

5 章 微分方程式

2 節 1 階微分方程式

A

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$$(1) \quad y' - \sqrt{y} = 0$$

$$\int \frac{1}{\sqrt{y}} dy = \int dx$$

$$2\sqrt{y} = x + c$$

$$4y = (x + c)^2$$

$$\therefore y = \frac{1}{4}(x + c)^2$$

$$(2) \quad xy' - y = 0$$

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$\log |y| = \log |x| + c$$

$$\log \left| \frac{y}{x} \right| = c$$

$$\frac{y}{x} = \pm e^c = c$$

$$\therefore y = cx$$

$$(3) \quad yy' = \sqrt{1 - y^2}$$

$$\int \frac{y}{\sqrt{1 - y^2}} dy = \int dx$$

$$-\sqrt{1 - y^2} = x + c$$

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$$(1) \quad yy' + x = 0 \quad (x = 1, \quad y = 1)$$

$$yy' = -x$$

$$\int y dy = -\int x dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$$

$$x = 1 \quad y = 1 \quad \text{を代入して} \quad \frac{1}{2} = -\frac{1}{2} + c \quad \therefore c = 1$$

$$\text{したがって} \quad x^2 + y^2 = 2$$

$$(2) \quad y' = e^{2x+y} \quad (x=0, \ y=0)$$

$$\int e^{-y} dy = \int e^{2x} dx$$

$$-e^{-y} = \frac{1}{2} e^{2x} + c$$

$$x=0, \ y=0 \text{ を代入して } -1 = \frac{1}{2} + c \quad \therefore \quad c = -\frac{3}{2}$$

$$\text{したがって} \quad e^{2x} + 2e^{-y} = 3$$

$$(3) \quad \cos x \cos y \frac{dy}{dx} = \sin x \sin y \quad \left(x=0, \ y=\frac{\pi}{2} \right)$$

$$\int \frac{\cos y}{\sin y} dy = \int \frac{\sin x}{\cos x} dx$$

$$\log |\sin y| = -\log |\cos x| + c$$

$$\log |\sin y \cos x| = c$$

$$\sin y \cos x = \pm e^c = c$$

$$\sin y \cos x = c$$

$$x=0, \ y=\frac{\pi}{2} \text{ を代入して } \sin \frac{\pi}{2} \cos 0 = c$$

$$1 = c$$

$$\text{したがって} \quad \sin y \cos x = 1$$

$$(1) \quad y' = \frac{2y}{x-y} = \frac{2 \frac{y}{x}}{1 - \frac{y}{x}}$$

$$\frac{y}{x} = u \quad \text{とおくと} \quad y' = u'x + u$$

$$u'x + u = \frac{2u}{1-u}$$

$$u'x = \frac{u+u^2}{1-u}$$

$$\int \frac{1-u}{u+u^2} du = \int \frac{1}{x} dx$$

$$\int \left(\frac{1}{u} - \frac{2}{1+u} \right) du = \int \frac{1}{x} dx$$

$$\log |u| - 2 \log |1+u| = \log |x| + c$$

$$\log \left| \frac{u}{(1+u)^2 x} \right| = c$$

$$\frac{u}{(1+u)^2 x} = \pm e^c = c$$

$$\frac{\frac{y}{x}}{\left(1 + \frac{y}{x}\right)^2} = c$$

$$\frac{y}{(x+y)^2} = c$$

$$\therefore y = c(x+y)^2$$

$$(3) \quad (xy' - y)e^{\frac{y}{x}} = x$$

$$xy' - y = xe^{-\frac{y}{x}}$$

$$xy' = xe^{-\frac{y}{x}} + y$$

$$y' = e^{-\frac{y}{x}} + \frac{y}{x}$$

$$\frac{y}{x} = u \quad \text{とおくと} \quad y' = u + xu'$$

$$u + xu' = e^{-u} + u$$

$$xu' = e^{-u}$$

$$\int e^u du = \int \frac{1}{x} dx$$

$$e^u = \log |x| + c$$

$$\therefore e^{\frac{y}{x}} = \log |x| + c$$

$$(2) \quad xy^2 y' = x^3 + y^3$$

$$y' = \frac{x^3 + y^3}{xy^2} = \frac{1 + \left(\frac{y}{x}\right)^3}{\left(\frac{y}{x}\right)^2}$$

