(x, y) \mid x^2 + y^2 < 1, \\ x > 0, y > 0\}$$

となる。極座標に変換すると

$$\begin{aligned} S &= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{\sqrt{r^2+1}}{\sqrt{r^2}} \cdot r \, dr \, d\theta \\ &= \frac{\pi}{2} \int_0^1 \sqrt{r^2+1} \, dr \\ &= \frac{\pi}{2} \left[\frac{1}{2} r \sqrt{r^2+1} \right. \\ &\quad \left. + \frac{1}{2} \log(r + \sqrt{r^2+1}) \right]_0^1 \\ &= \frac{\pi}{4} (\sqrt{2} + \log(1 + \sqrt{2})) \end{aligned}$$

4章の問題

$$\begin{aligned} \text{A-1 (1)} \quad & \int_0^1 \int_0^2 xy^2 \, dx \, dy \\ &= \int_0^1 y^2 \, dy \cdot \int_0^2 x \, dx \\ &= \left[\frac{y^3}{3} \right]_0^1 \cdot \left[\frac{x^2}{2} \right]_0^2 \\ &= \frac{1}{3} \cdot \frac{4}{2} \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{(2)} \quad & \int_1^2 \int_0^2 (2x+y) \, dy \, dx \\ &= \int_1^2 \left[2xy + \frac{y^2}{2} \right]_0^2 \, dx \\ &= \int_1^2 (4x+2) \, dx \\ &= \left[2x^2 + 2x \right]_1^2 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{A-2 (1)} \quad & \int_{-1}^0 \int_0^{x+1} f(x, y) \, dy \, dx \\ &+ \int_0^1 \int_0^{1-x} f(x, y) \, dy \, dx \end{aligned}$$

または

$$\begin{aligned} & \int_0^1 \int_{y-1}^{1-y} f(x, y) \, dx \, dy \\ \text{(2)} \quad & \int_0^1 \int_{1-x}^1 f(x, y) \, dy \, dx \end{aligned}$$

または

$$\begin{aligned} & \int_0^1 \int_{1-y}^1 f(x, y) \, dx \, dy \\ \text{(3)} \quad & \int_0^1 \int_0^{2-2x} f(x, y) \, dy \, dx \end{aligned}$$

または

$$\begin{aligned} & \int_0^2 \int_0^{\frac{2-y}{2}} f(x, y) \, dx \, dy \\ \text{A-3 (1)} \quad & \int_0^2 \int_1^2 xy^2 \, dx \, dy \end{aligned}$$

$$\begin{aligned} &= \int_0^2 y^2 \, dy \cdot \int_1^2 x \, dx \\ &= \left[\frac{y^3}{3} \right]_0^2 \cdot \left[\frac{x^2}{2} \right]_1^2 \\ &= \frac{8}{3} \cdot \frac{3}{2} \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{(2)} \quad & \int_2^4 \int_1^3 (x+y) \, dx \, dy \\ &= \int_2^4 \left[\frac{x^2}{2} + xy \right]_1^3 \, dy \\ &= \int_2^4 (4+2y) \, dy \\ &= \left[4y + y^2 \right]_2^4 = 20 \end{aligned}$$

$$\begin{aligned} \text{A-4 (1)} \quad & \int_1^2 \int_3^{2x+1} f(x, y) \, dy \, dx \\ &= \int_3^5 \int_{\frac{y-1}{2}}^2 f(x, y) \, dx \, dy \end{aligned}$$

$$\begin{aligned} \text{(2)} \quad & \int_0^1 \int_y^{\sqrt{y}} f(x, y) \, dx \, dy \\ &= \int_0^1 \int_{x^2}^x f(x, y) \, dy \, dx \end{aligned}$$

$$\begin{aligned} \text{(3)} \quad & \int_0^{\sqrt{2}} \int_{y^2}^2 f(x, y) \, dx \, dy \\ &= \int_0^2 \int_0^{\sqrt{x}} f(x, y) \, dy \, dx \end{aligned}$$

$$\text{A-5 (1)} \quad \int_0^\pi \int_0^2 r^2 \, dr \, d\theta$$

$$\text{(2)} \quad \int_0^{\frac{\pi}{2}} \int_0^a r^2 \sin \theta \, dr \, d\theta$$

