

1 章 微分法

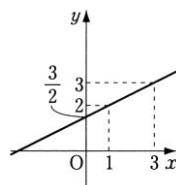
1 節 いろいろな関数表示の微分法

A

1

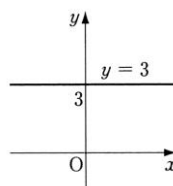
$$(1) \quad \begin{cases} x = 2t + 1 & \dots \textcircled{1} \\ y = t + 2 & \dots \textcircled{2} \end{cases}$$

$$\textcircled{1} - 2 \times \textcircled{2} \text{ より } x - 2y = -3 \quad \therefore y = \frac{1}{2}x + \frac{3}{2}$$



$$(2) \quad \begin{cases} x = 2t - 1 \\ y = 3 \end{cases}$$

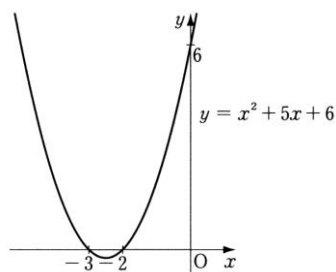
任意の t について y は常に 3 となる $\therefore y = 3$



$$(3) \quad \begin{cases} x = t - 2 & \dots \textcircled{1} \\ y = t^2 + t & \dots \textcircled{2} \end{cases}$$

$$\textcircled{1} \text{ より } t = x + 2$$

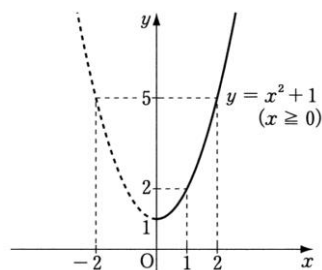
$$\textcircled{2} \text{ に代入し, 整理すると } y = x^2 + 5x + 6$$



$$(4) \quad \begin{cases} x = \sqrt{t} & \dots \textcircled{1} \\ y = t + 1 & \dots \textcircled{2} \end{cases}$$

$$\textcircled{1} \text{ の両辺を 2 乗して } \textcircled{2} \text{ に代入すると } y = x^2 + 1$$

ただし, $\textcircled{1}$ より $x \geq 0$



2

$$(1) \quad y = -2x + 5 \text{ に } x = t + 1 \text{ を代入}$$

$$\therefore \begin{cases} x = t + 1 \\ y = -2t + 3 \end{cases}$$

$$(2) \quad y = -2x + 5 \text{ に } x = -\frac{1}{2}t \text{ を代入}$$

$$\therefore \begin{cases} x = -\frac{1}{2}t \\ y = t + 5 \end{cases}$$

3

$$\begin{cases} x = t^2 + \frac{1}{t^2} & \dots \textcircled{1} \\ y = t - \frac{1}{t} & \dots \textcircled{2} \end{cases}$$

②の両辺を2乗すると $y^2 = \left(t - \frac{1}{t}\right)^2$

$$y^2 = t^2 - 2 \times t \times \frac{1}{t} + \left(\frac{1}{t}\right)^2$$

$$y^2 = t^2 - 2 + \frac{1}{t^2}$$

$$y^2 = x - 2 \quad (\textcircled{1} \text{を代入})$$

$$y = \pm \sqrt{x - 2}$$

4

$$(1) \quad \frac{dx}{dt} = (2t^2)' = 4t, \quad \frac{dy}{dt} = (t^2 - 1)' = 2t \quad \text{より} \quad \frac{dy}{dx} = \frac{2t}{4t} = \frac{1}{2}$$

$$(2) \quad \frac{dy}{dx} = \frac{3'}{(2t - 1)'} = \frac{0}{2} = 0$$

$$(3) \quad \frac{dy}{dx} = \frac{\left(t^2 + \frac{1}{t^2}\right)'}{\left(t - \frac{1}{t}\right)'} = \frac{2t - \frac{2}{t^3}}{1 + \frac{1}{t^2}} = \frac{2t^4 - 2}{t^3 + t} = 2 \frac{t^2 - 1}{t} = 2 \left(t - \frac{1}{t}\right)$$

$$(4) \quad \frac{dy}{dx} = \frac{(2 \sin t - 1)'}{(2 \cos t + 4)'} = \frac{2 \cos t}{-2 \sin t} = -\frac{1}{\tan t} = -\cot t$$