$$\frac{y}{x} = u \quad \text{とおくと} \quad y' = u + xu'$$

$$u + xu' = \frac{1+u^3}{u^2}$$

$$xu' = \frac{1+u^3}{u^2} - u = \frac{1}{u^2}$$

$$\int u^2 du = \int \frac{1}{x} dx$$

$$\frac{1}{3} u^3 = \log |x| + c$$

$$\left(\frac{y}{x}\right)^3 = 3 \log |x| + 3c$$

$$\therefore y^3 = 3x^3(\log |x| + c)$$

$$(4) \quad y' = \frac{y}{x} + \tan \frac{y}{x}$$

$$\frac{y}{x} = u \quad \text{とおくと} \quad y' = u + xu'$$

$$u + xu' = u + \tan u$$

$$\int \frac{1}{\tan u} du = \int \frac{1}{x} dx$$

$$\log |\sin u| = \log |x| + c$$

$$\frac{\sin u}{x} = \pm e^c = c$$

$$\therefore \sin \frac{y}{x} = cx$$

$$(1) \quad 2xy y' = y^2 - x^2$$

$$y' = \frac{y^2 - x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2 \frac{y}{x}}$$

$$\frac{y}{x} = u \quad \text{とおく} \quad y' = u + xu'$$

$$u + xu' = \frac{u^2 - 1}{2u}$$

$$xu' = \frac{u^2 - 1}{2u} - u = \frac{-1 - u^2}{2u}$$

$$\int \frac{2u}{1 + u^2} du = - \int \frac{1}{x} dx$$

$$\log |1 + u^2| = -\log |x| + c$$

$$\log |(1 + u^2)x| = c$$

$$(1 + u^2)x = \pm e^c = c$$

$$\left(1 + \left(\frac{y}{x}\right)^2\right)x = c$$

$$\frac{x^2 + y^2}{x} = c$$

$$x = 1, \quad y = -1 \quad \text{を代入すると} \quad c = 2$$

$$\therefore x^2 + y^2 = 2x \quad \therefore (x - 1)^2 + y^2 = 1$$

$$(2) \quad (x+y) + (x-y)y' = 0$$

$$y' = \frac{-x-y}{x-y} = \frac{-1-\frac{y}{x}}{1-\frac{y}{x}}$$

$$\frac{y}{x} = u \quad \text{とおくと} \quad y' = u + xu'$$

$$u + xu' = \frac{-1-u}{1-u}$$

$$xu' = \frac{u^2 - 2u - 1}{1-u}$$

$$\int \frac{1-u}{u^2 - 2u - 1} du = \int \frac{1}{x} dx$$

$$-\frac{1}{2} \log |u^2 - 2u - 1| = \log |x| + c$$

$$\log \left| x\sqrt{u^2 - 2u - 1} \right| = -c$$

$$x\sqrt{u^2 - 2u - 1} = \pm e^{-c} = c$$

$$x \sqrt{\left(\frac{y}{x}\right)^2 - \frac{2y}{x} - 1} = c$$

$$\sqrt{y^2 - 2xy - x^2} = c$$

$$y^2 - 2xy - x^2 = c^2 = c$$

$$x=0, \quad y=0 \quad \text{を代入して} \quad c=0$$

$$\therefore y^2 - 2xy - x^2 = 0$$

$$(3) \quad y' = \frac{y}{x} \left(1 + \log \frac{y}{x} \right) \quad (x=1, \quad y=1)$$

