３章　積分法　解答

１節　不定積分と定積分(1)

A 問題

123

(1) $\int 2x\,dx = 2\cdot\frac{1}{2}x^2 + C = x^2 + C$

(2) $\int dx = x + C$

(3) $\int(-6x+5)dx = -6\cdot\frac{1}{2}x^2 + 5x + C$
$= -3x^2 + 5x + C$

(4) $\int(3x^2 - 4x + 1)dx = 3\cdot\frac{1}{3}x^3 - 4\cdot\frac{1}{2}x^2 + x + C$
$= x^3 - 2x^2 + x + C$

(5) $\int x(x+3)dx = \int(x^2 + 3x)dx$
$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + C$

(6) $\int(x-2)(2x-3)dx = \int(2x^2 - 7x + 6)dx$
$= \frac{2}{3}x^3 - \frac{7}{2}x^2 + 6x + C$

(7) $\int(y-2)^2 dy = \int(y^2 - 4y + 4)dy$
$= \frac{1}{3}y^3 - 2y^2 + 4y + C$

(8) $\int(t+a)(t-a)dt = \int(t^2 - a^2)dt$
$= \frac{1}{3}t^3 - a^2 t + C$

124

(1) $\int(2x^2 + 5x + 3)dx + \int(x^2 - 5x - 2)dx$
$= \int\{(2x^2 + 5x + 3) + (x^2 - 5x - 2)\}dx$
$= \int(3x^2 + 1)dx = x^3 + x + C$

(2) $\int(x+1)^2 dx - \int(x-1)^2 dx$
$= \int\{(x+1)^2 - (x-1)^2\}dx$
$= \int 4x\,dx = 2x^2 + C$

125

(1) $\int\left(3x^5 + \sqrt[3]{x^2}\right)dx = \int\left(3x^5 + x^{\frac{2}{3}}\right)dx$
$= \frac{1}{2}x^6 + \frac{3}{5}x^{\frac{5}{3}} + C$
$= \frac{1}{2}x^6 + \frac{3}{5}x\sqrt[3]{x^2} + C$

(2) $\int\left(\sqrt{x} + 1\right)^2 dx = \frac{1}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}} + x + C$
$= \frac{1}{2}x^2 + \frac{4}{3}x\sqrt{x} + x + C$

(3) $\int(3x-1)^4 dx = \frac{1}{15}(3x-1)^5 + C$

(4) $\int\sqrt[3]{x}\left(\sqrt[3]{x} - 1\right)dx = \int\left(x^{\frac{2}{3}} - x^{\frac{1}{3}}\right)dx$
$= \frac{3}{5}x^{\frac{5}{3}} - \frac{3}{4}x^{\frac{4}{3}} + C$
$= \frac{3}{5}x\sqrt[3]{x^2} - \frac{3}{4}x\sqrt[3]{x} + C$

(5) $\displaystyle\int \frac{(x+2)^2}{x}\,dx = \int\left(x+4+\frac{4}{x}\right)dx$

$\displaystyle = \frac{1}{2}x^2 + 4x + 4\log|x| + C$

(6) $\displaystyle\int \frac{(x-1)^2}{\sqrt{x}}\,dx = \int\left(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx$

$\displaystyle = \frac{2}{5}x^{\frac{5}{2}} - \frac{4}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

$\displaystyle = \frac{2}{5}x^2\sqrt{x} - \frac{4}{3}x\sqrt{x} + 2\sqrt{x} + C$

126

(1) $\displaystyle\int (4\sin x - 5\cos x)\,dx = -4\cos x - 5\sin x + C$

(2) $\displaystyle\int (3\cos x + \sin 2x)\,dx = 3\sin x - \frac{1}{2}\cos 2x + C$

(3) $\displaystyle\int \frac{1+\cos^3 x}{\cos^2 x}\,dx = \int\left(\frac{1}{\cos^2 x} + \cos x\right)dx = \tan x + \sin x + C$

(4) $\displaystyle\int\left(1-\tan^2 x\right)dx = \int\left\{1-\left(\frac{1}{\cos^2 x} - 1\right)\right\}dx = 2x - \tan x + C$

127

(1) $\displaystyle\int \left(2e^x - x^3\right)dx = 2e^x - \frac{1}{4}x^4 + C$

(2) $\displaystyle\int \left(3^x\log 3 - 1\right)dx = 3^x - x + C$

(3) $\displaystyle\int e^{\frac{x}{2}}dx = 2e^{\frac{x}{2}} + C$

(4) $\displaystyle\int 2^{x+1}dx = \frac{2^{x+1}}{\log 2} + C$

128

(1) $\displaystyle\int \sqrt[3]{2x-5}\,dx = \int (2x-5)^{\frac{1}{3}}dx$

$\displaystyle = \frac{3}{8}(2x-5)^{\frac{4}{3}} + C$

$\displaystyle = \frac{3}{8}(2x-5)\sqrt[3]{2x-5} + C$

(2) $2x-1 = t$ とおくと $x = \dfrac{t+1}{2}$

$\dfrac{dx}{dt} = \dfrac{1}{2}$ より $dx = \dfrac{1}{2}dt$

$\displaystyle\int x(2x-1)^3 dx = \int \frac{t+1}{2}\cdot t^3\cdot\frac{1}{2}\,dt$

$\displaystyle = \frac{1}{4}\int\left(t^4 + t^3\right)dt$

$\displaystyle = \frac{1}{20}(2x-1)^5 + \frac{1}{16}(2x-1)^4 + C$

(3) $\displaystyle\int \frac{dx}{(2+3x)^3} = \int (2+3x)^{-3}dx$

$\displaystyle = -\frac{1}{6(2+3x)^2} + C$

(4) $x+2 = t$ とおくと $x = t-2$

$\dfrac{dx}{dt} = 1$ より $dx = dt$

$\displaystyle\int \frac{x}{(x+2)^2}\,dx = \int\frac{t-2}{t^2}\,dt = \int\left(\frac{1}{t} - \frac{2}{t^2}\right)dt$

$= \log|t| + 2t^{-1} + C$

$\displaystyle = \log|x+2| + \frac{2}{x+2} + C$

(5) $\displaystyle\int \sin\left(\frac{1}{3}x+\pi\right)dx = -3\cos\left(\frac{1}{3}x+\pi\right)+C$

(6) $\displaystyle\int \cos\frac{3}{2}\pi x\,dx = \frac{2}{3\pi}\sin\frac{3}{2}\pi x + C$

(7) $\displaystyle\int e^{-2x+1}dx = -\frac{1}{2}e^{-2x+1}+C$

(8) $\displaystyle\int 2^{4x-1}dx = \frac{2^{4x-1}}{4\log 2}+C$

(9) $\displaystyle\int 5^{1-x}dx = -\frac{5^{1-x}}{\log 5}+C$

129

(1) $\displaystyle\int \frac{x}{x^2+1}dx = \frac{1}{2}\int\frac{\left(x^2+1\right)'}{x^2+1}dx$

$\displaystyle= \frac{1}{2}\log\left(x^2+1\right)+C$

(2) $\displaystyle\int \frac{e^x}{e^x+1}dx = \int\frac{\left(e^x+1\right)'}{e^x+1}dx$

$\displaystyle= \log\left(e^x+1\right)+C$

(3) $\displaystyle\int \frac{\sin x}{1-\cos x}dx = \int\frac{\left(1-\cos x\right)'}{1-\cos x}dx$

$\displaystyle= \log\left|1-\cos x\right|+C$

(4) $\displaystyle\int \tan x\,dx = -\int\frac{\left(\cos x\right)'}{\cos x}dx$

$\displaystyle= -\log\left|\cos x\right|+C$

(5) $\displaystyle\int \frac{\sin 2x}{\sin^2 x+1}dx = \int\frac{\left(\sin^2 x+1\right)'}{\sin^2 x+1}dx$

$\displaystyle= \log\left(\sin^2 x+1\right)+C$

(6) $\displaystyle\int \frac{2^x\log 2-1}{2^x-x}dx = \int\frac{\left(2^x-x\right)'}{2^x-x}dx$

$\displaystyle= \log\left|2^x-x\right|+C$

130

(1) $\displaystyle\int xe^{2x}dx = \int x\left(\frac{1}{2}e^{2x}\right)'dx$

$\displaystyle= x\left(\frac{1}{2}e^{2x}\right)-\frac{1}{2}\int e^{2x}dx$

$\displaystyle= \frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$

(2) $\displaystyle\int \log 2x\,dx = \int x'\log 2x\,dx$

$\displaystyle= x\log 2x - \int x\cdot\frac{1}{2x}\times 2dx$

$\displaystyle= x\log 2x - x + C$

（別解） $\displaystyle\int \log 2x\,dx = \int\left(\log x+\log 2\right)dx$

$\displaystyle= \int \log x\,dx + \log 2\int dx$

$\displaystyle= x\log x - x + x\log 2 + C$

$\displaystyle= x\log 2x - x + C$

131

(1) $\displaystyle\int \frac{x-3}{x+1}\,dx = \int \frac{x+1-4}{x+1}\,dx$

$\displaystyle = \int \left(1-\frac{4}{x+1}\right)dx$

$= x - 4\log|x+1| + C$

(2) $\displaystyle\int \frac{x^2-1}{x^2+x}\,dx = \int \frac{(x+1)(x-1)}{x(x+1)}\,dx$