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$$(1) \quad \frac{dx}{dt} = 3 > 0 \quad \text{より} \quad t \text{の範囲はすべての実数}$$

$$(2) \quad \frac{dx}{dt} = 1 - \cos t$$

$$0 \leq t \leq 2\pi \quad \text{において} \quad \frac{dx}{dt} = 0 \quad \text{となるのは}$$

$$1 - \cos t = 0$$

$$\cos t = 1$$

$$t = 0, 2\pi$$

$$\text{よって, } \frac{dy}{dx} \text{の値が定まるのは } 0 < t < 2\pi$$

$$(3) \quad \frac{dx}{dt} = 2t - 3$$

$$\frac{dx}{dt} = 0 \text{ となるのは } 2t - 3 = 0 \text{ より } t = \frac{3}{2}$$

$$\text{よって, } \frac{dy}{dx} \text{ の値が定まるのは } t \neq \frac{3}{2}, t > 0$$

6

$$(1) \quad \frac{dx}{dt} = 14t = 0 \text{ のとき, すなわち } t = 0 \text{ のとき } \frac{dy}{dx} \text{ の値は定まらない。}$$

$$\text{よって, } t \neq 0 \text{ とすると } \frac{dy}{dx} = \frac{3t^2 - 12}{14t}$$

$$\frac{dy}{dx} = 0 \text{ より } 3t^2 - 12 = 0 \quad \therefore t = \pm 2$$

求める x は

$$t = 2 \text{ のとき } x_1 = 7 \times 2^2 + 2 = 30$$

$$t = -2 \text{ のとき } x_2 = 7 \times (-2)^2 + 2 = 30 \quad \therefore x = 30$$

$$(2) \quad \frac{dx}{dt} = -\sin t = 0 \text{ のとき, すなわち } t = 0, \pi, 2\pi \text{ のとき } \frac{dy}{dx} \text{ の値は定まらない。}$$

$$\text{よって, } t \neq 0, \pi, 2\pi \text{ とすると } \frac{dy}{dx} = -\frac{\cos t}{\sin t} = -\cot t$$

$$\text{このとき, } \frac{dy}{dx} = 0 \text{ となるのは } -\cot t = 0 \quad \therefore t = \frac{\pi}{2}, \frac{3}{2}\pi$$

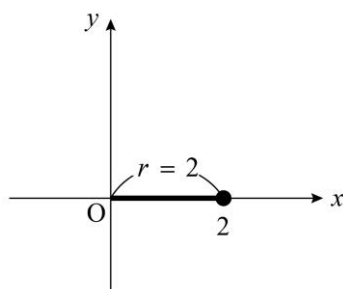
$$t = \frac{\pi}{2} \text{ の時 } x_1 = 0$$

$$t = \frac{3}{2}\pi \text{ の時 } x_2 = 0$$

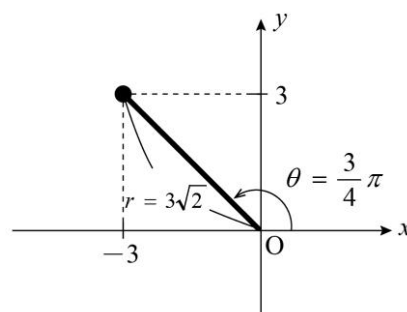
$$\therefore x = 0$$

7

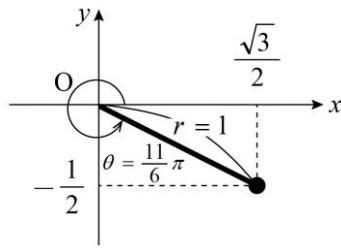
$$(1) \quad (2, 0) \longrightarrow (2, 0)$$



