２節　積分法の応用

A 問題

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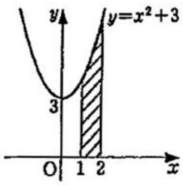
(1)　求める面積は右図の斜線部分だから

$$S=\int_1^2\left(x^2+3\right)dx$$

$$=\left[\frac{1}{3}x^3+3x\right]_1^2$$

$$=\left(\frac{8}{3}+6\right)-\left(\frac{1}{3}+3\right)$$

$$=\frac{7}{3}+3=\frac{16}{3}$$

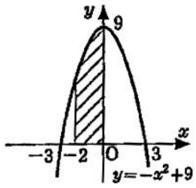



(2)　求める面積は右図の斜線部分だから

$$S=\int_{-2}^0\left(-x^2+9\right)dx$$

$$=\left[-\frac{1}{3}x^3+9x\right]_{-2}^0$$

$$=0-\left(\frac{8}{3}-18\right)=\frac{46}{3}$$

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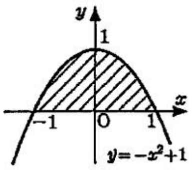
(1)　求める面積は右図の斜線部分だから

$$S=\int_{-1}^1\left(-x^2+1\right)dx$$

$$=\left[-\frac{1}{3}x^3+x\right]_{-1}^1$$

$$=\left(-\frac{1}{3}+1\right)-\left(\frac{1}{3}-1\right)=\frac{4}{3}$$




(2)　放物線と $x$ 軸との交点の $x$ 座標は

$-x^2+2x=0$　を解いて

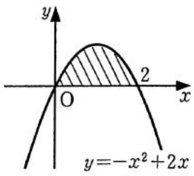
$-x\left(x-2\right)=0$　$\therefore$　$x=0,\ 2$

求める面積は右図の斜線部分だから

$$S=\int_0^2\left(-x^2+2x\right)dx$$

$$=\left[-\frac{1}{3}x^3+x^2\right]_0^2$$

$$=\left(-\frac{8}{3}+4\right)-0=\frac{4}{3}$$




（別解）　$S=\int_0^2\left(-x^2+2x\right)dx$

$$=-\int_0^2 x\left(x-2\right)dx$$

$$=\frac{\left(2-0\right)^3}{6}=\frac{4}{3}$$

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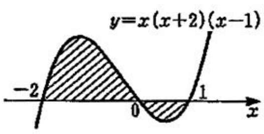
(1)　$y=x\left(x+2\right)\left(x-1\right)$

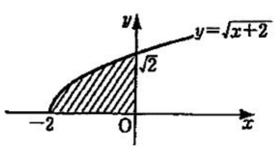
$$S=\int_{-2}^0\left(x^3+x^2-2x\right)dx$$

$$-\int_0^1\left(x^3+x^2-2x\right)dx=\frac{37}{12}$$



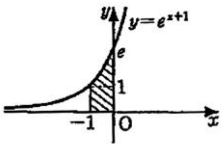

(2)　$S=\int_{-2}^0\sqrt{x+2}\,dx=\left[\frac{2}{3}\left(x+2\right)^{\frac{3}{2}}\right]_{-2}^0$

$$=\frac{4\sqrt{2}}{3}$$



(3) $S = \int_{-1}^{0} e^{x+1} dx$

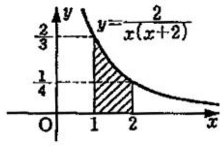
$= \left[ e^{x+1} \right]_{-1}^{0} = e - 1$




(4) $1 \leqq x \leqq 2$ において $y = \dfrac{2}{x(x+2)} > 0$ より

$S = \int_{1}^{2} \dfrac{2}{x(x+2)} dx = \int_{1}^{2} \left( \dfrac{1}{x} - \dfrac{1}{x+2} \right) dx$

$= \Big[ \log x - \log(x+2) \Big]_{1}^{2} = \log \dfrac{3}{2}$

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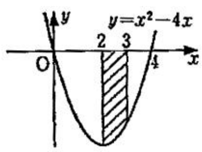
(1) 求める面積は右図の斜線部分だから

求める面積を $S$ とすると




$S = -\int_{2}^{3} \left( x^2 - 4x \right) dx = -\left[ \dfrac{1}{3} x^3 - 2x^2 \right]_{2}^{3}$

$= -\left\{ (9 - 18) - \left( \dfrac{8}{3} - 8 \right) \right\}$

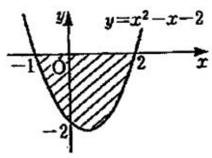
$= -\left( -1 - \dfrac{8}{3} \right) = \dfrac{11}{3}$

(2) 放物線と $x$ 軸との交点の $x$ 座標は

$x^2 - x - 2 = 0$ を解いて

$(x+1)(x-2) = 0$

$\therefore\ x = -1,\ 2$




求める面積は右図の斜線部分だから

$S = -\int_{-1}^{2} \left( x^2 - x - 2 \right) dx$

$= -\left[ \dfrac{1}{3} x^3 - \dfrac{1}{2} x^2 - 2x \right]_{-1}^{2}$

$= -\left\{ \dfrac{1}{3}(8+1) - \dfrac{1}{2}(4-1) - 2(2+1) \right\} = \dfrac{9}{2}$

（別解） $S = -\int_{-1}^{2} \left( x^2 - x - 2 \right) dx$

$= -\int_{-1}^{2} (x+1)(x-2) dx$

$= \dfrac{\{2 - (-1)\}^3}{6} = \dfrac{9}{2}$

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(1)　求める面積は下図の斜線部分だから

$$S=\int_1^2\{-(x+1)(x-2)\}dx$$

$$+\left[-\int_2^3\{-(x+1)(x-2)\}dx\right]$$

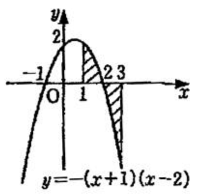
$$=\int_1^2(-x^2+x+2)dx+\int_2^3(x^2-x-2)dx$$

$$=\left[-\frac{1}{3}x^3+\frac{1}{2}x^2+2x\right]_1^2+\left[\frac{1}{3}x^3-\frac{1}{2}x^2-2x\right]_2^3$$