$\displaystyle = \int \left(1-\frac{1}{x}\right)dx$

$= x - \log|x| + C$

(3) $\displaystyle\int \frac{x^2-x+1}{x+1}\,dx = \int \left(x-2+\frac{3}{x+1}\right)dx$

$\displaystyle = \frac{1}{2}x^2 - 2x + 3\log|x+1| + C$

(4) $\displaystyle\int \frac{1}{x(x-2)}\,dx = \frac{1}{2}\int \left(\frac{1}{x-2}-\frac{1}{x}\right)dx$

$\displaystyle = \frac{1}{2}\left(\log|x-2| - \log|x|\right) + C$

$\displaystyle = \frac{1}{2}\log\left|\frac{x-2}{x}\right| + C$

(5) $\displaystyle\frac{3x-1}{(x+3)(x-1)} = \frac{a}{x+3} + \frac{b}{x-1}$ とおくと

$3x-1 = a(x-1) + b(x+3)$

これが x についての恒等式だから

$\displaystyle a = \frac{5}{2},\ \ b = \frac{1}{2}$

$\displaystyle\int \frac{3x-1}{(x+3)(x-1)}\,dx$

$\displaystyle = \int \left(\frac{5}{2}\cdot\frac{1}{x+3} + \frac{1}{2}\cdot\frac{1}{x-1}\right)dx$

$\displaystyle = \frac{5}{2}\log|x+3| + \frac{1}{2}\log|x-1| + C$

(6) $\displaystyle\frac{1}{x(x-1)^2} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$ とおくと

$1 = a(x-1)^2 + bx(x-1) + cx$

これが x についての恒等式だから

$a = 1,\ \ b = -1,\ \ c = 1$

$\displaystyle\int \frac{1}{x(x-1)^2}\,dx = \int \left\{\frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2}\right\}dx$

$\displaystyle = \log|x| - \log|x-1| - \frac{1}{x-1} + C$

$\displaystyle = \log\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C$

132

(1) $\displaystyle\int 2\cos^2 x\,dx = \int 2\cdot\frac{1+\cos 2x}{2}\,dx$

$\displaystyle = x + \frac{1}{2}\sin 2x + C$

(2) $\displaystyle\int \sin 5x\cos 3x\,dx = \frac{1}{2}\int (\sin 8x + \sin 2x)\,dx$

$\displaystyle = -\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + C$

(3) $\displaystyle\int \cos 4x\cos 2x\,dx = \frac{1}{2}\int (\cos 6x + \cos 2x)\,dx$

$\displaystyle = \frac{1}{12}\sin 6x + \frac{1}{4}\sin 2x + C$

(4) $\displaystyle\int \sin 3x\sin 2x\,dx = \frac{1}{2}\int (\cos 5x - \cos x)\,dx$

$\displaystyle = -\frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$

(5) $\displaystyle\int \frac{1}{\sqrt{x+1}+\sqrt{x-1}}\,dx$

$\displaystyle = \frac{1}{2}\int\left(\sqrt{x+1}-\sqrt{x-1}\right)dx$

$\displaystyle = \frac{1}{3}\left\{\sqrt{(x+1)^3}-\sqrt{(x-1)^3}\right\}+C$

(6) $e^x+2=t$ とおくと $\dfrac{dt}{dx}=e^x$

より $\displaystyle dx=\frac{1}{e^x}\,dt=\frac{1}{t-2}\,dt$

$\displaystyle \int\frac{1}{e^x+2}\,dx=\int\frac{1}{t(t-2)}\,dt$

$\displaystyle = \frac{1}{2}\int\left(\frac{1}{t-2}-\frac{1}{t}\right)dt$

$\displaystyle = \frac{1}{2}\left\{\log|t-2|-\log|t|\right\}+C$

$\displaystyle = \frac{1}{2}\log\left|\frac{t-2}{t}\right|+C$

$\displaystyle = \frac{1}{2}\log\frac{e^x}{e^x+2}+C$

B 問題

133

(1) $\displaystyle\int x\cos 3x\,dx=\int x\left(\frac{1}{3}\sin 3x\right)'dx$

$\displaystyle = x\left(\frac{1}{3}\sin 3x\right)-\frac{1}{3}\int\sin 3x\,dx$

$\displaystyle = \frac{1}{3}x\sin 3x+\frac{1}{9}\cos 3x+C$

(2) $\displaystyle\int\log_3 x\,dx=\int x'\log_3 x\,dx$

$\displaystyle = x\log_3 x-\int x\cdot\frac{1}{x\log 3}\,dx$

$\displaystyle = x\log_3 x-\frac{x}{\log 3}+C$

134

(1) $\displaystyle\int\frac{x-\sqrt[4]{x}}{\sqrt{x}}\,dx=\int\left(x^{\frac{1}{2}}-x^{-\frac{1}{4}}\right)dx$

$\displaystyle = \frac{2}{3}x^{\frac{3}{2}}-\frac{4}{3}x^{\frac{3}{4}}+C$

$\displaystyle = \frac{2}{3}x\sqrt{x}-\frac{4}{3}\sqrt[4]{x^3}+C$

(2) $\displaystyle\int\frac{\left(2\sqrt{x}-1\right)^3}{x}\,dx=\int\frac{8x^{\frac{3}{2}}-12x+6x^{\frac{1}{2}}-1}{x}\,dx$

$\displaystyle = \int\left(8x^{\frac{1}{2}}-12+6x^{-\frac{1}{2}}-x^{-1}\right)dx$

$\displaystyle = \frac{16}{3}x^{\frac{3}{2}}-12x+12x^{\frac{1}{2}}-\log|x|+C$

$\displaystyle = \frac{16}{3}x\sqrt{x}-12x+12\sqrt{x}-\log|x|+C$

(3) $\displaystyle\int\sqrt{x}\left(1-\frac{1}{\sqrt[3]{x}}\right)dx$

$\displaystyle = \int\left(x^{\frac{1}{2}}-x^{\frac{1}{6}}\right)dx=\frac{2}{3}x^{\frac{3}{2}}-\frac{6}{7}x^{\frac{7}{6}}+C$

$\displaystyle = \frac{2}{3}x\sqrt{x}-\frac{6}{7}x\sqrt[6]{x}+C$

(4) $\displaystyle\int\frac{x-1}{\sqrt{x}+1}\,dx=\int\frac{(x-1)\left(\sqrt{x}-1\right)}{\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)}\,dx$

$\displaystyle = \int\left(\sqrt{x}-1\right)dx$

$\displaystyle = \int\left(x^{\frac{1}{2}}-1\right)dx=\frac{2}{3}x^{\frac{3}{2}}-x+C$

$\displaystyle = \frac{2}{3}x\sqrt{x}-x+C$

(5) $\displaystyle\int(1+\tan x)\cos x\,dx=\int(\cos x+\sin x)\,dx$

$= \sin x-\cos x+C$

(6) $\displaystyle\int\frac{\sin^2 x}{1-\cos x}\,dx=\int\frac{1-\cos^2 x}{1-\cos x}\,dx$

$\displaystyle = \int(1+\cos x)\,dx=x+\sin x+C$

135

(1) $\int(\sin x+\cos x)^2 dx = \int(1+2\sin x\cos x)dx$
$= \int(1+\sin 2x)dx$
$= x-\dfrac{1}{2}\cos 2x + C$

(2) $\int \dfrac{x+\cos 2x+1}{x\cos^2 x}dx = \int \dfrac{x+2\cos^2 x}{x\cos^2 x}dx$
$= \int\left(\dfrac{1}{\cos^2 x}+\dfrac{2}{x}\right)dx$
$= \tan x+\log x^2 + C$

(3) $\int \dfrac{1}{\sin^2 2x}dx = -\dfrac{1}{2}\dfrac{1}{\tan 2x}+C$

(4) $\int \dfrac{1}{\cos^2(2-4x)}dx = -\dfrac{1}{4}\tan(2-4x)+C$

(5) $\int(e^x+e^{-x})^2 dx = \int(e^{2x}+2+e^{-2x})dx$
$= \dfrac{1}{2}e^{2x}+2x-\dfrac{1}{2}e^{-2x}+C$

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(1) $\int(2^x+1)^2 dx = \int(2^{2x}+2^{x+1}+1)dx$
$= \dfrac{2^{2x}}{2\log 2}+\dfrac{2^{x+1}}{\log 2}+x+C$

(2) $\int \dfrac{9^x-1}{3^x-1}dx = \int \dfrac{(3^x+1)(3^x-1)}{3^x-1}dx$
$= \int(3^x+1)dx = \dfrac{3^x}{\log 3}+x+C$