$$\frac{y}{x} = u \quad \text{とおくと} \quad y' = u + xu'$$

$$u + xu' = u(1 + \log u)$$

$$xu' = u \log u$$

$$\int \frac{1}{u \log u} du = \int \frac{1}{x} dx$$

$$\log |\log u| = \log |x| + c$$

$$\log \left| \frac{\log u}{x} \right| = c$$

$$\frac{\log u}{x} = \pm e^c = c$$

$$\log \frac{y}{x} = cx$$

$$\frac{y}{x} = e^{cx} \quad \therefore y = xe^{cx}$$

$$x=1, \quad y=1 \quad \text{を代入して} \quad 1 = e^c \quad \therefore c=0$$

$$\therefore y = x$$

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$$(1) \quad xy' + y = x^3$$

$$y' + \frac{1}{x}y = x^2$$

$$y' + \frac{1}{x}y = 0$$

$$y' = -\frac{1}{x}y$$

$$\int \frac{1}{y} dy = -\int \frac{1}{x} dx$$

$$\log |y| = -\log |x| + c$$

$$xy = c$$

$$c \text{ を } x \text{ の関数 } u \text{ とおく} \quad y = \frac{u}{x} \quad y' = \frac{u'}{x} - \frac{u}{x^2}$$

$$\frac{u'}{x} - \frac{u}{x^2} + \frac{u}{x^2} = x^2$$

$$u' = x^3$$

$$u = \frac{1}{4}x^4 + c$$

$$\therefore y = \frac{1}{x} \left(\frac{1}{4}x^4 + c \right) = \frac{1}{4}x^3 + \frac{c}{x}$$

$$(2) \quad x^2y' + 2xy = 1$$

$$y' + \frac{2}{x}y = \frac{1}{x^2}$$

$$y' + \frac{2}{x}y = 0$$

$$\int \frac{1}{y} dy = -\int \frac{2}{x} dx$$

$$\log |y| = -2 \log |x| + c$$

$$yx^2 = c$$

$$y = \frac{c}{x^2}$$

$$c \text{ を } x \text{ の関数 } u \text{ とおく}$$

$$y = \frac{u}{x^2}$$

$$y' = \frac{u'}{x^2} - \frac{2u}{x^3}$$

$$\frac{u'}{x^2} - \frac{2u}{x^3} + \frac{2u}{x^3} = \frac{1}{x^2}$$

$$u' = 1$$

$$u = x + c$$

$$\therefore y = \frac{1}{x^2}(x + c) = \frac{1}{x} + \frac{c}{x^2}$$

$$(3) \quad xy' \log x + y = \log x$$

$$y' + \frac{1}{x \log x} y = \frac{1}{x}$$

$$y' + \frac{1}{x \log x} y = 0$$

$$\int \frac{1}{y} dy = - \int \frac{1}{x \log x} dx$$

$$\log |y| = -\log |\log x| + c$$

$$y \log x = c$$

c を x の関数 u とおく

$$y = \frac{u}{\log x} \quad y' = -\frac{u}{x(\log x)^2} + \frac{u'}{\log x}$$

$$\frac{u'}{\log x} - \frac{c}{x(\log x)^2} + \frac{c}{x(\log x)^2} = \frac{1}{x}$$

$$u' = \frac{1}{x} \log x$$

$$u = \int \frac{1}{x} \log x dx$$

$$= (\log x)^2 - \int \frac{\log x}{x} dx \quad \text{より}$$

$$u = \int \frac{1}{x} \log x dx = \frac{1}{2} (\log x)^2 + c$$

$$\text{したがって} \quad y = \frac{1}{\log x} \left(\frac{1}{2} (\log x)^2 + c \right)$$