$$=-\frac{1}{3}(8-1)+\frac{1}{2}(4-1)+2(2-1)$$

$$+\frac{1}{3}(27-8)-\frac{1}{2}(9-4)-2(3-2)$$

$$=-\frac{7}{3}+\frac{3}{2}+2+\frac{19}{3}-\frac{5}{2}-2=3$$




(2)　求める面積は下図の斜線部分だから

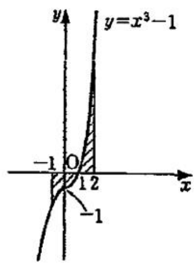
$$S=-\int_{-1}^1(x^3-1)dx+\int_1^2(x^3-1)dx$$

$$=-2\int_0^1(-1)dx+\left[\frac{1}{4}x^4-x\right]_1^2$$

$$=2\left[x\right]_0^1+(4-2)-\left(\frac{1}{4}-1\right)$$

$$=2(1-0)+3-\frac{1}{4}$$

$$=5-\frac{1}{4}=\frac{19}{4}$$

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(1)　交点の $x$ 座標は

$x^2-2x=-2x+1$　を解いて　$x^2-1=0$

$(x+1)(x-1)=0$　∴　$x=\pm1$

求める面積は下図の斜線部分だから

$$S=\int_{-1}^1\{(-2x+1)-(x^2-2x)\}dx$$

$$=\int_{-1}^1(-x^2+1)dx$$

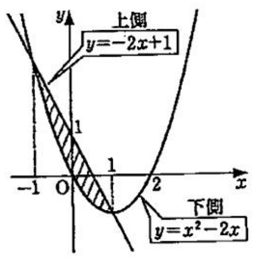
$$=\left[-\frac{1}{3}x^3+x\right]_{-1}^1$$

$$=\left(-\frac{1}{3}+1\right)-\left(\frac{1}{3}-1\right)=\frac{4}{3}$$

（別解）$S=\int_{-1}^1(-x^2+1)dx$

$$=\int_{-1}^1(x+1)(x-1)dx$$

$$=\frac{\{1-(-1)\}^3}{6}=\frac{4}{3}$$




(2)　交点の $x$ 座標は

$-x^2+3x+4=2x-2$　を解いて　$x^2-x-6=0$

$(x+2)(x-3)=0$　∴　$x=-2,\ 3$

求める面積は下図の斜線部分だから

$$S=\int_{-2}^3\{(-x^2+3x+4)-(2x-2)\}dx$$

$$=\int_{-2}^3(-x^2+x+6)dx$$

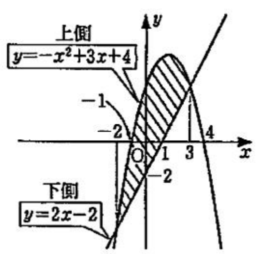
$$=\left[-\frac{1}{3}x^3+\frac{1}{2}x^2+6x\right]_{-2}^3$$

$$=\left(-9+\frac{9}{2}+18\right)-\left(\frac{8}{3}+2-12\right)=\frac{125}{6}$$

（別解）$S=\int_{-2}^3(-x^2+x+6)dx$

$$=-\int_{-2}^3(x+2)(x-3)dx$$

$$=\frac{\{3-(-2)\}^3}{6}=\frac{125}{6}$$



(3)　2 曲線の交点の $x$ 座標は

$x^2-1=-x^2-2x-1$ を解いて

$2x^2+2x=0$

$2x(x+1)=0 \quad \therefore \quad x=0,\ -1$

求める面積は下図の斜線部分だから

$$S=\int_{-1}^{0}\left\{\left(-x^2-2x-1\right)-\left(x^2-1\right)\right\}dx$$

$$=\int_{-1}^{0}\left(-2x^2-2x\right)dx$$

$$=\left[-\frac{2}{3}x^3-x^2\right]_{-1}^{0}$$

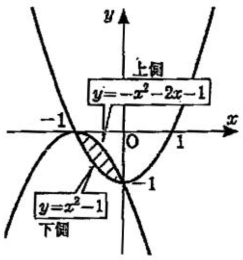
$$=0-\left(\frac{2}{3}-1\right)=\frac{1}{3}$$

（別解）$S=\int_{-1}^{0}\left(-2x^2-2x\right)dx$

$$=-2\int_{-1}^{0}\left(x^2+x\right)dx$$

$$=-2\int_{-1}^{0}(x+1)x\,dx$$

$$=\frac{2(0+1)^3}{6}=\frac{1}{3}$$




(4)　2 曲線の交点の $x$ 座標は

$2x^2+2x-3=-x^2+2x$ を解いて

$3x^2-3=0$

$3(x+1)(x-1)=0 \quad \therefore \quad x=\pm 1$

求める面積は下図の斜線部分だから

$$S=\int_{-1}^{1}\left\{\left(-x^2+2x\right)-\left(2x^2+2x-3\right)\right\}dx$$

$$=\int_{-1}^{1}\left(-3x^2+3\right)dx$$

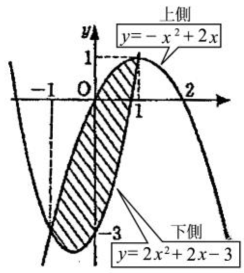
$$=\left[-x^3+3x\right]_{-1}^{1}$$

$$=(-1+3)-(1-3)=4$$

（別解）$S=\int_{-1}^{1}\left(-3x^2+3\right)dx$

$$=-3\int_{-1}^{1}(x+1)(x-1)dx$$

$$=\frac{3(1+1)^3}{6}=4$$

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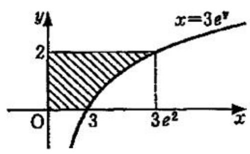
(1)　$S=\int_{0}^{2}3e^y dy$

$$=\left[3e^y\right]_{0}^{2}$$

$$=3e^2-3$$

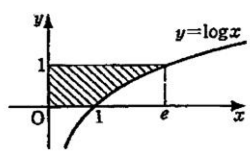



(2)　$y=\log x$ より $x=e^y$

$$S=\int_{0}^{1}e^y dy=\left[e^y\right]_{0}^{1}=e-1$$

（別解）$S=\int_{0}^{e}dx-\int_{1}^{e}\log x\,dx$

$$=\left[x\right]_{0}^{e}-\left[x\log x-x\right]_{1}^{e}=e-1$$



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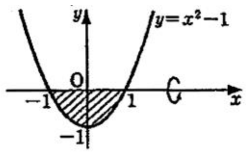
(1) $V = \pi\int_{-1}^{1}\left(x^2 - 1\right)^2 dx$