(3) $\int \dfrac{e^{3x}+1}{e^x+1}dx = \int \dfrac{(e^x+1)(e^{2x}-e^x+1)}{e^x+1}dx$
$= \int(e^{2x}-e^x+1)dx$
$= \dfrac{1}{2}e^{2x}-e^x+x+C$

137

(1) $x^2-1=t$ とおくと $\dfrac{dt}{dx}=2x$ より

$x\,dx = \dfrac{1}{2}dt$

$\int x\sqrt{x^2-1}\,dx = \dfrac{1}{2}\int\sqrt{t}\,dt = \dfrac{1}{3}t^{\frac{3}{2}}+C$
$= \dfrac{1}{3}(x^2-1)\sqrt{x^2-1}+C$

(2) $\cos x = t$ とおくと $\dfrac{dt}{dx}=-\sin x$

より $\sin x\,dx = -dt$

$\int \cos^4 x\sin x\,dx = -\int t^4 dt$
$= -\dfrac{1}{5}\cos^5 x + C$

(3) $\log x = t$ とおくと $\dfrac{dt}{dx} = \dfrac{1}{x}$ より

$$\frac{1}{x}\,dx = dt$$

$$\int \frac{1}{x\log x}\,dx = \int \frac{1}{t}\,dt = \log|t| + C$$

$$= \log|\log x| + C$$

(4) $\cos x = t$ とおくと $\dfrac{dt}{dx} = -\sin x$

より $\sin x\,dx = -dt$

$$\int \sin^3 x\,dx = \int \left(1 - \cos^2 x\right)\sin x\,dx$$

$$= -\int \left(1 - t^2\right)dt = -t + \frac{1}{3}t^3 + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

（別解）3 倍角の公式

$$\sin 3x = 3\sin x - 4\sin^3 x$$

より $\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

$$\int \sin^3 x\,dx = \frac{1}{4}\int \left(3\sin x - \sin 3x\right)dx$$

$$= \frac{1}{12}\cos 3x - \frac{3}{4}\cos x + C$$

(5) $\cos x = t$ とおくと $\dfrac{dt}{dx} = -\sin x$

より $\sin x\,dx = -dt$

$$\int \frac{\tan x}{\cos x}\,dx = \int \frac{\sin x}{\cos^2 x}\,dx = -\int \frac{1}{t^2}\,dt$$

$$= \frac{1}{t} + C = \frac{1}{\cos x} + C$$

(6) $e^x = t$ とおくと $\dfrac{dt}{dx} = e^x$ より

$$e^x dx = dt$$

$$\int \frac{e^x}{e^x + e^{-x}}\,dx = \int \frac{dt}{t + t^{-1}} = \frac{1}{2}\int \frac{2t}{t^2 + 1}\,dt$$

$$= \frac{1}{2}\log\left(e^{2x} + 1\right) + C$$

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(1) $$\int \frac{x}{\cos^2 x}\,dx = \int x\left(\tan x\right)'dx$$

$$= x\tan x - \int \tan x\,dx$$

$$= x\tan x + \int \frac{\left(\cos x\right)'}{\cos x}\,dx$$

$$= x\tan x + \log|\cos x| + C$$

(2) $$\int x^2\cos x\,dx = \int x^2\left(\sin x\right)'dx$$

$$= x^2\sin x - 2\int x\sin x\,dx$$

$$= x^2\sin x - 2\left\{x\left(-\cos x\right) + \int \cos x\,dx\right\}$$

$$= x^2\sin x + 2x\cos x - 2\sin x + C$$

(3) $$\int x^3e^x dx = \int x^3\left(e^x\right)'dx$$

$$= x^3e^x - 3\int x^2e^x dx$$

$$= x^3e^x - 3\left(x^2e^x - 2\int xe^x dx\right)$$

$$= x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x dx\right)$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

$$= \left(x^3 - 3x^2 + 6x - 6\right)e^x + C$$

(4) $$\int x\log\left(x^2+1\right)dx = \int \left(\frac{x^2+1}{2}\right)'\log\left(x^2+1\right)dx$$

$$= \frac{x^2+1}{2}\log\left(x^2+1\right) - \int \frac{x^2+1}{2}\cdot\frac{2x}{x^2+1}\,dx$$

$$= \frac{1}{2}\left(x^2+1\right)\log\left(x^2+1\right) - \int x\,dx$$

$$= \frac{1}{2}\left(x^2+1\right)\log\left(x^2+1\right) - \frac{1}{2}x^2 + C$$

139

(1) $\cos x = t$ とおくと $\dfrac{dt}{dx} = -\sin x$

より $\sin x\, dx = -dt$

$$\int \frac{\sin^5 x}{1+\cos x}\,dx = \int \frac{\sin x\left(1-\cos^2 x\right)^2}{1+\cos x}\,dx$$

$$= -\int \frac{\left(1+t^2\right)^2}{1+t}\,dt$$

$$= \int \left(-t^3+t^2+t-1\right)dt$$

$$= -\frac{1}{4}\cos^4 x + \frac{1}{3}\cos^3 x + \frac{1}{2}\cos^2 x - \cos x + C$$

(2) $\sin x = t$ とおくと $\dfrac{dt}{dx} = \cos x$

より $\cos x\, dx = dt$

$$\int \left(\sin x + \cos^2 x\right)\cos x\, dx = \int \left(\sin x + 1 - \sin^2 x\right)\cos x\, dx$$

$$= \int \left(t+1-t^2\right)dt$$

$$= \frac{t^2}{2} + t - \frac{t^3}{3} + C$$

$$= -\frac{1}{3}\sin^3 x + \frac{1}{2}\sin^2 x + \sin x + C$$

(3) $e^x + 1 = t$ とおくと $\dfrac{dt}{dx} = e^x$

より $e^x dx = dt$

$$\int \frac{e^x \cdot e^x}{\sqrt{e^x+1}}\,dx = \int \frac{t-1}{\sqrt{t}}\,dt = \int \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)dt$$

$$= \frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}\left(t-3\right)t^{\frac{1}{2}} + C$$

$$= \frac{2}{3}\left(e^x - 2\right)\sqrt{e^x+1} + C$$

(4) $\log x + 1 = t$ とおくと $\dfrac{dt}{dx} = \dfrac{1}{x}$

より $\dfrac{1}{x}\,dx = dt$

$$\int \frac{\log x}{\log x + 1}\cdot\frac{1}{x}\,dx = \int \frac{t-1}{t}\,dt = \int \left(1-\frac{1}{t}\right)dt$$

$$= t - \log\left|t\right| + C$$

$$= \log x + 1 - \log\left|\log x + 1\right| + C$$

$$= \log x - \log\left|\log x + 1\right| + C$$

(5) $$\int \frac{2x^2+x-1}{x^3+1}\,dx = \int \frac{\left(x+1\right)\left(2x-1\right)}{\left(x+1\right)\left(x^2-x+1\right)}\,dx$$

$$= \int \frac{\left(x^2-x+1\right)'}{x^2-x+1}\,dx$$

$$= \log\left(x^2-x+1\right) + C$$

140

(1) $\sin x = t$ とおくと $\dfrac{dt}{dx} = \cos x$ より

$\cos x\,dx = dt$

$$\int \frac{\cos x}{4-\sin^2 x}dx = \int \frac{dt}{4-t^2} = \frac{1}{4}\int\left(\frac{1}{t+2}-\frac{1}{t-2}\right)dt$$

$$= \frac{1}{4}\left\{\log|t+2| - \log|t-2|\right\} + C$$

$$= \frac{1}{4}\log\left|\frac{t+2}{t-2}\right| + C$$

$$= \frac{1}{4}\log\left|\frac{\sin x+2}{\sin x-2}\right| + C$$

$$= \frac{1}{4}\log\left(\frac{2+\sin x}{2-\sin x}\right) + C$$

(2) $\displaystyle\int \frac{1}{\cos x}dx = \int \frac{\cos x}{\cos^2 x}dx = \int \frac{\cos x}{1-\sin^2 x}dx$