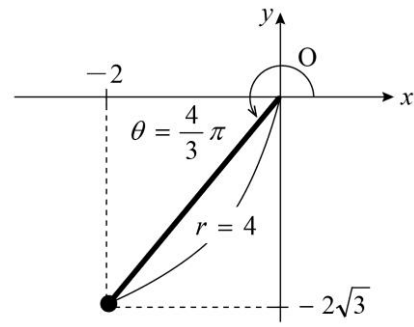
$$(2) \quad (-3, 3) \longrightarrow \left(3\sqrt{2}, \frac{3}{4}\pi \right)$$



$$(3) \left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \right) \longrightarrow \left(1, \frac{11}{6}\pi \right)$$

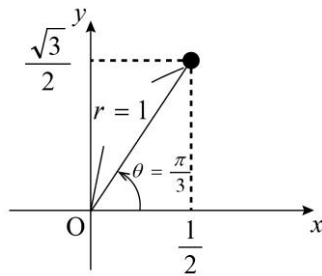


$$(4) (-2, -2\sqrt{3}) \longrightarrow \left(4, \frac{4}{3}\pi \right)$$

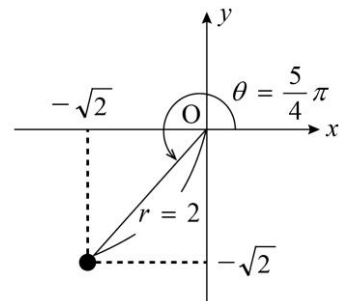


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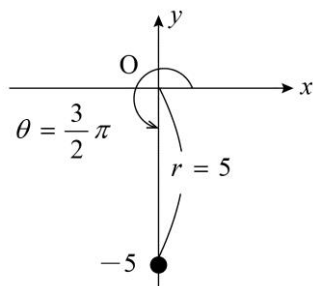
$$(1) \left(1, \frac{\pi}{3} \right) \longrightarrow \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$



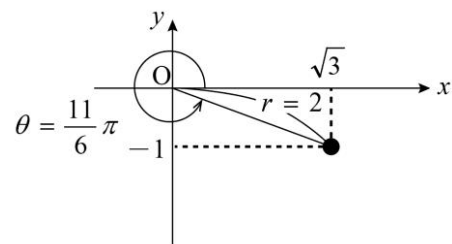
$$(2) \left(2, \frac{5}{4}\pi \right) \longrightarrow (-\sqrt{2}, -\sqrt{2})$$



$$(3) \left(5, \frac{3}{2}\pi \right) \longrightarrow (0, -5)$$



$$(4) \left(2, \frac{11}{6}\pi \right) \longrightarrow (\sqrt{3}, -1)$$



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(1) 極座標 (r, θ) と直交座標 (x, y) の間には次のような関係が成り立つ。

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \iff \begin{cases} \tan \theta = \frac{y}{x} \\ r = \sqrt{x^2 + y^2} \end{cases}$$

$$r \sin \theta = 3 \quad \text{より} \quad y = 3$$

$$(2) \quad \tan \theta = \frac{y}{x} \quad \text{より} \quad \tan \frac{\pi}{4} = \frac{y}{x}$$

$$1 = \frac{y}{x}$$

$$y = x$$

$$(3) \quad r = \sqrt{x^2 + y^2} \quad \text{より} \quad \sqrt{x^2 + y^2} = 2$$

$$x^2 + y^2 = 4$$

$$(4) \quad r = 6 \cos \theta \text{ の両辺に } r \text{ を掛けると } r^2 = 6r \cos \theta$$

$$r = \sqrt{x^2 + y^2}, \quad x = r \cos \theta \quad \text{より} \quad x^2 + y^2 = 6x$$

$$(x - 3)^2 + y^2 = 3^2$$

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(1) $r = f(\theta)$ が与えられたとき, 直交座標 (x, y) と極座標 (r, θ) の関係より

$$\begin{cases} x = f(\theta) \cos \theta \\ y = f(\theta) \sin \theta \end{cases}$$

$$r = 2 \quad \text{より} \quad \begin{cases} x = 2 \cos \theta \\ y = 2 \sin \theta \end{cases}$$

$$\text{このとき} \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{(2 \sin \theta)'}{(2 \cos \theta)'} = \frac{2 \cos \theta}{-2 \sin \theta} = -\cot \theta$$