$= 2\pi\int_{0}^{1}\left(x^4 - 2x^2 + 1\right)dx$

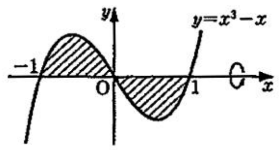
$= 2\pi\left[\frac{x^5}{5} - \frac{2}{3}x^3 + x\right]_0^1 = \frac{16}{15}\pi$




(2) $V = \pi\int_{-1}^{1}\left(x^3 - x\right)^2 dx$

$= 2\pi\int_{0}^{1}\left(x^6 - 2x^4 + x^2\right)dx$

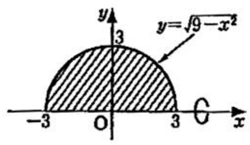
$= 2\pi\left[\frac{x^7}{7} - \frac{2}{5}x^5 + \frac{x^3}{3}\right]_0^1 = \frac{16}{105}\pi$




(3) $V = \pi\int_{-3}^{3}\left(\sqrt{9 - x^2}\right)^2 dx$

$= 2\pi\int_{0}^{3}\left(9 - x^2\right)dx$

$= 2\pi\left[9x - \frac{x^3}{3}\right]_0^3 = 36\pi$

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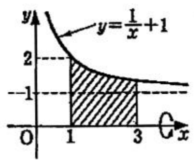
(1) $V = \pi\int_{1}^{3}\left(\frac{1}{x} + 1\right)^2 dx$

$= \pi\int_{1}^{3}\left(\frac{1}{x^2} + \frac{2}{x} + 1\right)dx$

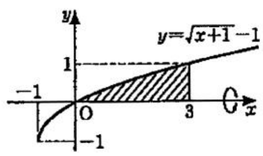
$= \pi\left[-\frac{1}{x} + 2\log x + x\right]_1^3 = \frac{8 + 6\log 3}{3}\pi$




(2) $V = \pi\int_{0}^{3}\left(\sqrt{x+1} - 1\right)^2 dx$

$= \pi\int_{0}^{3}\left(x + 2 - 2\sqrt{x+1}\right)dx$

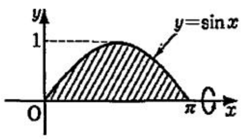
$= \pi\left[\frac{x^2}{2} + 2x - \frac{4}{3}\left(x+1\right)^{\frac{3}{2}}\right]_0^3 = \frac{7}{6}\pi$




(3) $V = \pi\int_{0}^{\pi}\sin^2 x\, dx$

$= \pi\int_{0}^{\pi}\frac{1 - \cos 2x}{2}dx$

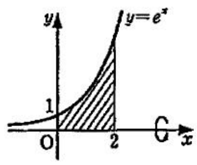
$= \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{1}{2}\pi^2$




(4) $V = \pi\int_{0}^{2}e^{2x}dx$

$= \pi\left[\frac{1}{2}e^{2x}\right]_0^2$

$= \frac{e^4 - 1}{2}\pi$



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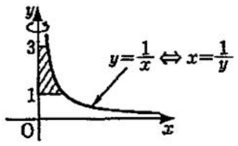
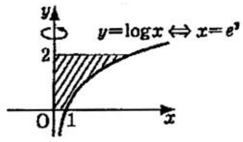
(1) $V = \pi\int_1^3 x^2 dy = \pi\int_1^3 \frac{1}{y^2}dy$

$= \pi\left[-\frac{1}{y}\right]_1^3 = \frac{2}{3}\pi$




(2) $V = \pi\int_0^2 x^2 dy = \pi\int_0^2 e^{2y}dy$

$= \pi\left[\frac{1}{2}e^{2y}\right]_0^2 = \frac{e^4-1}{2}\pi$




B 問題

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(1) $\cos x = \sin\frac{1}{2}x$ より

$2\sin^2\frac{1}{2}x + \sin\frac{1}{2}x - 1 = 0$

$\left(2\sin\frac{1}{2}x - 1\right)\left(\sin\frac{1}{2}x + 1\right) = 0$

$\therefore\ \sin\frac{1}{2}x = \frac{1}{2},\ -1$

$0 \leqq x \leqq 2\pi$ より $x = \frac{\pi}{3},\ \frac{5}{3}\pi$

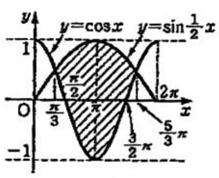
また，$\frac{\pi}{3} \leqq x \leqq \frac{5}{3}\pi$ で $\sin\frac{1}{2}x \geqq \cos x$ である。

よって $S = \int_{\frac{\pi}{3}}^{\frac{5}{3}\pi}\left(\sin\frac{1}{2}x - \cos x\right)dx$

$= \left[-2\cos\frac{1}{2}x - \sin x\right]_{\frac{\pi}{3}}^{\frac{5}{3}\pi} = 3\sqrt{3}$

（別解） グラフの対称性から

$S = 2\int_{\frac{\pi}{3}}^{\pi}\left(\sin\frac{1}{2}x - \cos x\right)dx = 3\sqrt{3}$




(2) $\sin 2x = \sin x$ より $\sin x\left(2\cos x - 1\right) = 0$

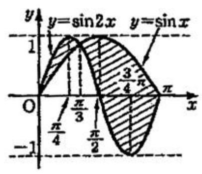
$\sin x = 0,\ \ \cos x = \frac{1}{2}$

$0 \leqq x \leqq \pi$ より $x = 0,\ \frac{\pi}{3},\ \pi$ よって

$S = \int_0^{\frac{\pi}{3}}\left(\sin 2x - \sin x\right)dx + \int_{\frac{\pi}{3}}^{\pi}\left(\sin x - \sin 2x\right)dx$

$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{3}}^{\pi}$

$= \frac{5}{2}$



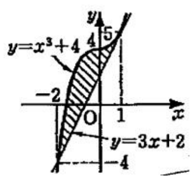

178 $y' = 3x^2$ より，接線の方程式は $y = 3x + 2$

$x^3 + 4 = 3x + 2$ より，共有点の $x$ 座標は

$x = 1,\ -2$

よって $S = \int_{-2}^{1}\left\{\left(x^3+4\right)-\left(3x+2\right)\right\}dx$

$= \left[\frac{x^4}{4} - \frac{3}{2}x^2 + 2x\right]_{-2}^{1} = \frac{27}{4}$



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(1) $y = e^x - e$ より $y' = e^x$