$\sin x = t$ とおけば $\dfrac{dt}{dx} = \cos x$ より

$\cos x\,dx = dt$

$$\int \frac{1}{\cos x}dx = \int \frac{1}{1-t^2}dt = \frac{1}{2}\int\left(\frac{1}{t+1}-\frac{1}{t-1}\right)dt$$

$$= \frac{1}{2}\left(\log|t+1| - \log|t-1|\right) + C$$

$$= \log\sqrt{\left|\frac{\sin x+1}{\sin x-1}\right|} + C$$

$$= \log\sqrt{\frac{1+\sin x}{1-\sin x}} + C$$

(3) $e^x + 1 = t$ とおくと $\dfrac{dt}{dx} = e^x$ より

$$dx = \frac{1}{e^x}dt = \frac{1}{t-1}dt$$

$$\int \frac{e^{3x}}{\left(e^x+1\right)^2}dx = \int \frac{(t-1)^3}{t^2(t-1)}dt$$

$$= \int \frac{\left(t^2-2t+1\right)(t-1)}{t^2(t-1)}dt = \int\left(1-\frac{2}{t}+\frac{1}{t^2}\right)dt$$

$$= t - 2\log|t| - \frac{1}{t} + C$$

$$= e^x - 2\log\left(e^x+1\right) - \frac{1}{e^x+1} + C$$

141

(i) $\displaystyle\int \cos^4 x\,dx = \int\left(\cos^2 x\right)^2 dx = \int\left(\frac{1+\cos 2x}{2}\right)^2 dx$

$$= \frac{1}{4}\int\left(1+2\cos 2x+\cos^2 2x\right)dx$$

$$= \frac{1}{4}\int\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

(ⅱ) $\int \cos^4 x\,dx = \int \cos^3 x \cos x\,dx = \int \cos^3 x (\sin x)' dx$

$$= \cos^3 x \sin x - \int (\cos^3 x)' \sin x\,dx$$

$$= \cos^3 x \sin x - \int (-3\cos^2 x \sin x) \sin x\,dx$$

$$= \cos^3 x \sin x + 3\int (\sin x \cos x)^2 dx$$

$$= \cos^3 x \sin x + \frac{3}{4}\int \sin^2 2x dx$$

$$= \cos^3 x \sin x + \frac{3}{4}\int \frac{1-\cos 4x}{2} dx$$

$$= \cos^3 x \sin x + \frac{3}{8}x - \frac{3}{32}\sin 4x + C$$

(参考) (ⅱ) の答えは次のように変形することによって

(ⅰ) の答えと同じであることがわかる。

$$\frac{3}{8}x + (\sin x \cos x)\cos^2 x - \frac{3}{32}\sin 4x + C$$

$$= \frac{3}{8}x + \frac{1}{2}\sin 2x \cdot \frac{1+\cos 2x}{2} - \frac{3}{32}\sin 4x + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{4}\sin 2x \cos 2x - \frac{3}{32}\sin 4x + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{8}\sin 4x - \frac{3}{32}\sin 4x + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

142

$\dfrac{x^2+4x-1}{(x^2+1)(x+2)} = \dfrac{ax+b}{x^2+1} + \dfrac{c}{x+2}$ の両辺に

$(x^2+1)(x+2)$ を掛けて

$$x^2+4x-1 = (ax+b)(x+2) + c(x^2+1)$$

$$= (a+c)x^2 + (2a+b)x + 2b + c$$

これが x についての恒等式だから

$a+c=1 \cdots$① $2a+b=4\cdots$② $2b+c=-1\cdots$③

①，③より $a-2b=2$ ②と連立して

$a=2,\ b=0$ ①から $c=-1$

$$\int \frac{x^2+4x-1}{(x^2+1)(x+2)} dx = \int \left(\frac{2x}{x^2+1} - \frac{1}{x+2}\right) dx$$

$$= \log(x^2+1) - \log|x+2| + C$$

$$= \log\left|\frac{x^2+1}{x+2}\right| + C$$

143

(1) $I = \int e^{-x} \sin x \, dx$ とおくと

$$I = \int \left(-e^{-x} \right)' \sin x \, dx = -e^{-x} \sin x + \int e^{-x} \cos x \, dx$$

ここで $\int e^{-x} \cos x \, dx = \int \left(-e^{-x} \right)' \cos x \, dx$

$$= -e^{-x} \cos x - \int e^{-x} \sin x \, dx$$

$$I = -e^{-x} \sin x + \left\{ -e^{-x} \cos x - \int e^{-x} \sin x \, dx \right\}$$

ゆえに $I = -e^{-x} \sin x - e^{-x} \cos x - I$

右辺の I を移項して両辺を 2 で割ると

$$I = -\frac{1}{2} e^{-x} \left(\sin x + \cos x \right) + C$$

(2) $\int \log\left(x + \sqrt{x^2+1} \right) dx = \int (x)' \log\left(x + \sqrt{x^2+1} \right) dx$

$$= x \log\left(x + \sqrt{x^2+1} \right) - \int x \left\{ \log\left(x + \sqrt{x^2+1} \right) \right\}' dx$$

ここで $\left\{ \log\left(x + \sqrt{x^2+1} \right) \right\}' = \dfrac{1 + \dfrac{x}{\sqrt{x^2+1}}}{x + \sqrt{x^2+1}}$

$$= \frac{1}{\sqrt{x^2+1}}$$

よって $\int \log\left(x + \sqrt{x^2+1} \right) dx = x \log\left(x + \sqrt{x^2+1} \right) - \int \dfrac{x}{\sqrt{x^2+1}} dx$

$$= x \log\left(x + \sqrt{x^2+1} \right) - \sqrt{x^2+1} + C$$

発展問題

144　点 $\left(x,\ \ y \right)$ における接線の傾きは $f'\left(x \right)$ であるから

$f'\left(x \right) = 6x^2 - 2x + 3$

よって　$f\left(x \right) = \int \left(6x^2 - 2x + 3 \right) dx$

$= 2x^3 - x^2 + 3x + C$

ここで，点 $\left(-1,\ \ 3 \right)$ を通るから　$f\left(-1 \right) = 3$　より

$-2 - 1 - 3 + C = 3 \quad \therefore \ C = 9$

したがって　$f\left(x \right) = 2x^3 - x^2 + 3x + 9$

$\therefore \ y = 2x^3 - x^2 + 3x + 9$

145　$I_n = \int x^n \left(e^x \right)' dx = x^n e^x - \int n x^{n-1} e^x dx$

$= x^n e^x - n \int x^{n-1} e^x dx$

ゆえに　$I_n = x^n e^x - n I_{n-1}$　（証明終）

１節　不定積分と定積分(2)

A 問題

146

(1) $\int_0^1 (3x^2-1)dx = \left[x^3-x\right]_0^1$

$= (1-1)-0 = 0$

(2) $\int_1^4 (x-2)(2x+1)dx = \int_1^4 (2x^2-3x-2)dx$

$= \left[\frac{2}{3}x^3-\frac{3}{2}x^2-2x\right]_1^4$

$= \frac{2}{3}(64-1)-\frac{3}{2}(16-1)-2(4-1)$

$= 42-\frac{45}{2}-6 = \frac{27}{2}$

(3) $\int_{-3}^2 (x-1)^2 dx = \int_{-3}^2 (x^2-2x+1)dx$

$= \left[\frac{1}{3}x^3-x^2+x\right]_{-3}^2$

$= \left(\frac{8}{3}-4+2\right)-(-9-9-3)$

$= \frac{65}{3}$

(4) $\int_2^1 (6x^2+2x-1)dx = \left[2x^3+x^2-x\right]_2^1$

$= (2+1-1)-(16+4-2)$

$= -16$

（別解）$\int_2^1 (6x^2+2x-1)dx = -\int_1^2 (6x^2+2x-1)dx$

$= -\left[2x^3+x^2-x\right]_1^2$

$= -\{(16+4-2)-(2+1-1)\}$

$= -16$

(5) $\int_1^3 (4x^3+2x)dx = \left[x^4+x^2\right]_1^3$

$= (81+9)-(1+1)$

$= 88$

(6) $\int_{-1}^2 (x^4+3x^2+2)dx = \left[\frac{1}{5}x^5+x^3+2x\right]_{-1}^2$

$= \left(\frac{32}{5}+8+4\right)-\left(-\frac{1}{5}-1-2\right)$

$= \frac{108}{5}$

147

(1) $\int_1^1 \left(2x^2-5x+3\right)dx = 0$

(2) $\int_{-2}^{-2} \left(x^3+x^2+x+1\right)dx = 0$

(3) $\int_{-2}^{2} \left(3x^2+2x+4\right)dx = 2\int_0^2 \left(3x^2+4\right)dx$

$= 2\left[x^3+4x\right]_0^2$

$= 2\{(8+8)-0\}$

$= 32$

(4) $\int_{-1}^{1} \left(x^2-3\right)\left(2x+5\right)dx$

$= \int_{-1}^{1} \left(2x^3+5x^2-6x-15\right)dx$

$= 2\int_0^1 \left(5x^2-15\right)dx$

$= 2\left[\frac{5}{3}x^3-15x\right]_0^1$

$= 2\left\{\left(\frac{5}{3}-15\right)-0\right\} = -\frac{80}{3}$

148

(1) $\int_0^2 \left(3x^2+2x-1\right)dx - \int_0^2 \left(3x^2-x\right)dx$

$= \int_0^2 \left\{\left(3x^2+2x-1\right)-\left(3x^2-x\right)\right\}dx$

$= \int_0^2 \left(3x-1\right)dx$

$= \left[\frac{3}{2}x^2-x\right]_0^2$

$= (6-2)-0 = 4$

(2) $\int_1^2 \left(x^3-2x^2\right)dx + \int_2^1 \left(x^3-x^2+5\right)dx$

$= \int_1^2 \left(x^3-2x^2\right)dx - \int_1^2 \left(x^3-x^2+5\right)dx$

$= \int_1^2 \left\{\left(x^3-2x^2\right)-\left(x^3-x^2+5\right)\right\}dx$

$= \int_1^2 \left(-x^2-5\right)dx$

$= \left[-\frac{1}{3}x^3-5x\right]_1^2$

$= -\frac{1}{3}(8-1)-5(2-1) = -\frac{22}{3}$

(3) $\int_1^2 \left(x^2-1\right)dx + \int_2^3 \left(x^2-1\right)dx$

$= \int_1^3 \left(x^2-1\right)dx$

$= \left[\frac{1}{3}x^3-x\right]_1^3$

$= (9-3)-\left(\frac{1}{3}-1\right) = \frac{20}{3}$

(4) $\int_{-2}^{4} \left(x^2+x\right)dx - \int_1^4 \left(x^2+x\right)dx$

$= \int_{-2}^{4} \left(x^2+x\right)dx + \int_4^1 \left(x^2+x\right)dx$

$= \int_{-2}^{1} \left(x^2+x\right)dx$

$= \left[\frac{1}{3}x^3+\frac{1}{2}x^2\right]_{-2}^1$

$= \frac{1}{3}(1+8)+\frac{1}{2}(1-4) = \frac{3}{2}$

149

(1) $\int_0^4 x\sqrt{x}\,dx = \int_0^4 x^{\frac{3}{2}}dx = \left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^4 = \frac{64}{5}$