$$(2) \quad r = 2a \cos \theta \quad \text{より} \quad \begin{cases} x = 2a \cos \theta \cdot \cos \theta = 2a \cos^2 \theta \\ y = 2a \cos \theta \cdot \sin \theta = a \sin 2\theta \end{cases}$$

$$\text{このとき} \quad \frac{dy}{dx} = \frac{(a \sin 2\theta)'}{(2a \cos^2 \theta)'} = \frac{2a \cos 2\theta}{4a \cos \theta \cdot (-\sin \theta)}$$

$$= \frac{2a \cos 2\theta}{-2a \sin 2\theta} = -\cot 2\theta$$

$$(3) \quad r = \sin 2\theta \quad \text{より} \quad \begin{cases} x = \sin 2\theta \cdot \cos \theta \\ y = \sin 2\theta \cdot \sin \theta \end{cases}$$

$$\text{このとき} \quad \frac{dy}{dx} = \frac{(\sin 2\theta \cos \theta)'}{(\sin 2\theta \sin \theta)'} = \frac{2 \cos 2\theta \cdot \sin \theta + \sin 2\theta \cos \theta}{2 \cos 2\theta \cos \theta - \sin 2\theta \sin \theta}$$

$$= \frac{\left(\frac{2 \cos 2\theta \sin \theta + \sin 2\theta \cos \theta}{\cos 2\theta \cos \theta}\right)}{\left(\frac{2 \cos 2\theta \cos \theta - \sin 2\theta \sin \theta}{\cos 2\theta \cos \theta}\right)}$$

$$= \frac{2 \tan \theta + \tan 2\theta}{2 - \tan 2\theta \tan \theta}$$

$$(1) \quad x^2 + y^2 - 5 = 0$$

$$y^2 = -x^2 + 5$$

$$y = \pm\sqrt{5-x^2}$$

$$y \geq 0 \quad \text{より} \quad y = \sqrt{5-x^2}$$

$$(2) \quad \text{問題を変更します}$$

$$\lceil x^2 + y^2 + 8x - 6y + 21 = 0 \rceil$$

$$x^2 + y^2 + 8x - 6y + 21 = 0$$

$$(x+4)^2 + (y-3)^2 = 4$$

$$(y-3)^2 = 4 - (x+4)^2$$

$$y-3 = \pm\sqrt{4-(x+4)^2}$$

$$y \geq 0 \quad \text{より} \quad y = 3 + \sqrt{4-(x+4)^2}$$

$$(3) \quad 4x - y^2 = 0$$

$$y^2 = 4x$$

$$y = \pm 2\sqrt{x}$$

$$y \geq 0 \quad \text{より} \quad y = 2\sqrt{x}$$

$$(4) \quad 4x^2 + 5y^2 = 20$$

$$5y^2 = -4x^2 + 20$$

$$y^2 = -\frac{4}{5}x^2 + 4$$

$$y = \pm\sqrt{4-\frac{4}{5}x^2}$$

$$y \geq 0 \quad \text{より} \quad y = \sqrt{4-\frac{4}{5}x^2}$$

$$(1) \quad y^2 = 4x$$

$$y = 2\sqrt{x} \quad (y \geq 0 \text{ より})$$

$$\frac{dy}{dx} = 2 \times \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

$$\text{別解} \quad y^2 = 4x$$

$$2y \frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{2}{y} = \frac{1}{\sqrt{x}}$$

$$(2) \quad x^2 + \frac{y^2}{4} = 1$$

$$4x^2 + y^2 = 4$$

$$y^2 = 4 - 4x^2$$

$$y = 2\sqrt{1-x^2} \quad (y \geq 0 \text{ より})$$

$$\frac{dy}{dx} = 2 \times \frac{1}{2} \frac{1}{\sqrt{1-x^2}} \times (-2x)$$

$$= -\frac{2x}{\sqrt{1-x^2}}$$

$$\text{別解} \quad x^2 + \frac{y^2}{4} = 1$$

$$2x + \frac{1}{2} y \cdot \frac{dy}{dx} = 0$$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{4x}{y} \\ &= \frac{-2x}{\sqrt{1-x^2}} \end{aligned}$$