$x = 1$ のとき $y' = e$ となるから

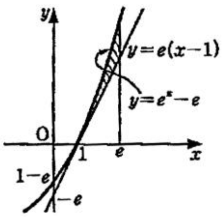
接線の方程式は $y = e(x-1)$

$$S = \int_1^e \left\{ \left(e^x - e\right) - e(x-1) \right\} dx$$

$$= \int_1^e \left(e^x - ex\right) dx = \left[ e^x - \frac{1}{2}ex^2 \right]_1^e$$

$$= \left( e^e - \frac{1}{2}e^3 \right) - \left( e - \frac{1}{2}e \right)$$

$$= e^e - \frac{1}{2}e^3 - \frac{1}{2}e$$




(2) $y = \log x$ より $y' = \dfrac{1}{x}$

$x = 1$ のとき $y' = 1$ となるから

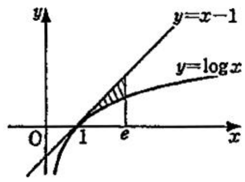
接線の方程式は $y = x - 1$

$$S = \int_1^e \left\{ (x-1) - \log x \right\} dx$$

$$= \left[ \frac{1}{2}x^2 - x \right]_1^e - \left[ x\log x - x \right]_1^e$$

$$= \left( \frac{1}{2}e^2 - e \right) - \left( \frac{1}{2} - 1 \right) - \left( e\log e - e + 1 \right)$$

$$= \frac{1}{2}e^2 - e - \frac{1}{2}$$




180

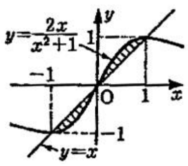
(1) グラフはともに原点に関して対称である。

$\dfrac{2x}{x^2+1} = x$ より $x(x+1)(x-1) = 0$

$\therefore\ x = 0,\ \pm 1$

$$S = 2\int_0^1 \left( \frac{2x}{x^2+1} - x \right) dx$$

$$= 2\left[ \log\left(x^2+1\right) - \frac{1}{2}x^2 \right]_0^1 = 2\log 2 - 1$$



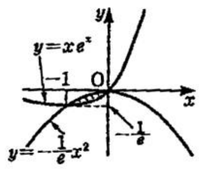

(2) $xe^x = -\dfrac{1}{e}x^2$ より $x\left(e^{x+1} + x\right) = 0$

$\therefore\ x = 0,\ -1$

$$S = \int_{-1}^0 \left( -\frac{1}{e}x^2 - xe^x \right) dx$$

$$= \left[ -\frac{1}{3e}x^3 \right]_{-1}^0 - \left[ xe^x \right]_{-1}^0 + \int_{-1}^0 e^x dx$$

$$= -\frac{1}{3e} - \frac{1}{e} + \left[ e^x \right]_{-1}^0 = 1 - \frac{7}{3e}$$



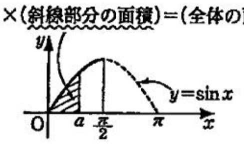

181 $2\displaystyle\int_0^a \sin x\,dx = \int_0^{\frac{\pi}{2}} \sin x\,dx$

$$2\left[ -\cos x \right]_0^a = \left[ -\cos x \right]_0^{\frac{\pi}{2}}$$

$$-2\cos a + 2 = 1 \quad \cos a = \frac{1}{2}$$

$0 < a < \dfrac{\pi}{2}$ より $a = \dfrac{\pi}{3}$

2×(斜線部分の面積)=(全体の面積)



182

切り口の断面積を $S(x)$ とおくと

$$S(x)=\frac{1}{2}\cdot\sin x\cdot\sin x\cdot\sin 60^\circ=\frac{\sqrt{3}}{4}\sin^2 x$$

よって $V=\int_0^\pi\frac{\sqrt{3}}{4}\sin^2 x\,dx$

$$=\frac{\sqrt{3}}{4}\int_0^\pi\frac{1-\cos 2x}{2}dx$$

$$=\frac{\sqrt{3}}{8}\left[x-\frac{1}{2}\sin 2x\right]_0^\pi=\frac{\sqrt{3}}{8}\pi$$

183

(1) $y=x^2-4\cdots①$，$y=3x\cdots②$ のグラフは下のようになり，①，②の $y<0$ の部分を $x$ 軸に対称に折り返して考える。

$x^2-4=3x$ より $(x+1)(x-4)=0$

$\therefore x=-1,\ 4$

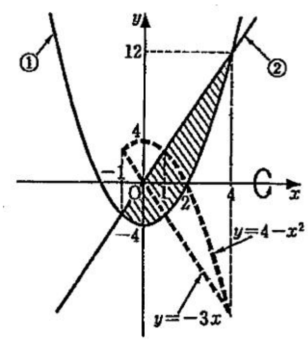
また，$y=x^2-4$ は $y$ 軸に関して対称だから

$$V=2\pi\int_0^1(4-x^2)^2dx-\pi\int_{-1}^0(-3x)^2dx$$

$$+\pi\int_1^4(3x)^2dx-\pi\int_2^4(x^2-4)^2dx$$

$$=2\pi\left[\frac{x^5}{5}-\frac{8}{3}x^3+16x\right]_0^1-\pi\left[3x^3\right]_{-1}^0$$

$$+\pi\left[3x^3\right]_1^4-\pi\left[\frac{x^5}{5}-\frac{8}{3}x^3+16x\right]_2^4=132\pi$$




(2) $y=\sin x\cdots①$，$y=\sin 2x\cdots②$ のグラフは下のようになり，②の $y<0$ の部分を $x$ 軸に対称に折り返して考える。

①，②の交点は

$\sin x=\sin 2x$ より $\sin x(2\cos x-1)=0$

$\sin x=0$，$\cos x=\frac{1}{2}$ $\frac{\pi}{3}\leqq x\leqq\pi$ より

$x=\frac{\pi}{3}$，$\pi$

$$V=\pi\int_{\frac{\pi}{3}}^{\frac{2}{3}\pi}\sin^2 x\,dx+\pi\int_{\frac{2}{3}\pi}^{\pi}\sin^2 2x\,dx$$