(2) $\int_1^e \frac{1}{x}\,dx = \left[\log x\right]_1^e = 1$

(3) $\int_{-1}^{0} \frac{x+3}{x+2}\,dx = \int_{-1}^{0}\left(\frac{1}{x+2}+1\right)dx$

$= \Big[\ \log|\,x+2\,|+x\Big]_{-1}^{0}$

$= \log 2 + 1$

(4) $\int_{4}^{5} \frac{1}{(x-3)(x-2)}\,dx = \int_{4}^{5}\left(\frac{1}{x-3}-\frac{1}{x-2}\right)dx$

$= \left[\log\left|\frac{x-3}{x-2}\right|\right]_{4}^{5}$

$= \log\frac{2}{3} - \log\frac{1}{2}$

$= \log\frac{4}{3}$

(5) $\int_{0}^{8} \frac{x-5}{\sqrt[3]{x}}\,dx = \int_{0}^{8}\left(x^{\frac{2}{3}} - 5x^{-\frac{1}{3}}\right)dx$

$= \left[\frac{3}{5}x^{\frac{5}{3}} - \frac{15}{2}x^{\frac{2}{3}}\right]_{0}^{8}$

$= -\frac{54}{5}$

(6) $\int_{0}^{1} \frac{dx}{\sqrt{x+1}-\sqrt{x}} = \int_{0}^{1}\left(\sqrt{x+1}+\sqrt{x}\,\right)dx$

$= \left[\frac{2}{3}(x+1)^{\frac{3}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right]_{0}^{1}$

$= \frac{4\sqrt{2}}{3}$

150

(1) $\int_{0}^{\frac{\pi}{4}} \sin^2 x\,dx = \frac{1}{2}\int_{0}^{\frac{\pi}{4}}(1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_{0}^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}$

(2) $\int_{0}^{\frac{\pi}{3}} \tan^2 x\,dx = \int_{0}^{\frac{\pi}{3}}\left(\frac{1}{\cos^2 x} - 1\right)dx$

$= \Big[\ \tan x - x\Big]_{0}^{\frac{\pi}{3}} = \sqrt{3} - \frac{\pi}{3}$

(3) $\int_{-2}^{2}\left(e^{x}+e^{-x}\right)^2 dx = \int_{-2}^{2}\left(e^{2x}+2+e^{-2x}\right)dx$

$= \left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x}\right]_{-2}^{2}$

$= e^4 - \frac{1}{e^4} + 8$

(4) $\int_{0}^{1}\left(5^x + e^x\right)dx = \left[\frac{5^x}{\log 5} + e^x\right]_{0}^{1}$

$= \frac{4}{\log 5} + e - 1$

(5) $\int_{0}^{\frac{\pi}{3}} \sin 2x\,dx = \left[-\frac{1}{2}\cos 2x\right]_{0}^{\frac{\pi}{3}} = \frac{3}{4}$

(6) $\int_{0}^{\pi} \sin^2\frac{x}{2}\,dx = \int_{0}^{\pi}\frac{1-\cos x}{2}\,dx$

$= \frac{1}{2}\Big[x - \sin x\Big]_{0}^{\pi} = \frac{\pi}{2}$

(7) $\int_{0}^{\frac{\pi}{4}}\left(\frac{1}{\cos^2 x} - \sin x\right)dx = \Big[\ \tan x + \cos x\Big]_{0}^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}$

(8) $\int_{0}^{\frac{\pi}{3}} \frac{\sin 2x}{\cos x}\,dx = \int_{0}^{\frac{\pi}{3}} \frac{2\sin x\cos x}{\cos x}\,dx$

$= \Big[\ -2\cos x\Big]_{0}^{\frac{\pi}{3}} = 1$

151

(1) $\int_{0}^{1}(3x+1)^2\,dx = \left[\frac{1}{9}(3x+1)^3\right]_{0}^{1} = 7$

(2) $\int_{1}^{3} \frac{1}{(x+1)^2}\,dx = \int_{1}^{3}(x+1)^{-2}\,dx$

$= \left[-\frac{1}{x+1}\right]_{1}^{3} = \frac{1}{4}$

(3) $\sqrt{1-x}=t$ とおくと

$x=1-t^2$ より $dx=(-2t)dt$

x	$0\to1$
t	$1\to0$

$$\int_0^1 (x+1)\sqrt{1-x}\,dx=\int_1^0 (2-t^2)t(-2t)dt$$
$$=\int_0^1 (4t^2-2t^4)dt$$
$$=\left[\frac{4}{3}t^3-\frac{2}{5}t^5\right]_0^1$$
$$=\frac{4}{3}-\frac{2}{5}=\frac{14}{15}$$

(4) $x^2-1=t$ とおくと $2x\dfrac{dx}{dt}=1$ より

$x\,dx=\dfrac{1}{2}dt$

x	$1\to3$
t	$0\to8$

$$\int_1^3 x\sqrt{x^2-1}\,dx=\frac{1}{2}\int_0^8 \sqrt{t}\,dt$$
$$=\frac{1}{3}\left[t^{\frac{3}{2}}\right]_0^8=\frac{16\sqrt{2}}{3}$$

(5) $x^2+1=t$ とおくと $2x\dfrac{dx}{dt}=1$ より

$x\,dx=\dfrac{1}{2}dt$

x	$-1\to0$
t	$2\to1$

$$\int_{-1}^0 x(x^2+1)^3dx=\frac{1}{2}\int_2^1 t^3dt$$
$$=\frac{1}{8}\left[t^4\right]_2^1=-\frac{15}{8}$$

(6) $x^3-3x^2+1=t$ とおくと

$3(x^2-2x)\dfrac{dx}{dt}=1$ より

$(x^2-2x)dx=\dfrac{1}{3}dt$

x	$0\to2$
t	$1\to-3$

$$\int_0^2 \frac{x^2-2x}{x^3-3x^2+1}dx=\frac{1}{3}\int_1^{-3}\frac{1}{t}dt$$
$$=\frac{1}{3}\Big[\log|t|\Big]_1^{-3}=\frac{1}{3}\log3$$

152

(1)
$$\int_0^{\frac{\pi}{2}}\sin3x\cos x\,dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin4x+\sin2x)dx$$
$$=\frac{1}{2}\left[-\frac{\cos4x}{4}-\frac{\cos2x}{2}\right]_0^{\frac{\pi}{2}}$$
$$=\frac{1}{2}$$

(2) $t=\cos x$ とおくと $\dfrac{dt}{dx}=-\sin x$ より

$\sin x\,dx=-dt$

x	$0\to\frac{\pi}{3}$
t	$1\to\frac{1}{2}$

$$\int_0^{\frac{\pi}{3}}\sin x\cos^2x\,dx=-\int_1^{\frac{1}{2}}t^2dt$$
$$=\int_{\frac{1}{2}}^1 t^2dt=\left[\frac{1}{3}t^3\right]_{\frac{1}{2}}^1=\frac{7}{24}$$

(3) $e^x=t$ とおくと $e^x\dfrac{dx}{dt}=1$ より

$e^xdx=dt$

x	$0\to1$
t	$1\to e$

$$\int_0^1\frac{e^{2x}}{e^x+1}dx=\int_1^e\frac{t}{t+1}dt$$
$$=\int_1^e\left(1-\frac{1}{t+1}\right)dt$$
$$=\Big[t-\log|t+1|\Big]_1^e$$
$$=e-1+\log\frac{2}{e+1}$$

(4) $e^x=t$ とおくと $e^x\dfrac{dx}{dt}=1$ より

$e^xdx=dt$

x	$1\to2$
t	$e\to e^2$

$$\int_1^2\frac{1}{e^x-1}dx=\int_1^2\frac{e^x}{e^x(e^x-1)}dx$$
$$=\int_e^{e^2}\frac{1}{t(t-1)}dt=\int_e^{e^2}\left(\frac{1}{t-1}-\frac{1}{t}\right)dt$$
$$=\left[\log\left|\frac{t-1}{t}\right|\right]_e^{e^2}=\log\frac{e^2-1}{e^2}-\log\frac{e-1}{e}$$
$$=\log\frac{e+1}{e}$$