$$\begin{aligned} \text{B-1 (1)} \quad & \iint_D (1+x+y) \, dx \, dy \\ &= \int_0^1 \int_0^{1-x} (1+x+y) \, dy \, dx \\ &= \int_0^1 \left[y + xy + \frac{y^2}{2} \right]_0^{1-x} \, dx \\ &= \int_0^1 \left((1-x) + x(1-x) + \frac{(1-x)^2}{2} \right) \, dx \\ &= \int_0^1 \left(1 - x^2 + \frac{1}{2}(x-1)^2 \right) \, dx \\ &= \left[x - \frac{1}{3}x^3 + \frac{1}{6}(x-1)^3 \right]_0^1 \\ &= 1 - \frac{1}{3} - \left(\frac{-1}{6} \right) \\ &= \frac{5}{6} \end{aligned}$$

$$\begin{aligned}
(2) \quad & \iint_D \frac{1}{\sqrt{1+x}} dx dy \\
&= \int_0^1 \int_{x^2}^x \frac{1}{\sqrt{1+x}} dy dx \\
&= \int_0^1 \frac{1}{\sqrt{1+x}} [y]_{x^2}^x dx \\
&= \int_0^1 \frac{1}{\sqrt{1+x}} (x - x^2) dx \\
&\text{ここで, } t = \sqrt{1+x} \text{ と置換すれば,} \\
&x = t^2 - 1, \quad dx = 2t dt, \quad x: 0 \rightarrow 1 \text{ の} \\
&\text{とき, } t: 1 \rightarrow \sqrt{2} \text{ であるから} \\
&= \int_1^{\sqrt{2}} \frac{1}{t} (t^2 - 1 - (t^2 - 1)^2) 2t dt \\
&= 2 \int_1^{\sqrt{2}} (t^2 - 1 - t^4 + 2t^2 - 1) dt \\
&= 2 \int_1^{\sqrt{2}} (-t^4 + 3t^2 - 2) dt \\
&= 2 \left[-\frac{1}{5}t^5 + t^3 - 2t \right]_1^{\sqrt{2}} \\
&= 2 \left(-\frac{4\sqrt{2}}{5} + 2\sqrt{2} - 2\sqrt{2}\frac{1}{5} - 1 + 2 \right) \\
&= 2 \left(-\frac{4\sqrt{2}}{5} + \frac{6}{5} \right) \\
&= \frac{12 - 8\sqrt{2}}{5}
\end{aligned}$$

$$\begin{aligned}
(3) \quad & \iint_D x^2 y dx dy \\
&= \int_0^1 \int_0^y x^2 y dx dy \\
&= \int_0^1 \left[\frac{x^3 y}{3} \right]_0^y dy \\
&= \int_0^1 \frac{y^4}{3} dy \\
&= \left[\frac{y^5}{15} \right]_0^1 \\
&= \frac{1}{15}
\end{aligned}$$

$$\begin{aligned}
(4) \quad & \iint_D e^{-(x^2+y^2)} dx dy \\
&= \int_0^{2\pi} \int_0^1 e^{-r^2} r dr d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^1 e^{-r^2} r dr \\
&= 2\pi \cdot \int_0^1 e^{-r^2} r dr \\
&= -\pi \int_0^1 e^{-r^2} (-r^2)' dr \\
&= -\pi \left[e^{-r^2} \right]_0^1
\end{aligned}$$

$$\begin{aligned}
&= -\pi (e^{-1} - e^0) \\
&= \pi (1 - e^{-1}) \\
(5) \quad & \iint_D \frac{1}{\sqrt{1-x^2-y^2}} dx dy \\
&= \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{1-r^2}} r dr d\theta \\
&= 2\pi \int_0^1 (1-r^2)^{-\frac{1}{2}} r dr \\
&= -\pi \int_0^1 (1-r^2)^{-\frac{1}{2}} (-2r) dr \\
&= -\pi \int_0^1 (1-r^2)^{-\frac{1}{2}} (1-r^2)' dr \\
&= -\pi \left[2(1-r^2)^{\frac{1}{2}} \right]_0^1 \\
&= -\pi(1-2) \\
&= 2\pi
\end{aligned}$$

$$(6) \quad I = \iint_D \{(x-y)^2 + (x+2y)^2\} dx dy$$

$$\text{ここで } \begin{cases} u = x-y \\ v = x+2y \end{cases}$$

と変換すると

$$dx dy = \frac{1}{3} du dv \text{ かつ } -2 \leq u \leq 2, \quad -1 \leq v \leq 1 \text{ となる。}$$