$$y = \frac{1}{2} \log x + \frac{c}{\log x}$$

$$(4) \quad y' - 3y = \sin x$$

$$y' - 3y = 0$$

$$\int \frac{1}{y} dy = 3 \int dx$$

$$\log |y| = 3x + c$$

$$y = ce^{3x}$$

c を x の関数 u とおく

$$y = ue^{3x}$$

$$y' = u'e^{3x} + 3ue^{3x}$$

$$u'e^{3x} + 3ue^{3x} - 3ue^{3x} = \sin x$$

$$u' = e^{-3x} \sin x$$

$$u = \int e^{-3x} \sin x dx$$

$$= -\frac{1}{10} e^{-3x} (3 \sin x + \cos x) + c$$

$$\therefore y = -\frac{1}{10} (3 \sin x + \cos x) + ce^{3x}$$

$$(1) \quad xy' + 3y + x = 0 \quad (x=1, y=1)$$

$$y' + \frac{3}{x}y = -1$$

$$y' + \frac{3}{x}y = 0$$

$$\int \frac{1}{y} dy = -\int \frac{3}{x} dx$$

$$\log|y| = -3\log|x| + c$$

$$yx^3 = c$$

$$y = \frac{c}{x^3}$$

$$c \text{ を } x \text{ の関数 } u \text{ とおく } y = \frac{u}{x^3}$$

$$y' = \frac{u'}{x^3} - \frac{3u}{x^4}$$

$$\frac{u'}{x^3} - \frac{3u}{x^4} + \frac{3u}{x^4} = -1$$

$$u' = -x^3$$

$$u = -\frac{1}{4}x^4 + c$$

$$\therefore y = \frac{1}{x^3} \left(-\frac{1}{4}x^4 + c \right)$$

$$= -\frac{1}{4}x + \frac{c}{x^3}$$

$$x=1, y=1 \text{ を代入すると } 1 = -\frac{1}{4} + c$$

$$\therefore c = \frac{5}{4}$$

$$\therefore y = -\frac{1}{4}x + \frac{5}{4x^3}$$

$$(2) \quad y' + y = e^x$$

$$y' + y = 0$$

$$\int \frac{1}{y} dy = -\int dx$$

$$\log|y| = -x + c$$

$$y = ce^{-x}$$

$$c \text{ を } x \text{ の関数 } u \text{ とおく } y = ue^{-x}, y' = u'e^{-x} - ue^{-x}$$

$$u'e^{-x} - ue^{-x} + ue^{-x} = e^x$$

$$u' = e^{2x}$$

$$u = \frac{1}{2}e^{2x} + c$$

$$\therefore y = \frac{1}{2}e^x + ce^{-x}$$

$$x=0, y=0 \text{ を代入すると}$$

$$0 = \frac{1}{2} + c \quad \therefore c = -\frac{1}{2}$$

$$\therefore y = \frac{1}{2}(e^x - e^{-x})$$

$$(3) \quad y' \cos x + y \sin x = 1$$

$$y' + \frac{\sin x}{\cos x} y = \frac{1}{\cos x}$$

$$y' + \frac{\sin x}{\cos x} y = 0$$

$$\int \frac{1}{y} dy = - \int \frac{\sin x}{\cos x} dx$$

$$\log |y| = \log |\cos x| + c$$

$$\frac{y}{\cos x} = c$$

$$y = c \cos x$$

c を x の関数 u とおく

$$y = u \cos x$$

$$y' = u' \cos x - u \sin x$$

$$u' \cos x - u \sin x + u \sin x = \frac{1}{\cos x}$$

$$u' = \frac{1}{\cos^2 x}$$

$$u = \tan x + c$$

$$\therefore y = \sin x + c \cos x$$

$$x = 0, \quad y = 1 \text{ を代入して } 1 = c$$

$$\therefore y = \sin x + \cos x$$

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点 P における接線 $Y - y = y'(X - x)$

点 R の x 座標は $0 - y = y'(X - x)$ より $X = x + \frac{y}{y'}$

$$\therefore \quad QR = \left| x - \left(x + \frac{y}{y'} \right) \right| = \left| \frac{y}{y'} \right| = 2$$

$$\therefore \quad \frac{y}{y'} = \pm 2$$

$$\int \frac{1}{y} dy = \pm \frac{1}{2} \int dx$$

$$\log |y| = \pm \frac{1}{2} x + c$$

$$y = ce^{\pm \frac{1}{2} x}$$

$$x = 0, \quad y = 2 \text{ を代入して } 2 = c$$

$$\therefore y = 2e^{\pm \frac{1}{2} x}$$

$$(1) \quad \frac{dy}{dx} = y^2 \left(xy - \frac{1}{xy} \right)$$

$$u = xy \quad \text{とおくと} \quad u' = y + xy' \quad \text{より} \quad y' = \frac{u' - y}{x} = \frac{u' - \frac{u}{x}}{x} = \frac{u'x - u}{x^2}$$