(5) $\log x = t$ とおくと $\dfrac{1}{x}\dfrac{dx}{dt} = 1$ より

$\dfrac{1}{x}dx = dt$

x	$e^2 \to e^3$
t	$2 \to 3$

$$\int_{e^2}^{e^3} \frac{1}{x\log x}dx = \int_2^3 \frac{1}{t}dt = \Big[\log|t|\Big]_2^3$$

$$= \log 3 - \log 2 = \log\frac{3}{2}$$

(6) $x^2 = t$ とおくと $2x\dfrac{dx}{dt} = 1$ より

$x\,dx = \dfrac{1}{2}dt$

x	$0 \to 1$
t	$0 \to 1$

$$\int_0^1 xe^{x^2}dx = \frac{1}{2}\int_0^1 e^t dt$$

$$= \frac{1}{2}\Big[e^t\Big]_0^1 = \frac{1}{2}(e-1)$$

153

(1) $$\int_0^{\pi} x\sin x\,dx = \int_0^{\pi} x(-\cos x)'dx$$

$$= \Big[x(-\cos x)\Big]_0^{\pi} + \int_0^{\pi}\cos x\,dx$$

$$= \pi + \Big[\sin x\Big]_0^{\pi} = \pi$$

(2) $$\int_{-1}^{0} xe^{-x}dx = \int_{-1}^{0} x(-e^{-x})'dx$$

$$= \Big[x(-e^{-x})\Big]_{-1}^{0} + \int_{-1}^{0} e^{-x}dx$$

$$= -e - \Big[e^{-x}\Big]_{-1}^{0} = -1$$

(3) $$\int_0^2 \log(x+1)dx = \int_0^2 (x+1)'\cdot\log(x+1)dx$$

$$= \Big[(x+1)\log(x+1)\Big]_0^2 - \int_0^2 \frac{x+1}{x+1}dx$$

$$= 3\log 3 - 2$$

(4) $$\int_1^e x\log x\,dx = \left[\frac{x^2}{2}\log x\right]_1^e - \int_1^e \frac{1}{2}x\,dx$$

$$= \frac{1}{2}e^2 - \frac{1}{4}\Big[x^2\Big]_1^e = \frac{1}{4}(e^2+1)$$

(5) $$\int_0^{\frac{\pi}{2}} (x+1)\cos x\,dx = \int_0^{\frac{\pi}{2}} (x+1)(\sin x)'dx$$

$$= \Big[(x+1)\sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= \frac{\pi}{2} + 1 + \Big[\cos x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

(6) $$\int_{\frac{1}{e}}^{1} x^2\log x\,dx = \int_{\frac{1}{e}}^{1}\left(\frac{1}{3}x^3\right)'\log x\,dx$$

$$= \left[\frac{1}{3}x^3\log x\right]_{\frac{1}{e}}^{1} - \frac{1}{3}\int_{\frac{1}{e}}^{1} x^2dx$$

$$= \frac{1}{3e^3} - \frac{1}{9}\Big[x^3\Big]_{\frac{1}{e}}^{1} = \frac{1}{9}\left(\frac{4}{e^3}-1\right)$$

154

(1) $$\int_1^3 x(x-3)^4dx$$

$$= \int_1^3 x\left\{\frac{1}{5}(x-3)^5\right\}'dx$$

$$= \left[x\cdot\frac{1}{5}(x-3)^5\right]_1^3 - \frac{1}{5}\int_1^3 (x-3)^5dx$$

$$= \frac{32}{5} - \frac{1}{30}\Big[(x-3)^6\Big]_1^3 = \frac{128}{15}$$

（参考） $$\int_1^3 x(x-3)^4dx$$

$$= \int_1^3 \{(x-3)+3\}(x-3)^4dx$$

$$= \int_1^3 (x-3)^5dx + 3\int_1^3 (x-3)^4dx$$

$$= \frac{1}{6}\Big[(x-3)^6\Big]_1^3 + \frac{3}{5}\Big[(x-3)^5\Big]_1^3 = \frac{128}{15}$$

(2) $\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)^3 dx$

$= \int_{\alpha}^{\beta}(x-\alpha)\left\{\frac{1}{4}(x-\beta)^4\right\}' dx$

$= \left[(x-\alpha)\cdot\frac{1}{4}(x-\beta)^4\right]_{\alpha}^{\beta} - \frac{1}{4}\int_{\alpha}^{\beta}(x-\beta)^4 dx$

$= 0 - \frac{1}{20}\left[(x-\beta)^5\right]_{\alpha}^{\beta} = \frac{1}{20}(\alpha-\beta)^5$

（参考） $\int_{\alpha}^{\beta}(x-\alpha)(x-\beta)^3 dx$

$= \int_{\alpha}^{\beta}\{(x-\beta)-(\alpha-\beta)\}(x-\beta)^3 dx$

$= \int_{\alpha}^{\beta}(x-\beta)^4 dx - (\alpha-\beta)\int_{\alpha}^{\beta}(x-\beta)^3 dx$

$= \frac{1}{5}\left[(x-\beta)^5\right]_{\alpha}^{\beta} - \frac{\alpha-\beta}{4}\left[(x-\beta)^4\right]_{\alpha}^{\beta}$

$= \frac{1}{20}(\alpha-\beta)^5$

155

(1) $F'(x) = x\sin x$

(2) $F'(x) = (2x\sin 2x)\times(2x)' = 4x\sin 2x$

(3) $F'(x) = (3x\sin 3x)\times(3x)' - (x\sin x)\times(x)'$

$= 9x\sin 3x - x\sin x$

(4) $F'(x) = \int_0^x \sin t\, dt + x\sin x$

$= \left[-\cos t\right]_0^x + x\sin x$

$= 1 - \cos x + x\sin x$

156 $\int_0^x tf(t)dt = x^3\sin x - 2x^2$

両辺を x で微分して　　$xf(x) = 3x^2\sin x + x^3\cos x - 4x$

両辺を x で割って　　$f(x) = 3x\sin x + x^2\cos x - 4$

B 問題

157

(1) $\int_1^4 \sqrt{x^2-4x+4}\,dx$

$= \int_1^4 \sqrt{(x-2)^2}\,dx$

$= \int_1^4 |x-2|\,dx$

$= \left[-\frac{1}{2}(x-2)^2\right]_1^2 + \left[\frac{1}{2}(x-2)^2\right]_2^4 = \frac{5}{2}$

(2) $\int_0^3 \sqrt{|x-2|}\,dx$

$= \int_0^2 \sqrt{2-x}\,dx + \int_2^3 \sqrt{x-2}\,dx$

$= \left[-\frac{2}{3}(3-x)^{\frac{3}{2}}\right]_0^2 + \left[\frac{2}{3}(x-2)^{\frac{3}{2}}\right]_2^3$

$= \frac{4\sqrt{2}}{3} + \frac{2}{3}$

(3) $\int_0^{\pi}|\cos 2x|\,dx$

$= \int_0^{\frac{\pi}{4}}\cos 2x\,dx - \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\cos 2x\,dx + \int_{\frac{3}{4}\pi}^{\pi}\cos 2x\,dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} - \left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{3}{4}\pi} + \left[\frac{1}{2}\sin 2x\right]_{\frac{3}{4}\pi}^{\pi} = 2$

(4) $\int_0^{\pi}|\sin x - \cos x|\,dx$

$= \int_0^{\frac{\pi}{4}}(\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\pi}(\sin x - \cos x)dx$

$= \left[\sin x + \cos x\right]_0^{\frac{\pi}{4}} + \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\pi}$

$= 2\sqrt{2}$

158

(1) $\displaystyle\int_0^3 \frac{x}{\sqrt{x+1}+1}dx = \int_0^3 \frac{x\left(\sqrt{x+1}-1\right)}{\left(\sqrt{x+1}+1\right)\left(\sqrt{x+1}-1\right)}dx$

$$= \int_0^3 \frac{x\left(\sqrt{x+1}-1\right)}{x}dx$$

$$= \int_0^3 \left(\sqrt{x+1}-1\right)dx = \left[\frac{2}{3}(x+1)^{\frac{3}{2}} - x\right]_0^3 = \frac{5}{3}$$

(2) $\displaystyle\int_0^1 \frac{x}{x^2+3}dx = \int_0^1 \frac{1}{2}\cdot\frac{\left(x^2+3\right)'}{x^2+3}dx$

$$= \left[\frac{1}{2}\log\left(x^2+3\right)\right]_0^1 = \frac{1}{2}\log\frac{4}{3}$$

(3) $\displaystyle\int_0^{\frac{\pi}{2}} \sin\frac{3}{2}x\cos\frac{x}{2}dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin 2x + \sin x)dx$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2x - \cos x\right]_0^{\frac{\pi}{2}} = 1$$