$$\begin{aligned}
I &= \frac{1}{3} \int_{-2}^2 \left(\int_{-1}^1 (u^2 + v^2) dv \right) du \\
&= \frac{4}{3} \int_0^2 \left(\int_0^1 (u^2 + v^2) dv \right) du \\
&= \frac{4}{3} \int_0^2 \left[u^2 v + \frac{1}{3} v^3 \right]_0^1 du \\
&= \frac{4}{3} \int_0^2 \left(u^2 + \frac{1}{3} \right) du \\
&= \frac{4}{3} \left[\frac{1}{3} u^3 + \frac{1}{3} u \right]_0^2 \\
&= \frac{4}{9} \left[u^3 + u \right]_0^2 \\
&= \frac{4}{9} \cdot (8 + 2) \\
&= \frac{40}{9}
\end{aligned}$$

$$(7) \quad I = \iint_D (x+y) e^{x-y} dx dy$$

$$\text{ここで } \begin{cases} u = x+y \\ v = x-y \end{cases}$$

と変換すると

$$dx dy = \frac{1}{2} du dv \text{ かつ } 0 \leq u \leq 1, \quad 0 \leq v \leq 1 \text{ となる。}$$

$$\begin{aligned}
I &= \frac{1}{2} \int_0^1 \int_0^1 u e^v du dv \\
&= \frac{1}{2} \int_0^1 u du \cdot \int_0^1 e^v dv \\
&= \frac{1}{2} \left[\frac{u^2}{2} \right]_0^1 \cdot \left[e^v \right]_0^1 \\
&= \frac{1}{2} \cdot \frac{1}{2} \cdot (e - 1) \\
&= \frac{e - 1}{4}
\end{aligned}$$

(8) $\iint_D (x^2 + y^2) dx dy$

$$\begin{cases} u = \frac{x}{a} \\ v = \frac{y}{b} \end{cases}$$

と変換すると

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$\therefore dx dy = ab du dv$ かつ

積分領域 D は $D = \{(u, v) \mid u^2 + v^2 \leq 1\}$ となるから

$$\begin{aligned}
I &= \iint_D ((au)^2 + (bv)^2) ab du dv \\
&= \iint_D (a^3 bu^2 + ab^3 v^2) du dv
\end{aligned}$$

極座標を導入して

$$\begin{aligned}
I &= 4 \int_0^{\frac{\pi}{2}} \int_0^1 (a^3 b r^2 \cos^2 \theta \\
&\quad + ab^3 r^2 \sin^2 \theta) r dr d\theta \\
&= 4 \int_0^{\frac{\pi}{2}} (a^3 b \cos^2 \theta \\
&\quad + ab^3 \sin^2 \theta) \left(\int_0^1 r^3 dr \right) d\theta \\
&= 4 \int_0^{\frac{\pi}{2}} (a^3 b \cos^2 \theta \\
&\quad + ab^3 \sin^2 \theta) \cdot \frac{1}{4} d\theta \\
&= \int_0^{\frac{\pi}{2}} (a^3 b \cos^2 \theta + ab^3 \sin^2 \theta) d\theta \\
&= a^3 b \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta \\
&\quad + ab^3 \int_0^{\frac{\pi}{2}} \sin^2 \theta d\theta \\
&= a^3 b \times \frac{\pi}{4} + ab^3 \times \frac{\pi}{4} \\
&= \frac{ab}{4} (a^2 + b^2) \pi
\end{aligned}$$

(9) $\iint_D \frac{x+y}{x^2+y^2} dx dy$

$$= \int_0^1 \int_0^x \frac{x+y}{x^2+y^2} dy dx$$

ここで、まず $\int_0^x \frac{x+y}{x^2+y^2} dy$ を計算す

$$\begin{aligned}
&\text{る} \\
&\int_0^x \frac{x+y}{x^2+y^2} dy \\
&= \int_0^x \frac{x}{x^2+y^2} dy + \int_0^x \frac{y}{x^2+y^2} dy \\
&= x \int \frac{1}{x^2+y^2} dy + \frac{1}{2} \int_0^x \frac{2y}{x^2+y^2} dy \\
&= x \left[\frac{1}{x} \tan^{-1} \left(\frac{y}{x} \right) \right]_0^x + \left[\frac{1}{2} \log(x^2+y^2) \right]_0^x \\
&= \tan^{-1}(1) - \tan^{-1}(0) + \frac{1}{2} (\log(2x^2) - \log(x^2)) \\
&= \frac{\pi}{4} + \frac{1}{2} \cdot \log \left(\frac{2x^2}{x^2} \right) \\
&= \frac{\pi}{4} + \frac{1}{2} \log 2
\end{aligned}$$