$$(3) \quad x^2 + y^2 + 2x = 0$$

$$(x+1)^2 + y^2 - 1 = 0$$

$$y^2 = 1 - (x+1)^2$$

$$y = \sqrt{1 - (x+1)^2} \quad (y \geq 0 \text{ より})$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1-(x+1)^2}} \times (-2x-2)$$

$$= -\frac{x+1}{\sqrt{1-(x+1)^2}}$$

$$\text{別解} \quad x^2 + y^2 + 2x = 0$$

$$2x + 2y \cdot \frac{dy}{dx} + 2 = 0$$

$$2y \frac{dy}{dx} = -2 - 2x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1-x}{y} \\ &= \frac{-x-1}{\sqrt{1-(x+1)^2}} \end{aligned}$$

$$(4) \quad x^2 - 4y^2 - 8y = 0$$

$$x^2 - 4(y+1)^2 + 4 = 0$$

$$(y+1)^2 = \frac{1}{4}x^2 + 1$$

$$y+1 = \sqrt{\frac{1}{4}x^2 + 1} \quad (y \geq 0 \text{ より } y+1 > 0)$$

$$y = \sqrt{\frac{1}{4}x^2 + 1} - 1$$

$$= \frac{1}{2}\sqrt{x^2 + 4} - 1$$

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{2} \frac{1}{\sqrt{x^2 + 4}} \times 2x$$

$$= \frac{x}{2\sqrt{x^2 + 4}}$$

$$\text{別解} \quad x^2 - 4y^2 - 8y = 0$$

$$2x - 8y \frac{dy}{dx} - 8 \frac{dy}{dx} = 0$$

$$(8y+8) \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{x}{4(y+1)}$$

$$= \frac{x}{2\sqrt{x^2 + 4}}$$

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$$(1) \quad x^2 + 4y^2 - 16 = 0$$

$$2x + 8y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{x}{4y}$$

$$(2) \quad x^3 + y^3 = 1$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{x^2}{y^2}$$

$$(3) \quad (x-3)^2 + (y-2)^2 = 1$$

$$2(x-3) + 2(y-2) \frac{dy}{dx} = 0$$

$$(y-2) \frac{dy}{dx} = 3-x \quad \therefore \frac{dy}{dx} = \frac{3-x}{y-2}$$

$$(4) \quad xe^x + ye^y = 1$$

$$(e^x + xe^x) + (e^y + ye^y) \frac{dy}{dx} = 0$$

$$(1+y)e^y \frac{dy}{dx} = -(1+x)e^x \quad \therefore \frac{dy}{dx} = -\frac{1+x}{1+y} e^{x-y}$$

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$$(1) \quad \sin x + \cos y = 1$$

$$\cos x - \sin y \cdot \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{\cos x}{\sin y}$$

$$(2) \quad xy = 4$$

$$y + x \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{y}{x}$$

$$(3) \quad x^2 + 3xy + y^2 = 1$$

$$2x + \left(3y + 3x \frac{dy}{dx}\right) + 2y \frac{dy}{dx} = 0$$

$$(2x + 3y) + (3x + 2y) \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{2x + 3y}{3x + 2y}$$

$$(4) \quad x + y = \log \frac{1}{xy}$$

$$1 + \frac{dy}{dx} = \frac{1}{\left(\frac{1}{xy}\right)} \times \frac{-\left(y + x \frac{dy}{dx}\right)}{(xy)^2}$$

$$1 + \frac{dy}{dx} = \frac{-1}{xy} \left(y + x \frac{dy}{dx}\right)$$

$$xy + xy \frac{dy}{dx} = -y - x \frac{dy}{dx}$$

$$(xy + x) \frac{dy}{dx} = -xy - y$$

$$\frac{dy}{dx} = -\frac{y(x+1)}{x(y+1)}$$

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$$(1) \quad r^2 = 9 \sin 2\theta$$

$$2r \frac{dr}{d\theta} = 9 \cdot 2 \cos 2\theta \quad \therefore \frac{dr}{d\theta} = \frac{9 \cos 2\theta}{r}$$