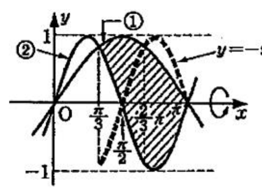
$$-\pi\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\sin^2 2x\,dx$$

$$=\frac{\pi}{2}\int_{\frac{\pi}{3}}^{\frac{2}{3}\pi}(1-\cos 2x)dx+\frac{\pi}{2}\int_{\frac{2}{3}\pi}^{\pi}(1-\cos 4x)dx$$

$$-\frac{\pi}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(1-\cos 4x)dx$$

$$=\frac{\pi}{2}\left\{\left[x-\frac{1}{2}\sin 2x\right]_{\frac{\pi}{3}}^{\frac{2}{3}\pi}+\left[x-\frac{1}{4}\sin 4x\right]_{\frac{2}{3}\pi}^{\pi}\right.$$

$$\left.-\left[x-\frac{1}{4}\sin 4x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}\right\}=\frac{2\pi^2+3\sqrt{3}\pi}{8}$$



184

(1) $2\sin x = 1$ とすると $\sin x = \dfrac{1}{2}$

$0 \leqq x \leqq \pi$ より $x = \dfrac{\pi}{6},\ \dfrac{5}{6}\pi$

グラフは直線 $x = \dfrac{\pi}{2}$ に関して対称

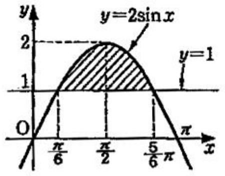
であるから，求める体積 $V$ は

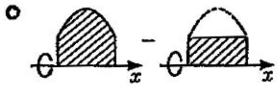
$$V = 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin x)^2 dx - 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1^2 dx$$

$$= 8\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1-\cos 2x}{2} dx - 2\pi\Big[\ x\ \Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 8\pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \frac{2}{3}\pi^2$$

$$= \frac{\left(2\pi + 3\sqrt{3}\right)\pi}{3}$$



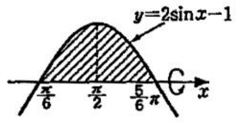

(2) $$V = 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin x - 1)^2 dx$$

$$= 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(4\sin^2 x - 4\sin x + 1\right)dx$$

$$= 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left\{2\left(1-\cos 2x\right) - 4\sin x + 1\right\}dx$$

$$= 2\pi\Big[-\sin 2x + 4\cos x + 3x\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(2\pi - 3\sqrt{3}\right)\pi$$




185 $y = x^2 \cdots$① とすると $y' = 2x$

①上の点 $\left(t,\ t^2\right)$ における接線の方程式は $y - t^2 = 2t(x - t)$

$\therefore\ y = 2tx - t^2 \cdots$②

これが点 $(1,\ -3)$ を通るためには

$-3 = 2t - t^2$

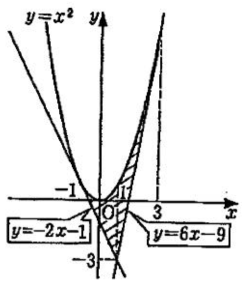
$t^2 - 2t - 3 = 0$

$(t+1)(t-3) = 0$

$\therefore\ t = -1,\ 3$

$t = -1$ のとき②から $y = -2x - 1$

$t = 3$ のとき②から $y = 6x - 9$




また求める面積は右図の斜線部分になるから

求める面積を $S$ とすると $S = \displaystyle\int_{-1}^{1}\left\{x^2 - (-2x-1)\right\}dx + \int_{1}^{3}\left\{x^2 - (6x-9)\right\}dx$

$$= \int_{-1}^{1}\left(x^2 + 2x + 1\right)dx + \int_{1}^{3}\left(x^2 - 6x + 9\right)dx$$

$$= 2\left[\frac{1}{3}x^3 + x\right]_0^1 + \left[\frac{1}{3}x^3 - 3x^2 + 9x\right]_1^3$$

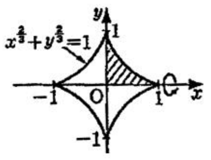
$$= 2\left\{\left(\frac{1}{3}+1\right) - 0\right\} + (9 - 27 + 27) - \left(\frac{1}{3} - 3 + 9\right)$$

$$= \frac{16}{3}$$

発展問題

186　曲線　$x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$　はアステロイドであり，

グラフは右図である。また




$y^{\frac{2}{3}}=1-x^{\frac{2}{3}}$　より　$y^2=\left(1-x^{\frac{2}{3}}\right)^3$

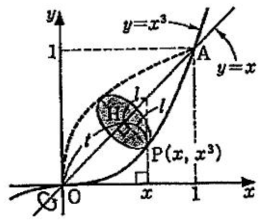
$$V=\pi\int_0^1\left(1-x^{\frac{2}{3}}\right)^3dx=\pi\int_0^1\left(1-3x^{\frac{2}{3}}+3x^{\frac{4}{3}}-x^2\right)dx$$

$$=\pi\left[x-\frac{9}{5}x^{\frac{5}{3}}+\frac{9}{7}x^{\frac{7}{3}}-\frac{1}{3}x^3\right]_0^1=\frac{16}{105}\pi$$

187

関数 $y=x^3$ と直線 $y=x$ の交点は

$\mathrm{A}(1,\ 1)$, また　$\mathrm{OA}=\sqrt{2}$　である。

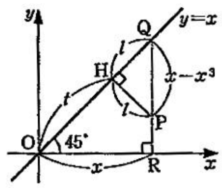



$0\leqq x\leqq 1$として，$y=x^3$ 上の点 $\mathrm{P}\left(x,\ x^3\right)$から

直線 $y=x$ に垂線 PH を下ろし $\mathrm{PH}=l$，$\mathrm{OH}=t$

とおくと、右図で△ PHQ は二等辺三角形で

$$l=\mathrm{PH}=\frac{\mathrm{PQ}}{\sqrt{2}}=\frac{x-x^3}{\sqrt{2}}$$




また，△ ORQ で $\dfrac{x}{t+l}=\cos 45^\circ=\dfrac{1}{\sqrt{2}}$　より

$$t=\sqrt{2}x-l=\sqrt{2}x-\frac{x-x^3}{\sqrt{2}}=\frac{x+x^3}{\sqrt{2}}$$

したがって　$\dfrac{dt}{dx}=\dfrac{1+3x^2}{\sqrt{2}}$

$$dt=\frac{1+3x^2}{\sqrt{2}}dt$$

| $t$ | $0\to\sqrt{2}$ |
|---|---|
| $x$ | $0\to\ 1$ |

よって　$V=\displaystyle\int_0^{\sqrt{2}}\pi l^2dt$

$$=\pi\int_0^1\left(\frac{x-x^3}{\sqrt{2}}\right)^2\cdot\frac{1+3x^2}{\sqrt{2}}dx$$