したがって、

$$\begin{aligned}
&\int_0^1 \int_0^x \frac{x+y}{x^2+y^2} dy dx \\
&= \int_0^1 \left(\frac{\pi}{4} + \frac{1}{2} \log 2 \right) dx \\
&= \left(\frac{\pi}{4} + \frac{1}{2} \log 2 \right) \int_0^1 dx \\
&= \left(\frac{\pi}{4} + \frac{1}{2} \log 2 \right) [x]_0^1 \\
&= \left(\frac{\pi}{4} + \frac{1}{2} \log 2 \right) (1 - 0) \\
&= \frac{\pi}{4} + \frac{1}{2} \log 2
\end{aligned}$$

(10) 極座標に変換する

$$\begin{aligned}
&\iint_D \frac{1}{(x^2+y^2)^2} dx dy \\
&= \int_0^{2\pi} \int_1^{+\infty} \frac{1}{r^4} \cdot r dr d\theta \\
&= 2\pi \int_1^{+\infty} r^{-3} dr \\
&= \lim_{R \rightarrow +\infty} 2\pi \int_1^R r^{-3} dr \\
&= \lim_{R \rightarrow +\infty} 2\pi \left[-\frac{1}{2r^2} \right]_1^R \\
&= \lim_{R \rightarrow +\infty} 2\pi \left(-\frac{1}{2R^2} + \frac{1}{2} \right) \\
&= \pi
\end{aligned}$$

B-2 まず $z = -a$ によって切られる

$\frac{x^2}{a^2} + \frac{y^2}{a^2} + z^2 = 1$ の切り口の方方程式を求めると

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + (-a)^2 = 1$$

$$\Rightarrow x^2 + y^2 = a^2 - a^4 = a^2(1 - a^2)$$

であるから、切り口は半径 $a\sqrt{1 - a^2}$ の円

であることが分かる。

$\frac{x^2}{a^2} + \frac{y^2}{a^2} + z^2 = 1$ を z について解けば

$$z = \pm \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}}$$

$$= \pm \sqrt{\frac{a^2 - (x^2 + y^2)}{a^2}}$$

$$= \pm \frac{\sqrt{a^2 - (x^2 + y^2)}}{a}$$

であるから、求める体積 V は

$$V = \iint_D \left\{ -a - \left(-\frac{\sqrt{a^2 - (x^2 + y^2)}}{a} \right) \right\} dx dy$$

$$D = \{(x, y) \mid x^2 + y^2 \leq a^2(1 - a^2)\}$$

ゆえに V は

$$V = \iint_D \left(\frac{\sqrt{a^2 - (x^2 + y^2)}}{a} - a \right) dx dy$$

極座標を導入すれば

$$V = 4 \int_0^{\frac{\pi}{2}} \int_0^{a\sqrt{1-a^2}} \left(\frac{\sqrt{a^2 - r^2}}{a} - a \right) r dr d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{a\sqrt{1-a^2}} \left(\frac{1}{a} r(a^2 - r^2)^{\frac{1}{2}} - ar \right) dr$$

$$= 4 \cdot \frac{\pi}{2} \cdot \left[-\frac{1}{3a} (a^2 - r^2)^{\frac{3}{2}} - \frac{a}{2} r^2 \right]_0^{a\sqrt{1-a^2}}$$

$$= 2\pi \cdot \frac{a^5 - 3a^3 + 2a^2}{6}$$

$$= \frac{\pi}{3} (a^5 - 3a^3 + 2a^2)$$

V を最大にする a を求めるために

$$g(a) = a^5 - 3a^3 + 2a^2 \text{ とおく.}$$

$$g'(a) = 5a^4 - 9a^2 + 4a$$

$$= a(5a^3 - 9a + 4)$$

$$= a(a - 1)(5a^2 + 5a - 4)$$

極値を求めるために $g'(a) = 0$ なる a を求め

る。 $a(a - 1)(5a^2 + 5a - 4) = 0$ から、

まず $a = 0$, $a = 1$, 次に $5a^2 + 5a - 4 = 0$

を解いて

$$a = \frac{-5 \pm \sqrt{25 - 4 \times 5 \times (-4)}}{10}$$

$$= \frac{-5 \pm \sqrt{105}}{10}$$

$0 < a \leq 1$ が a の満たすべき条件なので、

結局 $g'(a) = 0$ なる a は $a = 1$ と

$$a = \frac{-5 + \sqrt{105}}{10} \quad (\sqrt{105} \doteq 10.2) \text{ となる.}$$

増減表を書いて

a	0	...	$\frac{-5 + \sqrt{105}}{10}$...	1
$g'(a)$	\times	$+$	0	$-$	0
$g(a)$	\times	\nearrow	最大	\searrow	0

増減表から（最大値自体を求める必要はない） V を最大にする a の値は

$$a = \frac{-5 + \sqrt{105}}{10} \text{ である.}$$