(4) $\displaystyle\int_1^{\log 2} e^{2x}dx = \left[\frac{1}{2}e^{2x}\right]_1^{\log 2} = \frac{1}{2}\left(e^{\log 4} - e^2\right)$

$$= \frac{1}{2}\left(4 - e^2\right) = 2 - \frac{1}{2}e^2$$

(5) $\displaystyle\int_0^{\pi} \sin kx\cos lx\,dx$

$$= \frac{1}{2}\int_0^{\pi}(\sin(k+l)x + \sin(k-l)x)dx$$

(ｉ) $k \neq l$ のとき $\displaystyle\int_0^{\pi}\sin kx\cos lx\,dx = -\frac{1}{2}\left[\frac{\cos(k+l)x}{k+l} + \frac{\cos(k-l)x}{k-l}\right]_0^{\pi}$

次に，$k+l$ が偶数ならば $k-l$ も偶数なので

$$\int_0^{\pi}\sin kx\cos lx\,dx = -\frac{1}{2}\left\{\left(\frac{1}{k+l}+\frac{1}{k-l}\right)-\left(\frac{1}{k+l}+\frac{1}{k-l}\right)\right\} = 0$$

また，$k+l$ が奇数ならば $k-l$ も奇数なので

$$\int_0^{\pi}\sin kx\cos lx\,dx = -\frac{1}{2}\left\{\left(\frac{-1}{k+l}+\frac{-1}{k-l}\right)-\left(\frac{1}{k+l}+\frac{1}{k-l}\right)\right\}$$

$$= \frac{1}{k+l}+\frac{1}{k-l} = \frac{2k}{(k+l)(k-l)}$$

（ⅱ） $k = l$ のとき $\displaystyle\int_0^{\pi} \sin kx \cos lx \, dx = \frac{1}{2}\int_0^{\pi} \sin 2kx \, dx$

$$= \frac{1}{2}\left[-\frac{1}{2k}\cos 2kx\right]_0^{\pi} = 0$$

よって，$k \neq l$ で $k + l$ が偶数のとき，0

$k \neq l$ で $k + l$ が奇数のとき，$\dfrac{2k}{(k+l)(k-l)}$

$k = l$ のとき，0

159

(1) $x = 3\sin\theta$ とおくと $\dfrac{dx}{d\theta} = 3\cos\theta$ より

$dx = 3\cos\theta \, d\theta$

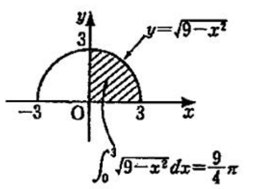
x	$0 \to 3$
t	$0 \to \frac{\pi}{2}$

$$\int_0^3 \sqrt{9 - x^2}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{9 - 9\sin^2\theta} \cdot 3\cos\theta \, d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta} \cdot \cos\theta \, d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \frac{9}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} = \frac{9}{4}\pi$$



(2) $x = 3\sin\theta$ とおくと $\dfrac{dx}{d\theta} = 3\cos\theta$ より

$dx = 3\cos\theta \, d\theta$

$x = 0$ のとき $\sin\theta = 0$

よって $\theta = 0$

x	$0 \to \frac{3}{2}$
θ	$0 \to \frac{\pi}{6}$

$x = \dfrac{3}{2}$ のとき $\sin\theta = \dfrac{1}{2}$

よって $\theta = \dfrac{\pi}{6}$

$$\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9 - x^2}} = \int_0^{\frac{\pi}{6}} \frac{3\cos\theta}{\sqrt{9 - 9\sin^2\theta}}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{3\cos\theta}{3\sqrt{1 - \sin^2\theta}}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} d\theta = \frac{\pi}{6}$$

(3) $x = \sqrt{2}\tan\theta$ とおくと $\dfrac{dx}{d\theta} = \dfrac{\sqrt{2}}{\cos^2\theta}$ より

$dx = \dfrac{\sqrt{2}}{\cos^2\theta}\, d\theta$

$x = 0$ のとき $\tan\theta = 0$

x	$0 \to \sqrt{6}$
θ	$0 \to \frac{\pi}{3}$

よって $\theta = 0$

$x = \sqrt{6}$ のとき $\tan\theta = \sqrt{3}$

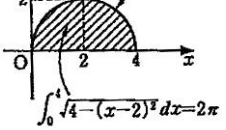
よって $\theta = \dfrac{\pi}{3}$

$$\int_0^{\sqrt{6}} \frac{1}{x^2 + 2}\, dx = \int_0^{\frac{\pi}{3}} \frac{1}{2\tan^2\theta + 2} \cdot \frac{\sqrt{2}}{\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{2(\tan^2\theta + 1)} \cdot \frac{\sqrt{2}}{\cos^2\theta}\, d\theta$$

$$= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{3}} \cos^2\theta \cdot \frac{1}{\cos^2\theta}\, d\theta$$

$$= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{3}} d\theta = \frac{\sqrt{2}}{6}\pi$$



(4) $\displaystyle\int_1^2 \frac{dx}{x^2 - 2x + 2} = \int_1^2 \frac{1}{(x-1)^2 + 1}\, dx$ より

$x - 1 = \tan\theta$ とおくと

$\dfrac{dx}{d\theta} = \dfrac{1}{\cos^2\theta}$ より $dx = \dfrac{1}{\cos^2\theta}\, d\theta$

x	$1 \to 2$
θ	$0 \to \frac{\pi}{4}$

だから

$$\int_1^2 \frac{1}{(x-1)^2 + 1}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{\tan^2\theta + 1} \cdot \frac{1}{\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta \cdot \frac{1}{\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{4}$$

(5) $\int_0^4 \sqrt{4x - x^2}\,dx = \int_0^4 \sqrt{4 - (x-2)^2}\,dx$ より $x - 2 = 2\sin\theta$ とおくと

$\dfrac{dx}{d\theta} = 2\cos\theta$ より $dx = 2\cos\theta\,d\theta$

x	$0 \to 4$
θ	$-\frac{\pi}{2} \to \frac{\pi}{2}$

だから

$$\int_0^4 \sqrt{4 - (x-2)^2}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4 - \sin^2\theta} \cdot 2\cos\theta\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sqrt{1 - \sin^2\theta} \cdot 2\cos\theta\,d\theta = 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\,d\theta = 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\pi$$

(6) $x = \tan\theta$ とおくと

$\dfrac{dx}{d\theta} = \dfrac{1}{\cos^2\theta}$ より $dx = \dfrac{1}{\cos^2\theta}\,d\theta$

x	$0 \to 1$
θ	$0 \to \frac{\pi}{4}$

だから

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{1}{1+\tan^2\theta}} \cdot \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos\theta \cdot \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\cos^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos\theta}{1 - \sin^2\theta}\,d\theta$$

ここで $t = \sin\theta$ とおくと

$\dfrac{dt}{d\theta} = \cos\theta$ より $\cos\theta\,d\theta = dt$

θ	$0 \to \frac{\pi}{4}$
t	$0 \to \frac{1}{\sqrt{2}}$

だから

$$\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{1 - \sin^2\theta}\,d\theta = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{1 - t^2}\,dt$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \frac{-1}{(t+1)(t-1)}\,dt = \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1}{t+1} - \frac{1}{t-1}\right)dt$$

$$= \frac{1}{2}\left[\log\left|\frac{t+1}{t-1}\right|\right]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{2}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \frac{1}{2}\log\left(\sqrt{2}+1\right)^2 = \log\left(\sqrt{2}+1\right)$$

（別解）　$\sqrt{1+x^2} = t - x$ とおけば

$0 \leqq x \leqq 1$ のとき $t > 1$ で

$x = \dfrac{t^2-1}{2t}$　　$\sqrt{1+x^2} = \dfrac{t^2+1}{2t}$

$\dfrac{dx}{dt} = \dfrac{t^2+1}{2t}$ より $dx = \dfrac{t^2+1}{2t^2}dt$

$t = x + \sqrt{x^2+1}$ であるから

x	$0 \to 1$
t	$1 \to 1+\sqrt{2}$

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \int_1^{1+\sqrt{2}} \frac{2t}{t^2+1} \cdot \frac{t^2+1}{2t^2}dt$$

$$= \int_1^{1+\sqrt{2}} \frac{1}{t}dt = \Big[\log|t|\Big]_1^{1+\sqrt{2}} = \log\left(1+\sqrt{2}\right)$$