$$=\frac{\pi}{2\sqrt{2}}\int_0^1\left(3x^8-5x^6+x^4+x^2\right)dx$$

$$=\frac{\pi}{2\sqrt{2}}\left[\frac{1}{3}x^9-\frac{5}{7}x^7+\frac{1}{5}x^5+\frac{1}{3}x^3\right]_0^1$$

$$=\frac{\pi}{2\sqrt{2}}\left(\frac{1}{3}-\frac{5}{7}+\frac{1}{5}+\frac{1}{3}\right)=\frac{4\sqrt{2}}{105}\pi$$

３章の問題

1

(1) $$\int \sqrt[3]{x}\log x\,dx = \int\left(\frac{3}{4}x^{\frac{4}{3}}\right)'\log x dx$$
$$= \frac{3}{4}x^{\frac{4}{3}}\log x - \int \frac{3}{4}x^{\frac{4}{3}}(\log x)'dx$$
$$= \frac{3}{4}x^{\frac{4}{3}}\log x - \int \frac{3}{4}x^{\frac{1}{3}}dx$$
$$= \frac{3}{4}x^{\frac{4}{3}}\log x - \frac{9}{16}x^{\frac{4}{3}} + C$$
$$= \frac{3}{4}x\sqrt[3]{x}\left(\log x - \frac{3}{4}\right) + C$$

(2) $$\int x2^x dx = \int\left(\frac{2^x}{\log 2}\right)'\cdot x\,dx$$
$$= \frac{x\cdot 2^x}{\log 2} - \int \frac{2^x}{\log 2}(x)'dx$$
$$= \frac{x\cdot 2^x}{\log 2} - \frac{1}{\log 2}\int 2^x dx$$
$$= \frac{x\cdot 2^x}{\log 2} - \frac{2^x}{(\log 2)^2} + C$$

(3) $$\int_0^{\pi}\sqrt{1-\cos x}\,dx = \int_0^{\pi}\sqrt{1-\left(1-2\sin^2\frac{x}{2}\right)}\,dx$$
$$= \int_0^{\pi}\sqrt{2}\sin\frac{x}{2}\,dx$$
$$= \sqrt{2}\left[-2\cos\frac{x}{2}\right]_0^{\pi} = 2\sqrt{2}$$

(4) $$\int_0^1 \frac{1}{e^x+1}dx \qquad e^x+1=t \quad とおくと$$

$e^x = t-1$

| $x$ | $0\to 1$ |
|---|---|
| $t$ | $0\to e+1$ |

$e^x dx = dt$

$$\therefore\ dx = \frac{dt}{e^x} = \frac{dt}{t-1}$$

$$(与式) = \int_2^{e+1}\frac{dt}{t(t-1)} = \int_2^{e+1}\left(\frac{1}{t-1}-\frac{1}{t}\right)dt$$
$$= \Big[\log|t-1| - \log|t|\Big]_2^{e+1}$$
$$= \log e - \log(e+1) + \log 2 = \log\frac{2e}{e+1}$$

2 $$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$
$$= 2\tan\frac{\theta}{2}\cdot\frac{1}{1+\tan^2\frac{\theta}{2}} = \frac{2x}{1+x^2}$$

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{1+\tan^2\frac{\theta}{2}} - 1$$
$$= \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}$$

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{1+\sin\theta+\cos\theta}$$

$\tan\dfrac{\theta}{2} = x$ とおくと $d\theta = 2\cos^2\dfrac{\theta}{2}dx$

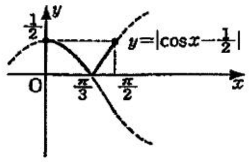
| $\theta$ | $0\to\frac{\pi}{2}$ |
|---|---|
| $x$ | $0\to 1$ |

$$= \frac{2}{1+x^2}dx$$

$$(与式) = \int_0^1 \frac{1}{1+\frac{2x}{1+x^2}+\frac{1-x^2}{1+x^2}}\cdot\frac{2}{1+x^2}dx$$
$$= \int_0^1\frac{1}{1+x}dx = \Big[\log|1+x|\Big]_0^1 = \log 2$$

3

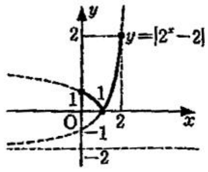
(1) $\displaystyle \int_0^{\frac{\pi}{2}} \left| \cos x - \frac{1}{2} \right| dx = \int_0^{\frac{\pi}{3}} \left( \cos x - \frac{1}{2} \right) dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left( \frac{1}{2} - \cos x \right) dx$




$$= \left[ \sin x - \frac{1}{2} x \right]_0^{\frac{\pi}{3}} + \left[ \frac{1}{2} x - \sin x \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= \left( \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) + \left( \frac{\pi}{4} - 1 \right) - \left( \frac{\pi}{6} - \frac{\sqrt{3}}{2} \right) = \sqrt{3} - 1 - \frac{\pi}{12}$$

(2) $\displaystyle \int_0^2 \left| 2^x - 2 \right| dx = \int_0^1 \left( 2 - 2^x \right) dx + \int_1^2 \left( 2^x - 2 \right) dx$




$$= \left[ 2x - \frac{2^x}{\log 2} \right]_0^1 + \left[ \frac{2^x}{\log 2} - 2x \right]_1^2$$

$$= \left( 2 - \frac{2}{\log 2} \right) + \frac{1}{\log 2} + \left( \frac{4}{\log 2} - 4 \right) - \left( \frac{2}{\log 2} - 2 \right) = \frac{1}{\log 2}$$

4

(1) $\displaystyle \int_0^1 x \cos \pi x \, dx = \int_0^1 x \left( \frac{1}{\pi} \sin \pi x \right)' dx$

$$= \left[ x \frac{1}{\pi} \sin \pi x \right]_0^1 - \frac{1}{\pi} \int_0^1 \sin \pi x \, dx$$

$$= 0 + \frac{1}{\pi^2} \left[ \cos \pi x \right]_0^1 = -\frac{2}{\pi^2}$$

(2) $\displaystyle I = \int_0^1 \left\{ \cos^2 \pi x - \left( ax + b \right) \right\}^2 dx$

$$= \int_0^1 \left\{ \cos^2 \pi x - 2\left( ax + b \right) \cos \pi x + \left( ax + b \right)^2 \right\} dx$$