$$\therefore \quad \frac{u'}{x} - \frac{u}{x^2} = \left(\frac{u}{x} \right)^2 \left(u - \frac{1}{u} \right)$$

$$\frac{u'}{x} = \frac{u^3}{x^2} \quad \therefore \quad u' = \frac{u^3}{x}$$

$$(2) \quad \int \frac{1}{u^3} du = \int \frac{1}{x} dx$$

$$-\frac{1}{2} u^{-2} = \log |x| + c$$

$$\frac{1}{u^2} = -2 \log |x| + c$$

$$\frac{1}{x^2 y^2} = -2 \log |x| + c$$

$$\frac{1}{y^2} = x^2 (-\log x^2 + c)$$

$$\therefore y^2 = \frac{1}{x^2 (-\log x^2 + c)} \quad (c \text{ は任意定数})$$

$$(1) \quad y' = xe^{x+y}$$

$$\int e^{-y} dy = \int xe^x dx$$

$$-e^{-y} = xe^x - e^x + c$$

$$\therefore e^{-y} = e^x(1-x) + c \quad (c \text{ は任意定数})$$

$$(2) \quad x^2 y' = y^2 - xy$$

$$y' = \left(\frac{y}{x}\right)^2 - \frac{y}{x}$$

$$\frac{y}{x} = u \quad \text{とおくと} \quad y' = xu' + u$$

$$xu' + u = u^2 - u$$

$$xu' = u^2 - 2u$$

$$\int \frac{1}{u^2 - 2u} du = \int \frac{1}{x} dx$$

$$\frac{1}{2} \left(\int \left(\frac{-1}{u} + \frac{1}{u-2} \right) du \right) = \int \frac{1}{x} dx$$

$$\frac{1}{2} (-\log|u| + \log|u-2|) = \log|x| + c$$

$$\log \left| \frac{u-2}{u} \right| = 2\log|x| + 2c$$

$$\log \left| \frac{u-2}{ux^2} \right| = c$$

$$\frac{u-2}{ux^2} = \pm e^c = c$$

$$\frac{\frac{y}{x} - 2}{\frac{y}{x} \cdot x^2} = c$$

$$\text{より } y = \frac{2x}{1-cx^2} \quad (c \text{ は任意定数})$$

$$(3) \quad (1+x^2) \frac{dy}{dx} = xy + 1$$

$$\frac{dy}{dx} = \frac{x}{1+x^2} y + \frac{1}{1+x^2}$$

$$\frac{dy}{dx} = \frac{x}{1+x^2} y$$

$$\int \frac{1}{y} dy = \int \frac{x}{1+x^2} dx$$

$$\log|y| = \frac{1}{2} \log|1+x^2| + c$$

$$\log \left| \frac{y}{\sqrt{1+x^2}} \right| = c$$

$$y = c\sqrt{1+x^2}$$

c を x の関数 u とおく

$$y = u\sqrt{1+x^2}$$

$$y' = u'\sqrt{1+x^2} + \frac{xu}{\sqrt{1+x^2}}$$

$$u'\sqrt{1+x^2} + \frac{xu}{\sqrt{1+x^2}} = \frac{x}{1+x^2} u\sqrt{1+x^2} + \frac{1}{1+x^2}$$

$$u' = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

$$u = \frac{x}{\sqrt{1+x^2}} + c$$

$$\therefore y = \left(\frac{x}{\sqrt{1+x^2}} + c \right) \sqrt{1+x^2}$$

$$y = x + c\sqrt{1+x^2} \quad (c \text{ は任意定数})$$

$$(1) \quad x(y^2 - 1) + y(x^2 - 1) \frac{dy}{dx} = 0 \quad (x = 2, \quad y = 2)$$