160

(1) $\sin^2 x\cos^3 x = \sin^2 x\left(1-\sin^2 x\right)\cos x$

より $\sin x = t$ とおくと

$\dfrac{dt}{dx} = \cos x$ より $\cos x\,dx = dt$

x	$0 \to \frac{\pi}{2}$
t	$0 \to 1$

だから

$$\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \sin^2 x\left(1-\sin^2 x\right)\cos x\,dx$$

$$= \int_0^1 t^2\left(1-t^2\right)dt = \int_0^1 \left(t^2-t^4\right)dt$$

$$= \left[\frac{1}{3}t^3 - \frac{1}{5}t^5\right]_0^1 = \frac{2}{15}$$

(2) $$\int_0^{\frac{\pi}{4}} \frac{dx}{1+\sin x} = \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{1-\sin^2 x}dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{\cos^2 x}dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right)dx$$

$$= \left[\tan x - \frac{1}{\cos x}\right]_0^{\frac{\pi}{4}} = 2-\sqrt{2}$$

161

(1) $$\int_0^{\frac{\pi}{2}} x\sin^2 x\,dx = \int_0^{\frac{\pi}{2}} x \cdot \frac{1-\cos 2x}{2}dx$$

$$= \int_0^{\frac{\pi}{2}} x \cdot \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)' dx$$

$$= \left[x \cdot \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(x - \frac{1}{2}\sin 2x\right)dx$$

$$= \frac{\pi^2}{8} - \frac{1}{2}\left[\frac{1}{2}x^2 + \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{2}} = \frac{1}{16}\pi^2 + \frac{1}{4}$$

(2) $\int_0^1 (1-x^2)e^x dx = \int_0^1 (1-x^2)(e^x)' dx$

$$= \Big[(1-x^2)e^x \Big]_0^1 + 2\int_0^1 xe^x dx$$

$$= -1 + 2\int_0^1 xe^x dx$$

また $\int_0^1 xe^x dx = \int_0^1 x(e^x)' dx$

$$= \Big[xe^x \Big]_0^1 - \int_0^1 e^x dx = e - \Big[e^x \Big]_0^1 = 1$$

よって $\int_0^1 (1-x^2)e^x dx = -1 + 2\cdot 1 = 1$

(3) $\int_1^e x(\log x)^2 dx = \int_1^e \left(\dfrac{x^2}{2}\right)' (\log x)^2 dx$

$$= \left[\frac{x^2}{2}(\log x)^2 \right]_1^e - \int_1^e \frac{x^2}{2}\cdot 2(\log x)\frac{1}{x} dx$$

$$= \frac{1}{2}e^2 - \int_1^e x\log x\, dx$$

$$= \frac{1}{2}e^2 - \int_1^e \left(\frac{x^2}{2}\right)' \log x\, dx$$

$$= \frac{1}{2}e^2 - \left[\frac{x^2}{2}\cdot \log x \right]_1^e + \int_1^e \frac{x^2}{2}\cdot\frac{1}{x} dx$$

$$= \frac{1}{2}e^2 - \frac{1}{2}e^2 + \left[\frac{1}{4}x^2 \right]_1^e = \frac{1}{4}(e^2 - 1)$$

(4) $\int_0^1 x\log(x^2+1)dx = \left[\dfrac{x^2}{2}\log(x^2+1) \right]_0^1 - \int_0^1 \dfrac{x^2}{2}\cdot\dfrac{2x}{x^2+1} dx$

$$= \frac{1}{2}\log 2 - \int_0^1 \left(x - \frac{x}{x^2+1} \right) dx$$

$$= \frac{1}{2}\log 2 - \left[\frac{x^2}{2} - \frac{1}{2}\log(x^2+1) \right]_0^1 = \log 2 - \frac{1}{2}$$

（別解）置換積分で求めることもできる。

$x^2+1=t$ とおくと

x	$0\to 1$
t	$1\to 2$

$\dfrac{dt}{dx} = 2x$ より $x\,dx = \dfrac{1}{2}dt$

$\int_0^1 x\log(x^2+1)dx = \int_1^2 \log t \cdot \dfrac{1}{2} dt$

$$= \frac{1}{2}\int_1^2 \log t\, dt = \frac{1}{2}\Big[t\log t - t \Big]_1^2 = \log 2 - \frac{1}{2}$$

162　〔部分積分〕

$$I=\int_0^1\left\{\frac{1}{2}\log\left(x^2+1\right)\right\}'\log\left(x^2+1\right)dx$$

$$=\left[\frac{1}{2}\log\left(x^2+1\right)\cdot\log\left(x^2+1\right)\right]_0^1$$

$$-\int_0^1\frac{1}{2}\log\left(x^2+1\right)\cdot\frac{2x}{x^2+1}dx$$

$$=\frac{1}{2}(\log 2)^2-\int_0^1\frac{x}{x^2+1}\log\left(x^2+1\right)dx$$

したがって

$$I=\frac{1}{2}(\log 2)^2-I$$

$2I=\frac{1}{2}(\log 2)^2$　よって　$I=\frac{1}{4}(\log 2)^2$

〔置換積分〕

$\log\left(x^2+1\right)=t$　とおくと　$\frac{2x}{x^2+1}\cdot\frac{dx}{dt}=1$　より

$$\frac{x}{x^2+1}dx=\frac{1}{2}dt$$

x	$0\to 1$
t	$0\to\log 2$

$$I=\int_0^1\log\left(x^2+1\right)\cdot\frac{x}{x^2+1}dx$$

$$=\int_0^{\log 2}t\left(\frac{1}{2}dt\right)=\frac{1}{2}\int_0^{\log 2}t\,dt$$

$$=\frac{1}{2}\left[\frac{1}{2}t^2\right]_0^{\log 2}=\frac{1}{4}(\log 2)^2$$

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$$f'(x)=\left(x-\frac{\pi}{2}\right)\sin x$$

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	π	$\cdots$	2π
$f'(x)$	0	$-$	0	$+$	0	$-$	0
$f(x)$	0	↘	極小	↗	極大	↘	-2π

$$f(x)=\left[\left(t-\frac{\pi}{2}\right)(-\cos t)\right]_0^x+\int_0^x\cos t\,dt$$

$$=-\left(x-\frac{\pi}{2}\right)\cos x-\frac{\pi}{2}+\left[\sin t\right]_0^x$$

$$=-\left(x-\frac{\pi}{2}\right)\cos x+\sin x-\frac{\pi}{2}$$

極大値 $f(\pi)=0$, 極小値 $f\left(\frac{\pi}{2}\right)=1-\frac{\pi}{2}$

164

(1)　$F(x)=\int_0^x(x-t)\cos t\,dt$

$=x\int_0^x\cos t\,dt-\int_0^x t\cos t\,dt$　より

$$F'(x)=\int_0^x\cos t\,dt+x\cos x-x\cos x$$

$$=\left[\sin t\right]_0^x=\sin x$$

(2)　$F(x)=\int_0^x t\sin(x-t)dt$　において

$x-t=u$　とおくと　$t=x-u$

$\frac{du}{dt}=-1$　より　$dt=(-1)du$

t	$0\to x$
u	$x\to 0$

$$F(x)=\int_x^0(x-u)\sin u\cdot(-1)du$$

$$=\int_0^x(x-u)\sin u\,du$$

$$=x\int_0^x\sin u\,du-\int_0^x u\sin u\,du$$

よって　$F'(x)=\int_0^x\sin u\,du+x\sin x-x\sin x$

$$=\left[-\cos u\right]_0^x=-\cos x+1$$

(3) $F(x)=\int_0^x e^t \log\frac{x+1}{t+1}dt$

$=\int_0^x e^t\{\log(x+1)-\log(t+1)\}dt$

$=\log(x+1)\int_0^x e^t dt-\int_0^x e^t\log(t+1)dt$

よって　$F'(x)=\frac{1}{x+1}\int_0^x e^t dt+e^x\log(x+1)$

$-e^x\log(x+1)$

$=\frac{1}{x+1}\left[e^t\right]_0^x=\frac{e^x-1}{x+1}$

(4) $\int_1^2 te^{x-t}dt=e^x\int_1^2 te^{-t}dt$

ここで $\int_1^2 te^{-t}dt=\left[t\left(-e^{-t}\right)\right]_1^2+\int_1^2 e^{-t}dt$

$=-2e^{-2}+e^{-1}+\left[-e^{-t}\right]_1^2=\frac{2}{e}-\frac{3}{e^2}$

よって $F(x)=\cos x+\left(\frac{2}{e}-\frac{3}{e^2}\right)e^x$

ゆえに $F'(x)=-\sin x+\left(\frac{2}{e}-\frac{3}{e^2}\right)e^x$

165 $f(x)=x\int_0^x \sin^2 t\,dt-\int_0^x t\sin^2 t\,dt$ より

$f'(x)=\int_0^x \sin^2 t\,dt+x\sin^2 x-x\sin^2 x$

$=\int_0^x \sin^2 t\,dt$

ゆえに　$f''(x)=\sin^2 x$

166

$\int_0^x (x-t)f(t)dt=a\cos x-2\cdots$①

$x\int_0^x f(t)dt-\int_0^x tf(t)dt=a\cos x-2$

両辺を x で微分すると

$\int_0^x f(t)dt+xf(x)-xf(x)=-a\sin x$

$\int_0^x f(t)dt=-a\sin x$

さらに両辺を x で微分すると

$f(x)=-a\cos x\cdots$②

また，①において　$x=0$　とおくと

$0=a\cos 0-2$　より　$a=2$

$a=2$ を②に代入して　$f(x)=-2\cos x$