ここで $\displaystyle \int_0^1 \cos^2 \pi x \, dx = \int_0^1 \frac{1 + \cos 2\pi x}{2} dx$

$$= \frac{1}{2} \left[ x + \frac{1}{2\pi} \sin 2\pi x \right]_0^1 = \frac{1}{2}$$

$$\int_0^1 2\left( ax + b \right) \cos \pi x \, dx = 2a \int_0^1 x \cos \pi x \, dx + 2b \int_0^1 \cos \pi x \, dx$$

$$= 2a \cdot \left( -\frac{2}{\pi^2} \right) + 2b \left[ \frac{1}{\pi} \sin \pi x \right]_0^1 = -\frac{4a}{\pi^2}$$

$$\int_0^1 \left( ax + b \right)^2 dx = \int_0^1 \left( a^2 x^2 + 2abx + b^2 \right) dx$$

$$= \left[ \frac{a^2}{3} x^3 + abx^2 + b^2 x \right]_0^1 = \frac{a^2}{3} + ab + b^2$$

よって $\displaystyle I = \frac{1}{2} + \frac{4a}{\pi^2} + \frac{a^2}{3} + ab + b^2$

(3) $I=\left(b+\dfrac{a}{2}\right)^2+\dfrac{a^2}{12}+\dfrac{4a}{\pi^2}+\dfrac{1}{2}$

$=\left(b+\dfrac{a}{2}\right)^2+\dfrac{1}{12}\left(a+\dfrac{24}{\pi^2}\right)^2+\dfrac{1}{2}-\dfrac{48}{\pi^4}$

これより $b+\dfrac{a}{2}=0$ かつ $a+\dfrac{24}{\pi^2}=0$ のとき，最小値をとる。

したがって $a=-\dfrac{24}{\pi^2}$，$b=\dfrac{12}{\pi^2}$ のとき 最小値 $\dfrac{1}{2}-\dfrac{48}{\pi^4}$

5

(1) $f(x)=x^2-\int_0^x(x-t)f'(t)dt$

$=x^2-x\int_0^x f'(t)dt+\int_0^x tf'(t)dt$

$f(0)=0^2-0+\int_0^0 tf'(t)dt=0$

$f'(x)=2x-\int_0^x f'(t)dt-xf'(x)+xf'(x)$

$=2x-\Big[f(t)\Big]_0^x=2x-f(x)$（証明終）

(2) $\left(e^x f(x)\right)'=e^x f(x)+e^x f'(x)$

$=e^x f(x)+e^x\left(2x-f(x)\right)$

$=e^x f(x)+2xe^x-e^x f(x)=2xe^x$ （証明終）

(3) $\left(e^x f(x)\right)'=2xe^x$ の両辺を $x$ で積分すると

$\int\left(e^x f(x)\right)'dx=\int 2xe^x\,dx$

$e^x f(x)=2\int\left(e^x\right)'x\,dx$

$=2e^x x-2\int e^x\cdot 1dx=2xe^x-2e^x+C$

$\therefore\ f(x)=2x-2+Ce^{-x}$

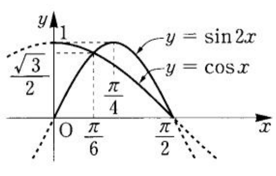
$f(0)=0$ より $f(0)=-2+Ce^0=0$

$\therefore\ C=2$

よって $f(x)=2x-2+2e^{-x}$

6

(1)




(2) $\int_0^{\frac{\pi}{2}}\sin 2x\,dx=\left[-\dfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$

$=-\dfrac{1}{2}(-1-1)=1$

(3) $\sin 2x=\cos x$ より $2\sin x\cos x=\cos x$

$\cos x(2\sin x-1)=0$ $\therefore\sin x=\dfrac{1}{2}$，$\cos x=0$

$0\leqq x\leqq\dfrac{\pi}{2}$ だから $x=\dfrac{\pi}{6}$，$\dfrac{\pi}{2}$

よって，交点は $\left(\dfrac{\pi}{6},\ \dfrac{\sqrt{3}}{2}\right)$，$\left(\dfrac{\pi}{2},\ 0\right)$

(4) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(\sin 2x-\cos x)dx=\left[-\dfrac{1}{2}\cos 2x-\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

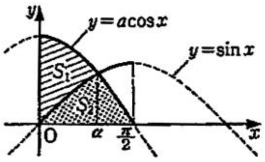
$=\left(\dfrac{1}{2}-1\right)-\left(-\dfrac{1}{4}-\dfrac{1}{2}\right)=\dfrac{1}{4}$

(5) $\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sin^2 2x\,dx-\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2 x\,dx=\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(\frac{1-\cos 4x}{2}-\frac{1+\cos 2x}{2}\right)dx$

$$=\pi\left[-\frac{1}{8}\sin 4x-\frac{1}{4}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$=\pi\left(\frac{\sqrt{3}}{16}+\frac{\sqrt{3}}{8}\right)=\frac{3\sqrt{3}}{16}\pi$$

7　2 つの曲線は右図のようになる。この 2 曲線の交点を　$x=\alpha$　とすると　$a\cos\alpha=\sin\alpha$




$0<\alpha<\frac{\pi}{2}$　より　$\sin\alpha>0,\ \cos\alpha>0$

よって　$a\cos\alpha=\sin\alpha$　と

$\sin^2\alpha+\cos^2\alpha=1$　を連立して解いて

$\sin\alpha=\frac{a}{\sqrt{1+a^2}}$，$\cos\alpha=\frac{1}{\sqrt{1+a^2}}$　$\cdots$①