$$\int \frac{y}{y^2 - 1} dy = - \int \frac{x}{x^2 - 1} dx$$

$$\frac{1}{2} \log |y^2 - 1| = - \frac{1}{2} \log |x^2 - 1| + c$$

$$\log \left| \sqrt{y^2 - 1} \sqrt{x^2 - 1} \right| = c$$

$$\sqrt{(x^2 - 1)(y^2 - 1)} = \pm e^c = c$$

$$(x^2 - 1)(y^2 - 1) = c$$

$$x = 2, \quad y = 2 \text{ を代入して } c = 9$$

$$\therefore (x^2 - 1)(y^2 - 1) = 9$$

$$(2) \quad xy' \cos \frac{y}{x} = y \cos \frac{y}{x} + x \quad \left(x = 1, \quad y = \frac{\pi}{2} \right)$$

$$y' \cos \frac{y}{x} = \frac{y}{x} \cos \frac{y}{x} + 1$$

$$\frac{y}{x} = u \quad \text{とおくと} \quad y' = u + xu'$$

$$(u + xu') \cos u = u \cos u + 1$$

$$xu' \cos u = 1$$

$$\int \cos u \, du = \int \frac{1}{x} \, dx$$

$$\sin u = \log |x| + c$$

$$\sin \frac{y}{x} = \log |x| + c$$

$$x = 1, \quad y = \frac{\pi}{2} \text{ を代入して}$$

$$\sin \frac{\pi}{2} = \log 1 + c \quad 1 = c$$

$$\therefore \sin \frac{y}{x} = \log |x| + 1$$

$$(3) \quad y' + \frac{\cos x}{\sin x} y = \frac{1}{\cos^2 x} \quad \left(x = \frac{\pi}{4}, \quad y = 2 \right)$$

$$y' = \frac{\cos x}{\sin x} y = 0$$

$$\int \frac{1}{y} dy = - \int \frac{\cos x}{\sin x} dx$$

$$\log |y| = -\log |\sin x| + c$$

$$\log |y \sin x| = c$$

$$y \sin x = \pm e^c = c$$

$$y = \frac{c}{\sin x}$$

c を x の関数 u とおく

$$y = \frac{u}{\sin x}, \quad y' = \frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x}$$

$$\frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x} + \frac{\cos x}{\sin x} \frac{u}{\sin x} = \frac{1}{\cos^2 x}$$

$$u' = \frac{\sin x}{\cos^2 x}$$

$$u = \frac{1}{\cos x} + c$$

$$\therefore y = \frac{1}{\sin x} \left(\frac{1}{\cos x} + c \right) = \frac{1}{\sin x \cos x} + \frac{c}{\sin x}$$

$x = \frac{\pi}{4}, \quad y = 2$ を代入すると

$$2 = \frac{1}{\frac{1}{2}} + \sqrt{2} c \quad c = 0$$

$$\therefore y = \frac{1}{\sin x \cos x}$$

$$(1) \quad y' = ay - 4x$$

$$y' = ay$$

$$\int \frac{1}{y} dy = \int a dx$$

$$\log|y| = ax + c$$

$$y = ce^{ax}$$

c を x の関数 u として

$$y = ue^{ax}, \quad y' = u'e^{ax} + aue^{ax}$$

$$u'e^{ax} + aue^{ax} = aue^{ax} - 4x$$

$$u' = -4e^{-ax}x$$

$$u = \frac{4}{a} e^{-ax} x - \frac{4}{a} \int e^{-ax} dx$$

$$= \frac{4}{a} e^{-ax} \left(x + \frac{1}{a} \right) + c$$

$$\therefore y = \frac{4}{a^2} (ax + 1) + ce^{ax} \quad (c \text{ は任意定数})$$

(2)

$$\begin{cases} y = \frac{4}{a^2} (ax + 1) + ce^{ax} \\ y' = \frac{4}{a} + cae^{ax} \end{cases}$$