$$(2) \quad r = \sqrt{4 \cos 2\theta}$$

$$\frac{dr}{d\theta} = \frac{1}{2} \cdot \frac{1}{\sqrt{4 \cos 2\theta}} \cdot 4 \cdot 2(-\sin 2\theta) \quad \therefore \frac{dr}{d\theta} = -\frac{2 \sin 2\theta}{\sqrt{\cos 2\theta}}$$

$$(3) \quad r^2 = \cos^2 2\theta - \sin^2 2\theta$$

$$r^2 = \cos 4\theta$$

$$2r \frac{dr}{d\theta} = -4 \sin 4\theta \quad \therefore \frac{dr}{d\theta} = -\frac{2 \sin 4\theta}{r}$$

B

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$$(1) \quad t = 1 \text{ のとき } x = 2, \quad y = 0 \quad \text{より}$$

求める接線の接点は $(2, 0)$

$$\frac{dy}{dx} = \frac{(t^2 - t)'}{(t + 1)'} = \frac{2t - 1}{1} = 2t - 1$$

$$t = 1 \text{ のとき } \frac{dy}{dx} = 1 \quad \text{より} \quad \text{接線の傾きは } 1$$

$$y - y' = m(x - x') \quad \text{より} \quad y - 0 = 1 \cdot (x - 2) \quad \therefore \quad y = x - 2$$

$$(2) \quad t = 2 \text{ のとき } x = \frac{2}{5}, \quad y = \frac{4}{5} \quad \text{より}$$

$$\text{求める接線の接点は} \quad \left(\frac{2}{5}, \frac{4}{5} \right)$$

$$\frac{dy}{dx} = \frac{\left(\frac{t^2}{1+t^2} \right)'}{\left(\frac{t}{1+t^2} \right)'} = \frac{\frac{2t}{(1+t^2)^2}}{\frac{1-t^2}{(1+t^2)^2}} = \frac{2t}{1-t^2}$$

$$t = 2 \text{ のとき } \frac{dy}{dx} = \frac{4}{1-4} = -\frac{4}{3}$$

$$\text{よって接線の傾きは} \quad -\frac{4}{3}$$

$$\begin{aligned} \text{以上より接線の方程式は} \quad y - \frac{4}{5} &= -\frac{4}{3} \left(x - \frac{2}{5} \right) \\ y &= -\frac{4}{3}x + \frac{4}{3} \end{aligned}$$

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$$y^2 = 8x$$

$$2y \cdot \frac{dy}{dx} = 8 \quad \therefore \frac{dy}{dx} = \frac{4}{y}$$

$$y - 4 = 1 \cdot (x - 2)$$

$$\therefore y = x + 2$$

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$$x^2 - xy + y^2 = 1$$

$$2x - y - x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{2x - y}{x - 2y}$$

$$y - 1 = \frac{1}{2}(x - 0) \quad \therefore y = \frac{1}{2}x + 1$$

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$$(1) \quad x^3 + xy + y^2 = a^2$$

$$3x^2 + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = -\frac{3x^2 + y}{x + 2y} \quad \therefore f'(x) = -\frac{3x^2 + y}{x + 2y}$$

$$(2) \quad \sin x + \sin y - \sin(x + y) = 0$$

$$\cos x + \cos y \frac{dy}{dx} - \cos(x + y) \cdot \left(1 + \frac{dy}{dx} \right) = 0$$

$$\frac{dy}{dx} = \frac{-\cos x + \cos(x + y)}{\cos y - \cos(x + y)} \quad \therefore f'(x) = \frac{-\cos x + \cos(x + y)}{\cos y - \cos(x + y)}$$

$$(3) \quad e^x + e^y = e^{x+y}$$

$$e^x + e^y \cdot \frac{dy}{dx} = e^{x+y} \left(1 + \frac{dy}{dx} \right)$$

$$(e^y - e^{x+y}) \frac{dy}{dx} = e^{x+y} - e^x$$

$$\frac{dy}{dx} = - \frac{e^{x+y} - e^x}{e^{x+y} - e^y} \quad \therefore f'(x) = - \frac{e^{x+y} - e^x}{e^{x+y} - e^y}$$