$S_1=S_2$　となればよいから

$$\int_0^{\alpha}(a\cos x-\sin x)dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}a\cos x\,dx$$

$$\left[a\sin x+\cos x\right]_0^{\alpha}=\frac{1}{2}\left[a\sin x\right]_0^{\frac{\pi}{2}}$$

$$a\sin\alpha+\cos\alpha-1=\frac{a}{2}$$

$$2a\sin\alpha+2\cos\alpha=a+2$$

①を代入して　$\frac{2a^2}{\sqrt{1+a^2}}+\frac{2}{\sqrt{1+a^2}}=a+2$

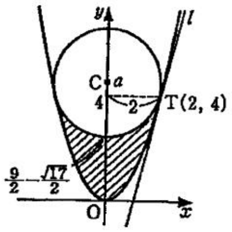
$$2\sqrt{a^2+1}=a+2$$

両辺2乗して　$3a^2-4a=0$

$a(3a-4)=0$　$a>0$　だから　$a=\frac{4}{3}$

8　右図のように，球とワイングラスの接点 T の座標は $\mathrm{T}(2,\ 4)$ である。

球の中心を $\mathrm{C}(0,\ a)$，T における接線を $l$ とすると $\mathrm{CT} \perp l$ である。




$y' = 2x$ より $l$ の傾きは 4　CT の傾きは $\dfrac{4-a}{2-0}$

$\therefore\ 4\cdot\dfrac{4-a}{2} = -1$ より $a = \dfrac{9}{2}$

球の半径は　$\mathrm{CT} = \sqrt{2^2 + \left(4-\dfrac{9}{2}\right)^2} = \sqrt{\dfrac{17}{4}}$

図の円の方程式（球を中心で切った断面）は

$x^2 + \left(y-\dfrac{9}{2}\right)^2 = \dfrac{17}{4}$　求める体積は

$$V = \pi\int_0^4 y\,dy - \pi\int_{\frac{9}{2}-\frac{\sqrt{17}}{2}}^{4}\left\{\frac{17}{4} - \left(y-\frac{9}{2}\right)^2\right\}dy$$

$$= \pi\left[\frac{1}{2}y^2\right]_0^4 - \pi\left[\frac{17}{4}y - \frac{1}{3}\left(y-\frac{9}{2}\right)^3\right]_{\frac{9}{2}-\frac{\sqrt{17}}{2}}^{4}$$

$$= 8\pi - \pi\left\{\frac{17}{4}\left(4-\frac{9}{2}+\frac{\sqrt{17}}{2}\right) - \frac{1}{3}\left(-\frac{1}{8}+\frac{17\sqrt{17}}{8}\right)\right\}$$

$$= \pi\left(8+\frac{17}{8}-\frac{17\sqrt{17}}{8}-\frac{1}{24}+\frac{17\sqrt{17}}{24}\right) = \frac{\pi}{12}\left(121-17\sqrt{17}\right)$$

9

(1)　楕円を $x$ 軸のまわりに回転してできる立体の体積を $V_1$，円柱（半径 $b$，高さ $2a$）の体積を $V_2$ とすると

$$V_1 = \pi\int_{-2}^{2} y^2\,dx = 2\pi\int_0^2\left(1-\frac{x^2}{4}\right)dx$$

$$= 2\pi\left[x-\frac{x^3}{12}\right]_0^2 = 2\pi\left(2-\frac{2}{3}\right) = \frac{8}{3}\pi$$

$V_2 = \pi b^2\cdot 2a = 2ab^2\pi$

ここで，点 $(a,\ b)$ は $\dfrac{x^2}{4}+y^2=1$ 上の点だから

$\dfrac{a^2}{4}+b^2=1$　$\therefore\ b^2 = 1-\dfrac{a^2}{4}$ より

$$V_2 = 2a\left(1-\frac{a^2}{4}\right)\pi = 2\left(a-\frac{a^3}{4}\right)\pi$$

よって　$V = V_1 - V_2 = \dfrac{8}{3}\pi - 2\left(a-\dfrac{a^3}{4}\right)\pi$

$V = \left(\dfrac{8}{3}-2a+\dfrac{a^3}{2}\right)\pi$　$(0 < a < 2)$

(2)　$V' = \left(-2+\dfrac{3}{2}a^2\right)\pi$

$$= \frac{3}{2}\left(a-\frac{2\sqrt{3}}{3}\right)\left(a+\frac{2\sqrt{3}}{3}\right)\pi$$

| $a$ | 0 | … | $\frac{2\sqrt{3}}{3}$ | … | 2 |
|---|---|---|---|---|---|
| $V'$ | | $-$ | 0 | $+$ | |
| $V$ | | ↘ | 極小 | ↗ | |

$a = \dfrac{2\sqrt{3}}{3}$　のとき

$$V = \left(\frac{8}{3}-\frac{4\sqrt{3}}{3}+\frac{4\sqrt{3}}{9}\right)\pi$$

$$= \frac{8\left(3-\sqrt{3}\right)}{9}\pi$$

増減表より最小値は　$a = \dfrac{2\sqrt{3}}{3}$　のとき

$\dfrac{8\left(3-\sqrt{3}\right)}{9}\pi$

10

(1)　2 曲線の交点の $x$ 座標は

$e^{x} = ne^{-x}$　より

$e^{2x} = n \iff 2x = \log n$

$\therefore\ x = \dfrac{1}{2}\log n$

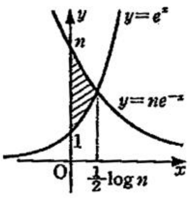
$$S_n = \int_0^{\frac{1}{2}\log n}\left(ne^{-x} - e^{x}\right)dx$$

$$= \Big[-ne^{-x} - e^{x}\Big]_0^{\frac{1}{2}\log n}$$

$$= \left(-ne^{-\frac{1}{2}\log n} - e^{\frac{1}{2}\log n}\right) - \left(-n-1\right)$$

$$= -n\cdot n^{-\frac{1}{2}} - n^{\frac{1}{2}} + n + 1 = n - 2\sqrt{n} + 1$$

$$\therefore\ S_n = \left(\sqrt{n}-1\right)^2$$




(2)　$\lim_{n\to\infty}\left(S_{n+1} - S_n\right)$

$$= \lim_{n\to\infty}\left\{\left(\sqrt{n+1}-1\right)^2 - \left(\sqrt{n}-1\right)^2\right\}$$

$$= \lim_{n\to\infty}\left(\sqrt{n+1}+\sqrt{n}-2\right)\left(\sqrt{n+1}-\sqrt{n}\right)$$

$$\frac{\left(\sqrt{n+1}+\sqrt{n}-2\right)\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}$$

と変形すると

$$= \lim_{n\to\infty}\left(1 - \frac{2}{\sqrt{n+1}+\sqrt{n}}\right) = 1$$