に $x = 0, y = 2, y' = 1$ を代入

$$\begin{cases} 2 = \frac{4}{a^2} + c \\ 1 = \frac{4}{a} + ca \end{cases} \quad \text{より } a = \frac{1}{2}, \quad c = -14$$

$$\therefore y = 16 \left(\frac{1}{2} x + 1 \right) - 14 e^{\frac{x}{2}}$$

$$= 8x - 14 e^{\frac{x}{2}} + 16$$

$$a = \frac{1}{2}$$

$$y = 8x - 14 e^{\frac{x}{2}} + 16$$

$$(1) \quad 2xyy' - y^2 + x = 0$$

$$y' - \frac{1}{2x} y = -\frac{1}{2} y^{-1}$$

$$z = y^2 \text{ とおくと, 例題12より } z' - \frac{1}{x} z = -1$$

$$z' = \frac{1}{x} z$$

$$\int \frac{1}{z} dz = \int \frac{1}{x} dx$$

$$\log|z| = \log|x| + c$$

$$\frac{z}{x} = c$$

$$z = cx$$

c を x の関数 u として

$$z = ux, \quad z' = u'x + u$$

$$u'x + u - u = -1$$

$$u' = -\frac{1}{x}$$

$$u = -\log|x| + c$$

$$\therefore z = -x \log|x| + cx$$

$$\therefore y^2 = -x \log|x| + cx \quad (c \text{ は任意定数})$$

$$(2) \quad y' + y = y^2(\cos x + \sin x)$$

$$z = y^{-1} \text{ とおくと, 例題12より } z' - z = -(\cos x + \sin x)$$

$$z' - z = 0 \text{ の一般解は } z = ce^x$$

c を x の関数 u として

$$z = ue^x, \quad z' = u'e^x + ue^x$$

$$u'e^x + ue^x - ue^x = -(\cos x + \sin x)$$

$$u' = -e^{-x}(\cos x + \sin x)$$

$$u = e^{-x}(\cos x + \sin x) - \int e^{-x}(-\sin x + \cos x) dx$$

$$= e^{-x}(\cos x + \sin x)$$

$$- \left(-e^{-x}(-\sin x + \cos x) + \int e^{-x}(-\cos x - \sin x) dx \right)$$

$$\therefore 2u = 2e^{-x} \cos x$$

$$\therefore u = e^{-x} \cos x + c$$

$$z = \cos x + ce^x$$

$$\text{したがって } y = \frac{1}{\cos x + ce^x} \quad (c \text{ は任意定数})$$

$$(1) \quad x^2 y' + 2xy = 1$$

$$y' + \frac{2}{x} y = \frac{1}{x^2}$$

$$y' + \frac{2}{x} y = 0$$

$$\int \frac{1}{y} dy = -\int \frac{2}{x} dx$$

$$\log |y| = -2 \log |x| + c$$

$$yx^2 = c$$

$$y = \frac{c}{x^2}$$

$$c \text{ を } x \text{ の関数 } u \text{ において } y = \frac{u}{x^2}, \quad y' = \frac{u'}{x^2} - \frac{2u}{x^3}$$

$$\frac{u'}{x^2} - \frac{2u}{x^3} + \frac{2u}{x^3} = \frac{1}{x^2}$$

$$u' = 1$$

$$\therefore u = x + c$$

$$y = \frac{1}{x^2} (x + c) = \frac{1}{x} + \frac{c}{x^2} \quad (c \text{ は任意定数})$$

問題の式は線形方程式なので特異解は存在しない。

$$\text{したがって } \lim_{x \rightarrow \infty} \left(\frac{1}{x} + \frac{c}{x^2} \right) = 0$$

$$(2) \quad y(1) = y(2) \text{ より}$$

$$1 + c = \frac{1}{2} + \frac{c}{4}$$

$$\therefore c = -\frac{2}{3}$$

$$\therefore y = \frac{1}{x} - \frac{2}{3x^2}$$

$$(1) \quad y' = \frac{Ax + By + C}{ax + by + c}$$

$x = X + x_0, y = Y + y_0$ を行おうと

$$\frac{dY}{dX} = \frac{AX + BY + Ax_0 + By_0 + C}{aX + bY + ax_0 + by_0 + c}$$

$$\frac{A}{a} \neq \frac{B}{b} \text{ のとき } \begin{cases} Ax_0 + By_0 + C = 0 \\ ax_0 + by_0 + c = 0 \end{cases}$$

を満たす x_0, y_0 が存在するので

$$\frac{dY}{dX} = \frac{AX + BY}{aX + bY} \text{ と同次形になる。}$$

$$(2) \quad \frac{A}{a} = \frac{B}{b} \neq \frac{C}{c} \text{ のとき}$$

$$\frac{A}{a} = \frac{B}{b} = k \text{ とおくと}$$

$$\frac{dy}{dx} = \frac{k(ax + by) + C}{ax + by + c}$$

$u = ax + by$ を行おうと

$$y' = \frac{u' - a}{b} \text{ より}$$

$$\frac{1}{b} \left(\frac{du}{dx} - a \right) = \frac{ku + C}{u + c} \text{ と変数分離形になる。}$$

$$(3) \quad \begin{cases} 2x + y - 3 = 0 \\ x + 2y - 5 = 0 \end{cases} \text{ より } x = \frac{1}{3}, y = \frac{7}{3}$$

$x = X + \frac{1}{3}, y = Y + \frac{7}{3}$ とすると

$$\frac{dY}{dX} = \frac{2X + Y}{X + 2Y} = \frac{2 + \frac{Y}{X}}{1 + 2\frac{Y}{X}}$$

$$\frac{Y}{X} = u \text{ とおくと } Y = Xu \quad Y' = u + Xu'$$

$$u + Xu' = \frac{2 + u}{1 + 2u}$$

$$Xu' = \frac{2 + u}{1 + 2u} - u = \frac{2 - 2u^2}{1 + 2u}$$

$$\int \frac{1 + 2u}{1 - u^2} du = \int \frac{2}{X} dX$$

$$\int \left(\frac{-\frac{1}{2}}{(1+u)} + \frac{\frac{3}{2}}{(1-u)} \right) du = \int \frac{2}{X} dX$$

$$-\frac{1}{2} \log |1 + u| - \frac{3}{2} \log |1 - u| = 2 \log |X| + c$$

$$\log |(1 + u)(1 - u)^3 X^4| = c$$

$$\therefore (1 + u)(1 - u)^3 X^4 = \pm e^c = c$$

$$\left(1 + \frac{Y}{X} \right) \left(1 - \frac{Y}{X} \right)^3 X^4 = c$$

$$(X + Y)(X - Y)^3 = c$$

$$X = x - \frac{1}{3}, Y = y - \frac{7}{3} \text{ より}$$

$$\left(x + y - \frac{8}{3} \right) (x - y + 2)^3 = c$$

(c は任意定数)

$y = u + 1$ とおくと例題 13 より

$$u' = \{2(2x-1) - (4x-1)\}u = (2x-1)u^2$$

$$u' + u = (2x-1)u^2$$

$z = u^{-1}$ とおくと例題 12 より

$$z' - z = -(2x-1)$$

$$z' - z = 0 \text{ の一般解は } z = ce^x$$

c を x の関数 v とおくと $z = ve^x$

$$z' = v'e^x + ve^x$$

$$v'e^x + ve^x - ve^x = -2x + 1$$

$$v' = e^{-x}(-2x+1)$$

$$v = -e^{-x}(-2x+1) + \int e^{-x}(-2) dx$$

$$= -e^{-x}(-2x+1) + 2e^{-x} + c$$

$$= ze^{-x}x + e^{-x} + c$$

$$\text{したがって } z = 2x + 1 + ce^x$$

$$\text{ゆえに } u = \frac{1}{2x+1+ce^x}$$

$$\text{したがって } y = \frac{1}{2x+1+ce^x} + 1 \quad (c \text{ は任意定数})$$